

Title: Mixed-state entanglement as a diagnostic for quasiparticles and finite-T topological order

Speakers: Tarun Grover

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Abstract: Quantum entanglement of pure states has led to new insights into a wide variety of topics. Entanglement of mixed states is however less well understood. In this talk I will focus on a few themes where mixed-state entanglement leads to new insights that are difficult to obtain otherwise. I will mainly focus on two topics: (i) Characterizing finite-temperature topological order (ii) Detecting presence/absence of quasiparticles. Time permitting, I will also discuss a relation between mixed-state entanglement and information obtained from projective measurements, and its application to characterizing multi-component systems, such as quantum disentangled liquids and Kondo lattice systems.

Mixed-state entanglement as a diagnostic for quasiparticles and finite-T topological order

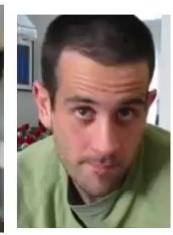
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Ben-Zion
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Lu, TG: 1808.04381 (Phys. Rev. B 2019), 1907.01569, 2008.11727.
Lu, Hsieh, TG: 1912.03313 (Phys. Rev. Lett. 2020),
Ben-Zion, McGreevy, TG: 1912.01027 (Phys. Rev. B 2020),
Wu, Lu, Chung, Kao, TG: 1912.03313 (Phys. Rev. Lett. 2020)

Funding: NSF, Sloan Research Fellowship.

**Does Quantum mechanics matter at
non-zero temperature?**



Motivation #1: Topological Order at Finite Temperature

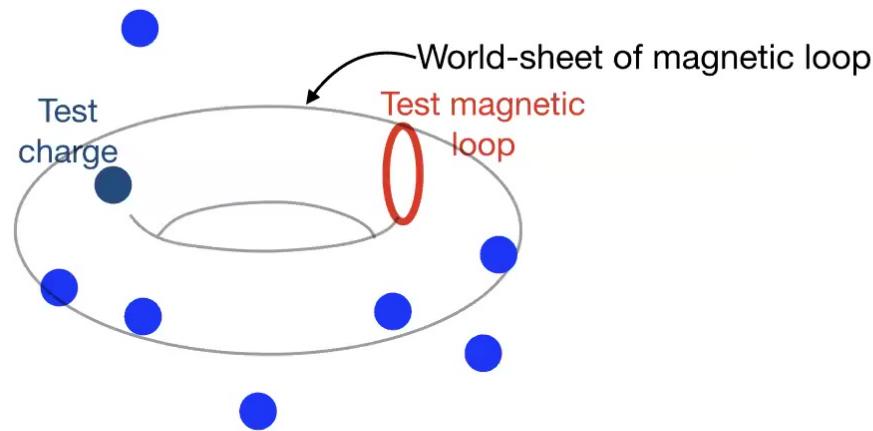
Consider 3d toric code:

At $T = 0$, one can implement braiding between a test magnetic loop and a test electric charge.



Does braiding survive at $T > 0$?

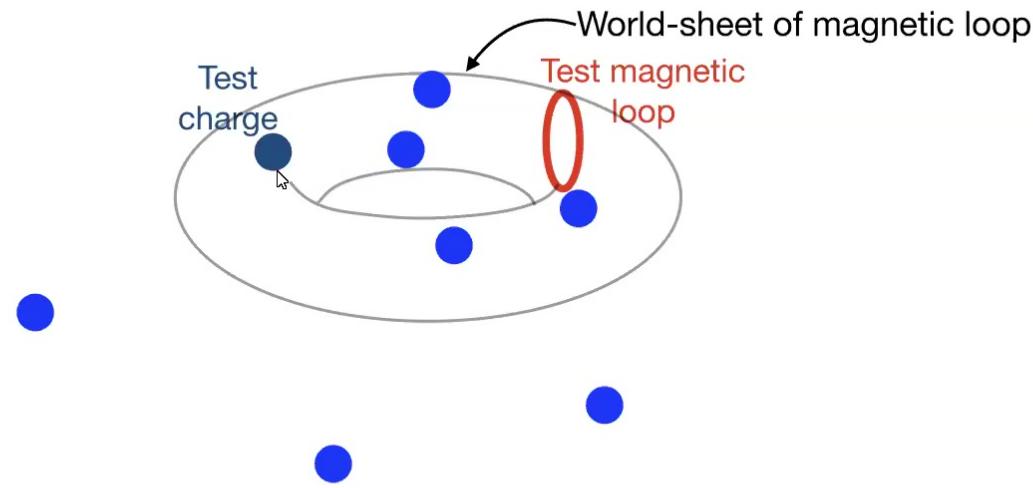
At $T > 0$, finite density of electric charges makes braiding hard to implement.¹⁴



[Dennis, Kitaev, Landahl, Preskill (2001);
Yoshida 2011; Hastings 2011]

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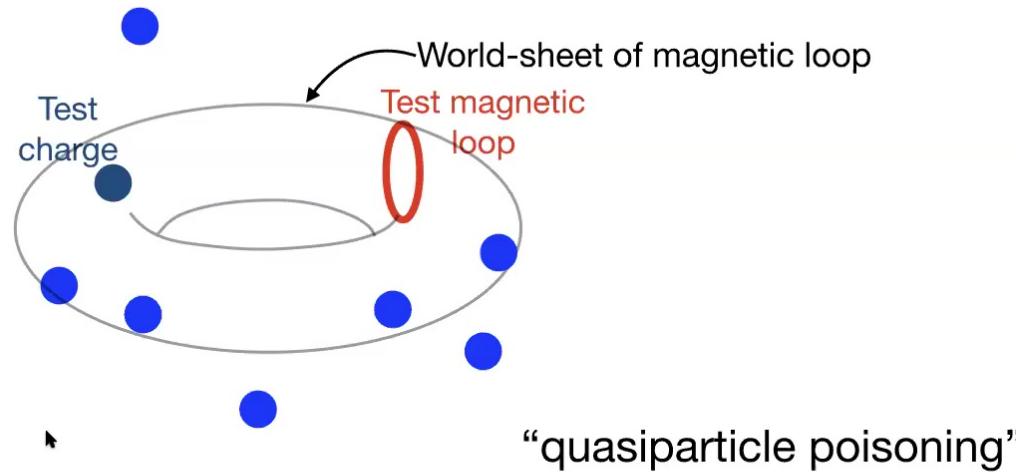
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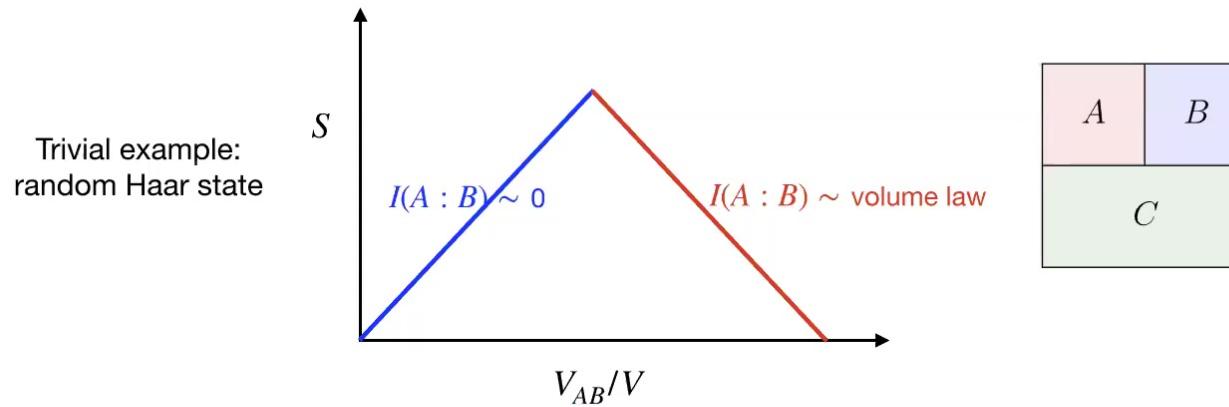


Scalable designs for quasiparticle-poisoning-protected topological quantum computation with Majorana zero modes

Torsten Karzig,¹ Christina Knapp,² Roman M. Lutchyn,¹ Parsa Bonderson,¹ Matthew B. Hastings,¹ Chetan Nayak,^{1,2} Jason Alicea,^{3,4} Karsten Flensberg,⁵ Stephan Plugge,^{5,6} Yuval Oreg,⁷ Charles M. Marcus,⁵ and Michael H. Freedman^{1,8}

Motivation #2: Detecting absence of thermalization.

When a system does not thermalize, expectation that the entanglement between two subsystems A, B must be “large”.



How to quantify this effect solely using quantum correlations?
Implications for localized, integrable and scar states?

These questions motivate understanding
many-body entanglement in *mixed states*.



Zeroth Order question:

When is a mixed state unentangled (“separable”)?

Separability Criterion for Mixed States

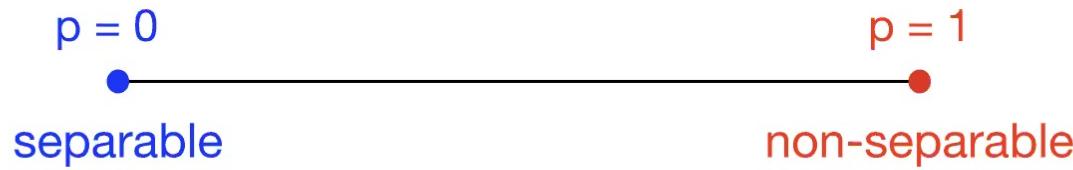
If $\rho = \sum_i p_i \rho_{i,A} \otimes \rho_{i,B}$ with $p_i > 0$
in *some* basis, then the expectation value of any
operator can be reproduced by an ensemble of
unentangled pure states. Therefore, such states
are unentangled or “separable”.

[Werner 1989]

Example:

Consider $\rho = p |\psi_{EPR}\rangle\langle\psi_{EPR}| + (1 - p) \frac{\mathbb{I}}{4}$,

where $|\psi_{EPR}\rangle = \frac{1}{\sqrt{2}} (| \uparrow \rangle | \downarrow \rangle - | \downarrow \rangle | \uparrow \rangle)$



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$$p = 0 \qquad \qquad \qquad 1/3 \qquad \qquad \qquad p = 1$$



separable

non-separable

$$\rho = \sum_i p_i \rho_{i,A} \otimes \rho_{i,B}$$

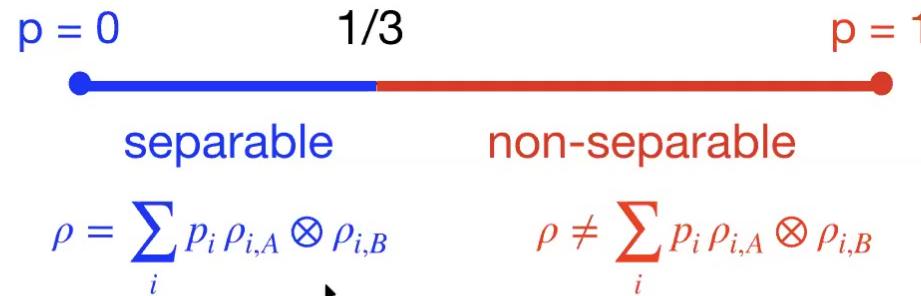
$$\rho \neq \sum_i p_i \rho_{i,A} \otimes \rho_{i,B}$$



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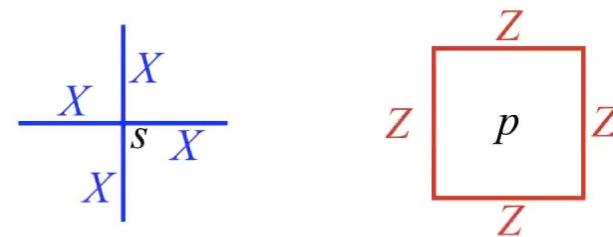


Many-body analogs of such transitions?

Separability in Toric codes

2d toric code at finite-T: $\rho = e^{-\beta H}/Z$

$$H = -\lambda_A \sum_s A_s - \lambda_B \sum_p B_p$$



Separability in Toric codes

$$\begin{aligned} \text{2d toric code at finite-T: } \rho &= e^{-\beta H}/Z \\ &= e^{-\beta E_n} |E_n\rangle\langle E_n|/Z \end{aligned}$$

Each eigenstate $|E_n\rangle$ is topologically ordered.

Not immediately obvious whether ρ is short-ranged entangled.

Let's rewrite ρ as

$$\rho = \frac{1}{Z} e^{-\beta H} = \frac{1}{Z} e^{-\beta H/2} \sum_m |m\rangle \langle m| e^{-\beta H/2} = \sum_m p_m |\phi_m\rangle \langle \phi_m|$$

Consider $\{ |m\rangle\}$ = set of all product states in the X-basis

$$|\phi_m\rangle \sim \sum_{\text{loop configs.}} e^{-\text{Area enclosed by loop} \times \log(\tanh(\beta\lambda_B/2))} |C\rangle$$

$\Rightarrow |\phi_m\rangle$ not topologically ordered at any non-zero T
since large loops suppressed.

Similar arguments lead to the following conclusion

$$e^{-\beta H}/Z = \sum_m p_m |\phi_m\rangle\langle\phi_m|$$

Not topologically ordered, area-law ground-state

whenever $T > \min(T_A, T_B)$ where T_A, T_B correspond to the critical temperatures of the classical Hamiltonians

$$-\lambda_A \sum_s A_s, -\lambda_B \sum_p B_p$$

Dimension	T_A	T_B
2D	$\frac{O(\lambda_A)}{\log L}$	$\frac{O(\lambda_B)}{\log L}$
3D	$\frac{O(\lambda_A)}{\log L}$	$O(\lambda_B)$
4D	$O(\lambda_A)$	$O(\lambda_B)$

[Tsung-Cheng Lu, Hsieh, TG 2019]

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How to quantify
“topological
non-separability”?

[Tsung-Cheng Lu, Hsieh, TG 2019]

von Neumann entropy not a measure of mixed-state entanglement

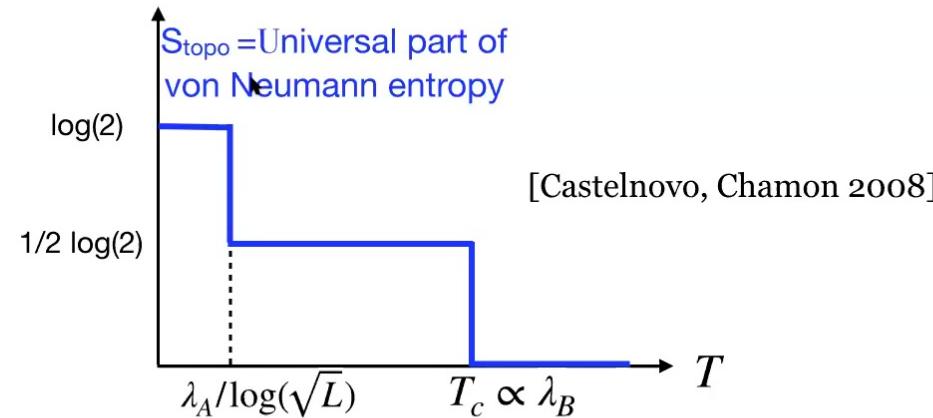
For gapped ground states, non-zero topological entanglement entropy implies topological order. [Levin, Wen 2005; Kitaev, Preskill 2005]

$$S_A = \alpha |\partial A| - S_{\text{topo}}$$

However, even Gibbs state of classical Hamiltonian, has a large (volume-law) von Neumann entropy, despite being clearly separable.

⇒ S_{topo} likely not a good measure of mixed-state entanglement.

Consider von Neumann entropy in 3d toric code...



But, as just mentioned, density matrix is short-ranged entangled when $T > T^* = \min(T_A, T_B) = 0$ in both 2d and 3d.

Dimension	T_A	T_B
2D	$O(\lambda_A) / \log L$	$O(\lambda_B) / \log L$
3D	$O(\lambda_A) / \log L$	$O(\lambda_B)$
4D	$O(\lambda_A)$	$O(\lambda_B)$

Quantifying Mixed-State Entanglement

“Entanglement Negativity”: $E_N = \log \left(|\rho_{\rightarrow}^{T_B}|_1 \right)$ [Eisert, Plenio 1999; Vidal, Werner 2001]

- Always zero for separable states (but can be zero also for non-separable states).
- Entanglement monotone [Plenio 2005].
- “Physical meaning”: upper bounds the rate of conversion of the mixed state to Bell pairs (“distillation rate”).

Several condensed matter applications: Negativity of CFTs (Calabrese, Cardy, Tonni 2012), Ground state of toric code and TQFTs (Lee, Vidal 2013; Castelnovo 2013; Wen, Matsuura, Ryu 2016), Characterizing SPTs (Shapourian, Shiozaki, Ryu 2017), Probe of chaos and scrambling (Kudler-Flam et al 2019),...

“Area Law” of Entanglement Negativity

[Sherman, Devakul, Hastings, Singh 2015]

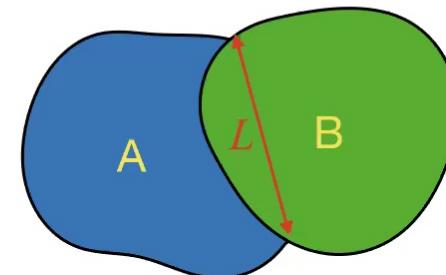
For thermal states of local Hamiltonians in d dimensions

$$E_N = \alpha L^{d-1} + \dots$$

Contrast this with von Neumann entropy
of thermal states:

$$S_A \sim L^d + \dots$$

The extensive part of von Neumann entropy is classical entropy,
and negativity is oblivious to it (as it should).



“Area Law” of Entanglement Negativity

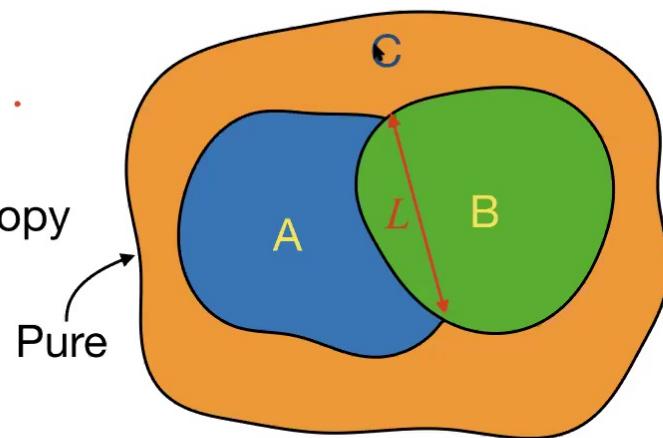
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Local Vs non-local part of negativity

Decompose E_N as:

$$E_N = E_{N,\text{local}} + E_{N,\text{nonlocal}}$$

$$\begin{aligned} E_{N,\text{local}} &= \sum_i E_{N,i}^\blacktriangleright = \int_{\partial A} F(\{\kappa, \partial_i \kappa, \dots\}) \\ &= \alpha_{d-1} L_A^{d-1} + \alpha_{d-3} L_A^{d-3} + \dots \end{aligned}$$

[TG, Turner,
Vishwanath (2011)]

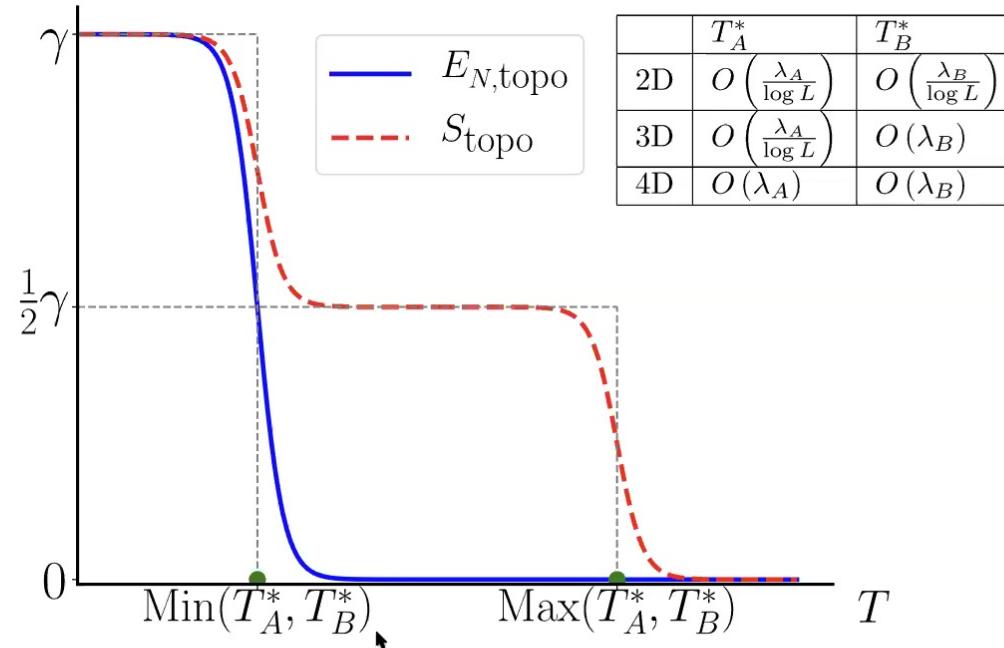
Let's study non-local part of negativity as a candidate order parameter for finite-T topological order.

$$E_N = \alpha L^{d-1} - E_{N,topo}$$

Focus on toric code in $d = 2, 3, 4$.

(Area-law coefficient α for 2D toric code studied in Hart, Castelnovo 2018.)

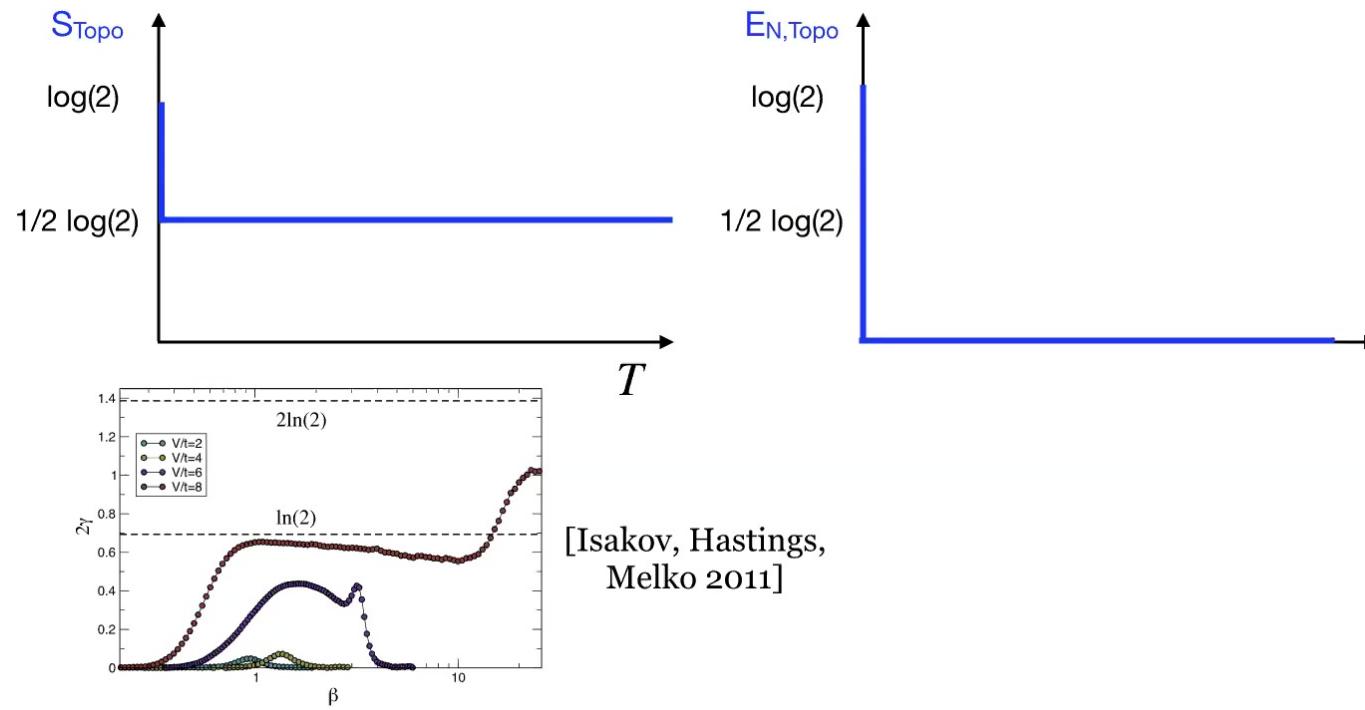
Main Result



[Lu, Hsieh, TG 2019]

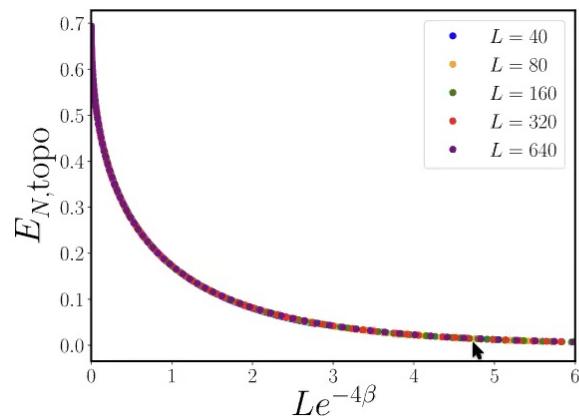
2D toric code in the limit $\lambda_B \rightarrow \infty, L \rightarrow \infty$

Contrast between Topological entropy & Topological negativity



2D toric code in the limit $\lambda_B \rightarrow \infty$.

Topological negativity can be calculated analytically:



$$E_{N,\text{topo}} = -\log \left\{ \frac{1}{2} + \frac{\left(\frac{L}{2}+1\right)}{2(x^{1/2} + x^{-1/2})^L} \left[\frac{1}{x} {}_2F_1(1, -\frac{L}{2} + 1; \frac{L}{2} + 2; -\frac{1}{x}) - x {}_2F_1(1, -\frac{L}{2} + 1; \frac{L}{2} + 2; -x) \right] \right\}$$

${}_2F_1(a, b; c; d)$: hypergeometric function $x = \tanh(\beta \lambda_A)$

Let's first suppress plaquette excitations

$$\lambda_A = O(1) \quad \lambda_B \rightarrow \infty$$

Point-like charges at finite T, similar to 2D Toric code.

Topological negativity is identical to that of
2D toric code in the limit $\lambda_B \rightarrow \infty$.

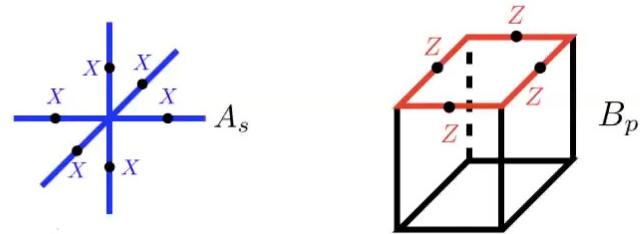


No topological order at finite T.

Consistent with Yoshida 2011: 3D Toric code only has a
“classical memory” at finite-T.

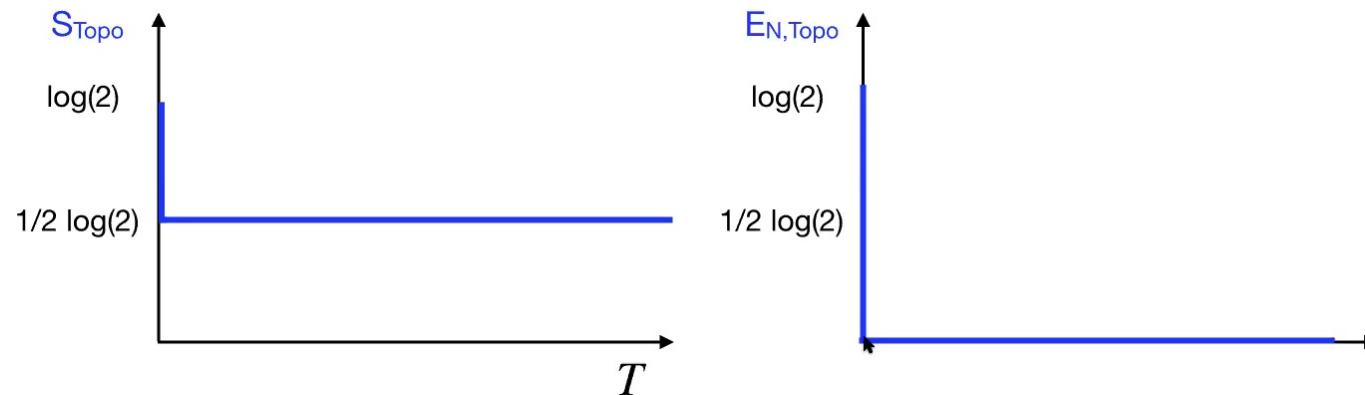
3d toric code

$$H_T = -\lambda_A \sum_s A_s - \lambda_B \sum_p B_p$$



2D toric code in the limit $\lambda_B \rightarrow \infty, L \rightarrow \infty$

Contrast between Topological entropy & Topological negativity



Conclusion:

No quantum liquid at $T > 0$. Decoherence due to electric charges.

Next, let's suppress point charges.

$$\lambda_A \rightarrow \infty \quad \lambda_B = O(1)$$

Renyi negativity can be calculated in a low-T expansion:

$$R_n = \alpha_n L^2 - E_{N,\text{topo}}$$

even n

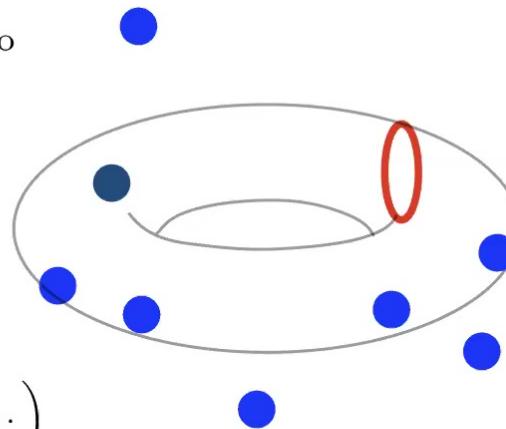
$$\alpha_n = \log 2 - \frac{1}{n-2} \left(\binom{n}{2} e^{-16\beta\lambda_B} + 2 \binom{n}{2} e^{-24\beta\lambda_B} + \dots \right)$$

odd n

$$\alpha_n = \log 2 - \frac{n}{n-1} \left(e^{-8\beta\lambda_B} + 2e^{-12\beta\lambda_B} - \frac{5}{2}e^{-16\beta\lambda_B} + \dots \right)$$

$$E_{N,\text{topo}} = \log 2$$

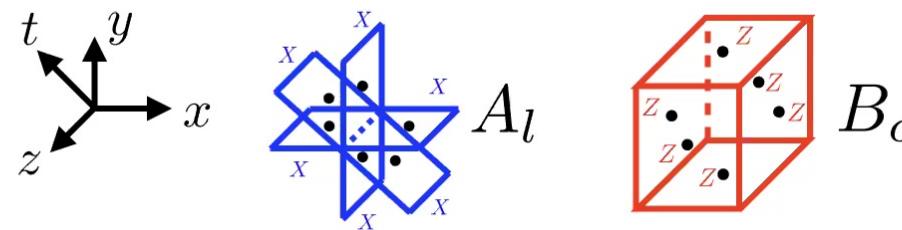
quantum liquid for $0 < T < T_c (\sim \lambda_B)$.



Finally, consider 4D toric code.

It has no point like excitations.

$$H_T = -\lambda_A \sum_l A_l - \lambda_B \sum_c B_c$$



As shown in Dennis, Kitaev, Landahl, Preskill (2001),
4D Toric code supports “finite-T quantum memory”.

Natural to expect non-zero topological entanglement negativity
for $0 < T < T_c$ where $T_c \propto \min(\lambda_A, \lambda_B)$.

4D toric code

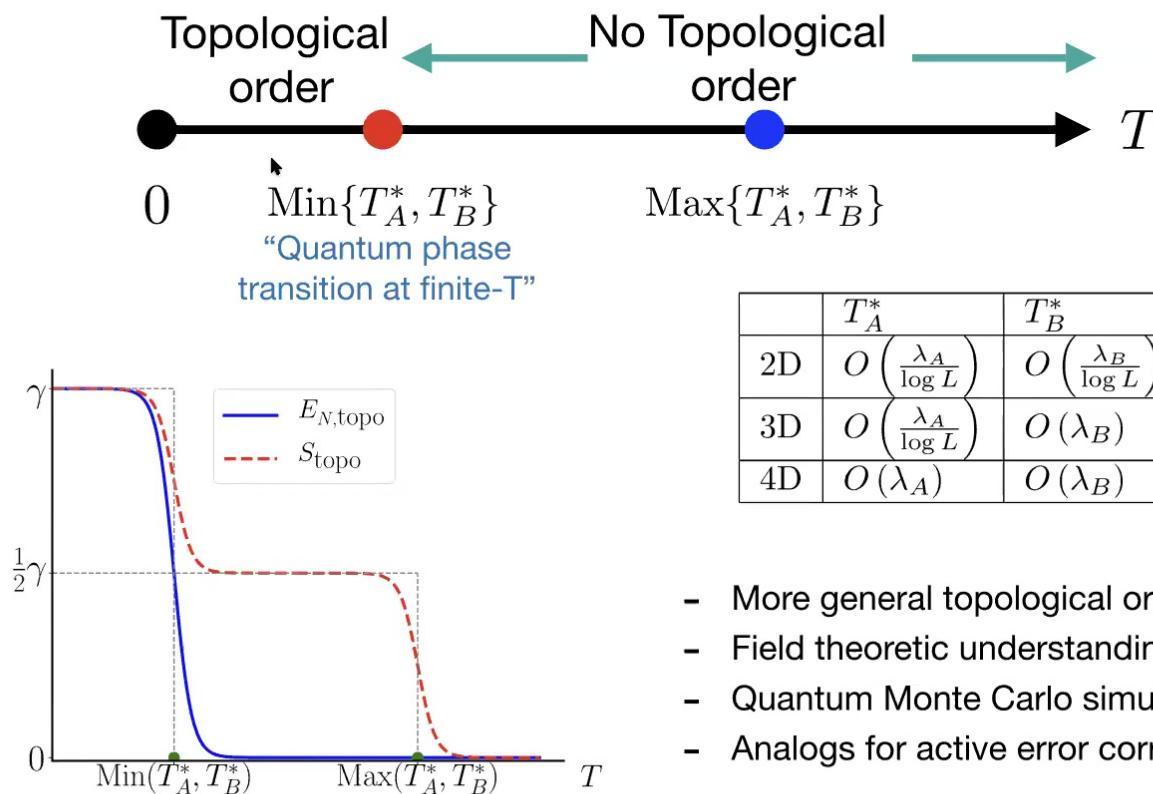
Renyi negativity of even $n \geq 4$

$$R_n = \alpha L^3 - E_{N,\text{topo}} \quad \text{Low-T expansion}$$

$$\begin{aligned} \alpha = & 2 \log 2 - \frac{3n(n-1)}{2(n-2)} (e^{-16\beta\lambda_A} + e^{-16\beta\lambda_B}) \\ & - \frac{6n(n-1)}{(n-2)} (e^{-24\beta\lambda_A} + e^{-24\beta\lambda_B}) \end{aligned}$$

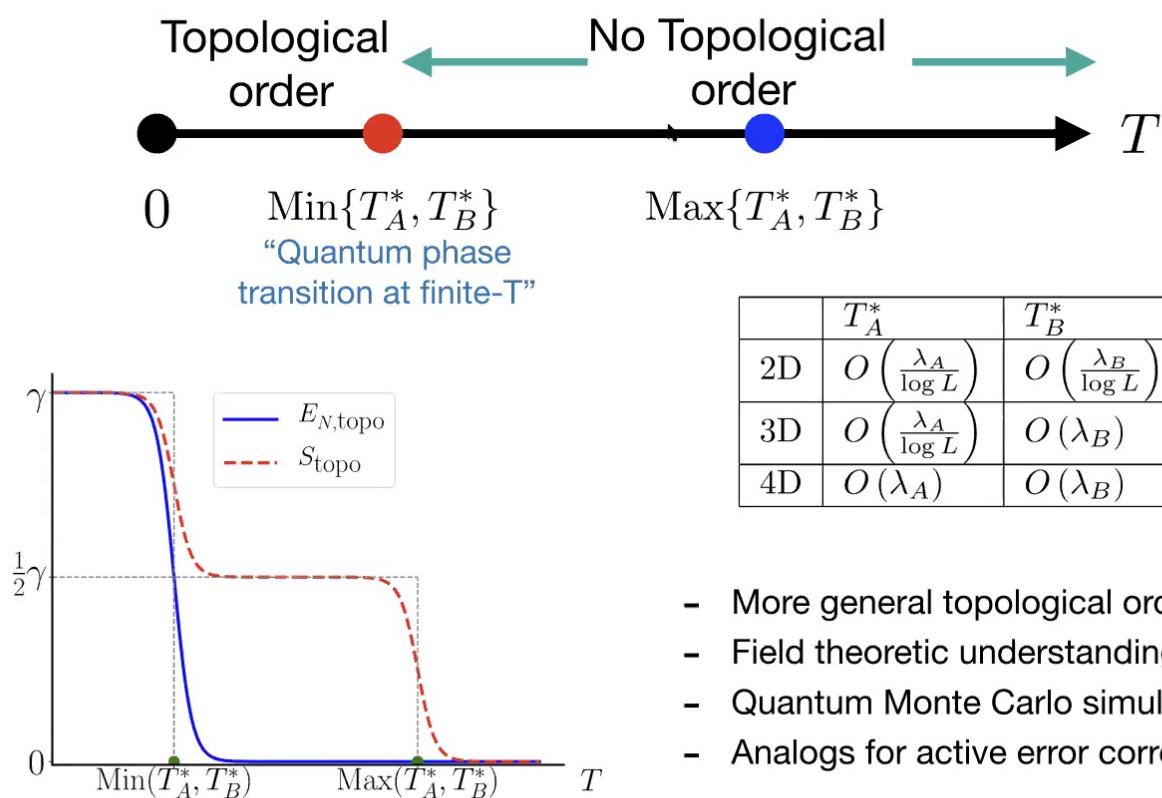
$$E_{N,\text{topo}} = 2 \log 2$$

Summary



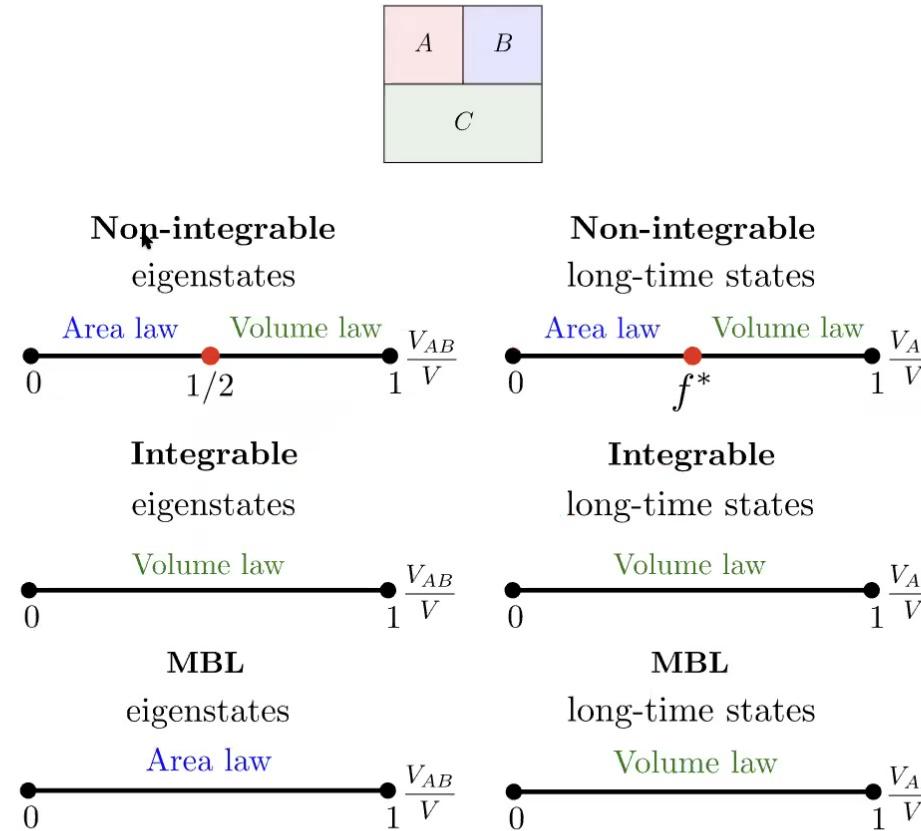
- More general topological orders?
- Field theoretic understanding?
- Quantum Monte Carlo simulations?
- Analogs for active error correction?

Summary



- More general topological orders?
- Field theoretic understanding?
- Quantum Monte Carlo simulations?
- Analogs for active error correction?

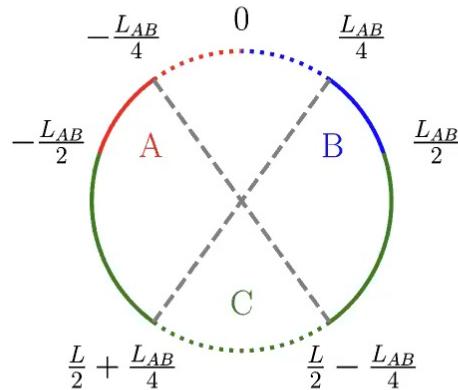
Summary of Results



[Lu, Grover 2020]

Quasiparticle picture for integrable systems

A = red,
B = blue,
C = green.



For quasiparticles to generate entanglement between A,B, one of them belongs to A, and the other to B.

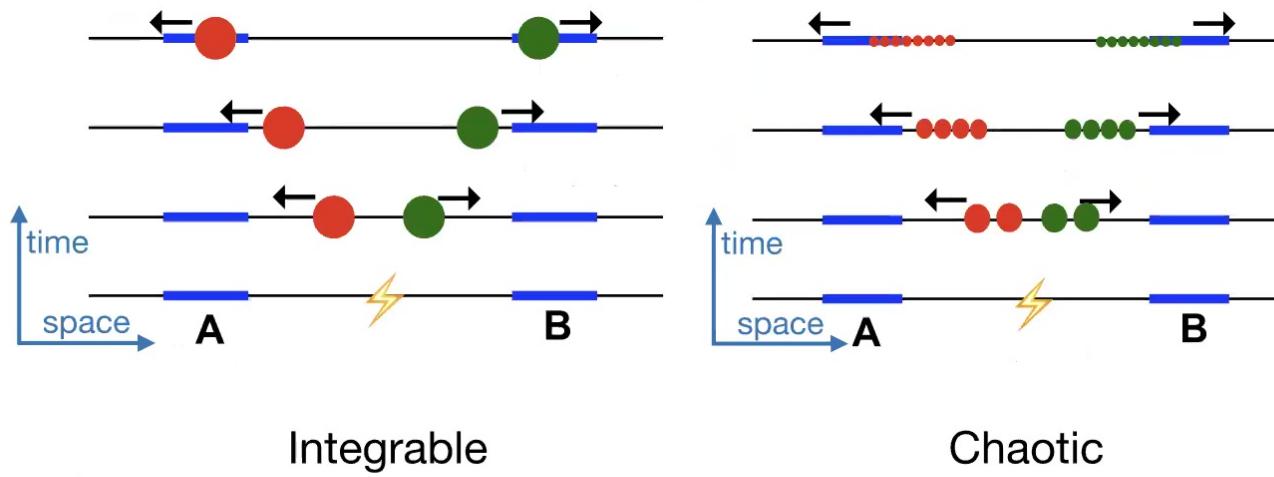
of quasiparticles that A and B can share $\propto L_{AB}$

Fraction of time these quasiparticles can entangle $\propto L_{AB}/L$

\Rightarrow Time averaged entanglement $\sim (L_{AB}/L) L_{AB} = \text{volume law}$

[Cardy, Calabrese 2005]

Quasiparticle picture breaks down in chaotic systems

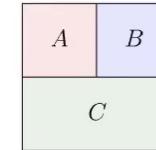


Integrable

Chaotic

Intuition for volume law negativity in a chaotic system when $V_{AB} > V_C$

Bulk properties of chaotic eigenstates resemble that of a “tripartite ergodic state”:



$$|E\rangle = \sum_{E_a^A + E_b^B + E_c^C \in (E - \frac{1}{2}\Delta, E + \frac{1}{2}\Delta)} \psi(a, b, c) |E_a^A\rangle \otimes |E_b^B\rangle \otimes |E_c^C\rangle$$

[Tsung-Cheng Lu,
TG 2017]

where $\psi(a, b, c)$ = random complex numbers and

$|E_i^X\rangle$ = eigenstate of Hamiltonian in region X

“Many-body Berry state”

[Berry 77;
Srednicki 1994]

Detecting Scar States in a “Quantum Disentangled Liquid” using negativity

A “Quantum disentangled liquid” (QDL) is a two-component system made of Heavy ($\{R_\alpha\}$) and Light ($\{r_\alpha\}$) particles, with a caricature finite energy density eigenstates,

$$\Psi(\{R_\alpha\}, \{r_\beta\}) = \psi(\{R_\alpha\}) \Phi^R(\{r_\beta\})$$

key property: $\psi(\{R_\alpha\})$ = Volume law
 $\Phi^R(\{r_\beta\})$ = Area law

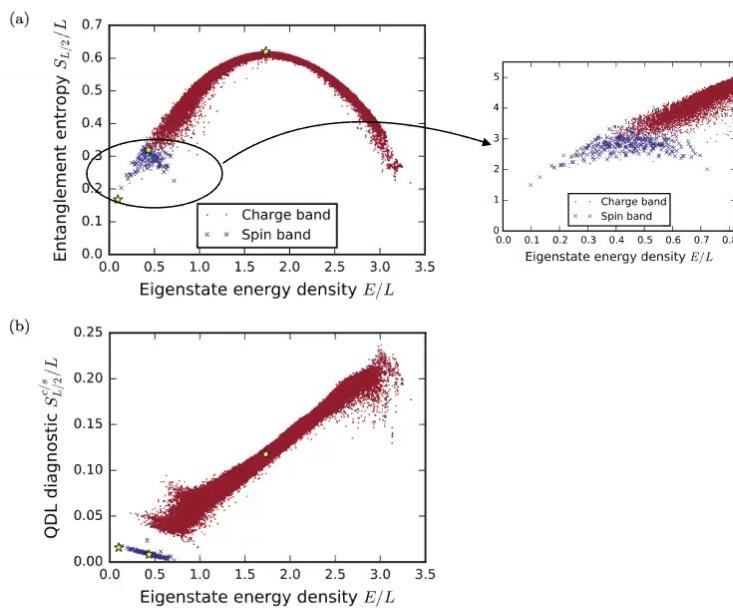
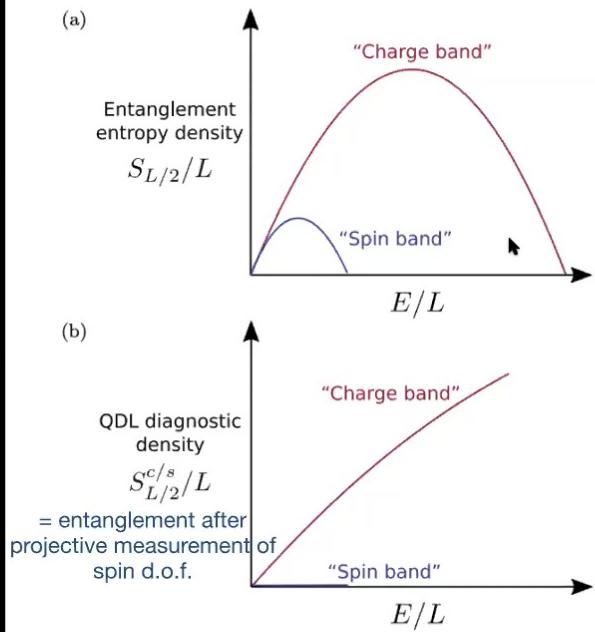
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Evidence of exponentially many such states in a extended Hubbard model
with R = spin, r = charge.

[Garrison, Mishmash, Fisher 2017]

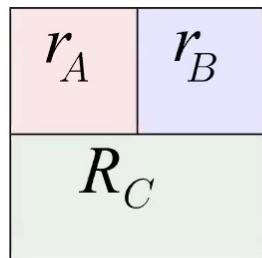
$$H = H_{\text{Hubbard}} + H_V$$
$$H_{\text{Hubbard}} = -t \sum_{\langle ij \rangle} \sum_{\sigma} \left(c_{i\sigma}^\dagger c_{j\sigma} + \text{H.c.} \right) + U \sum_i n_{i\uparrow} n_{i\downarrow}$$
$$H_V = V \sum_{\langle ij \rangle} n_i n_j$$



[Garrison, Mishmash, Fisher 2017]

Detecting QDL using negativity

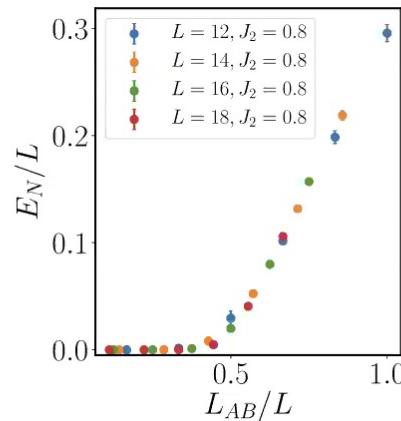
Due to QDL's key property, the reduced density matrix of r-particles is essentially separable even when $C < AB$!



$$\rho(r_A) = \sum_R p(R) \rho^R(r_A)$$

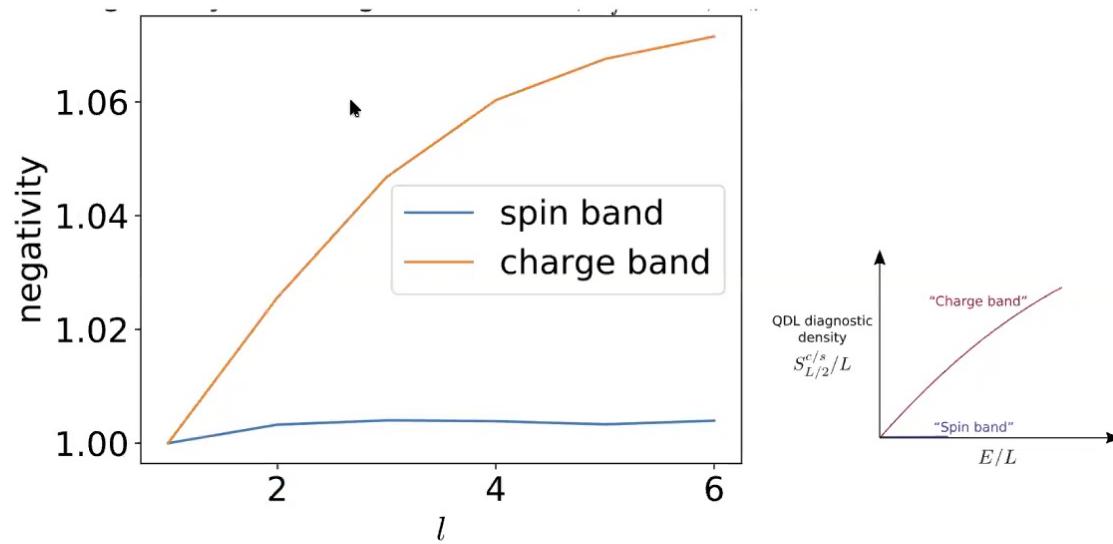
Each of $\rho^R(r_A)$ is only area-law entangled.

In contrast, in a truly chaotic system, tracing out less than one-half d.o.f. → generates volume law entanglement.



[McGreevy,
Ben-Zion, TG
(2019)]

Negativity after tracing out spins. (Spin Hilbert space > Charge Hilbert Space)



Negativity serves as an “operator agnostic” diagnostic
for the scar states in extended Hubbard model.
Area law for scar states, volume law for chaotic states.

Diagnosing “Gapless topological order” using entanglement



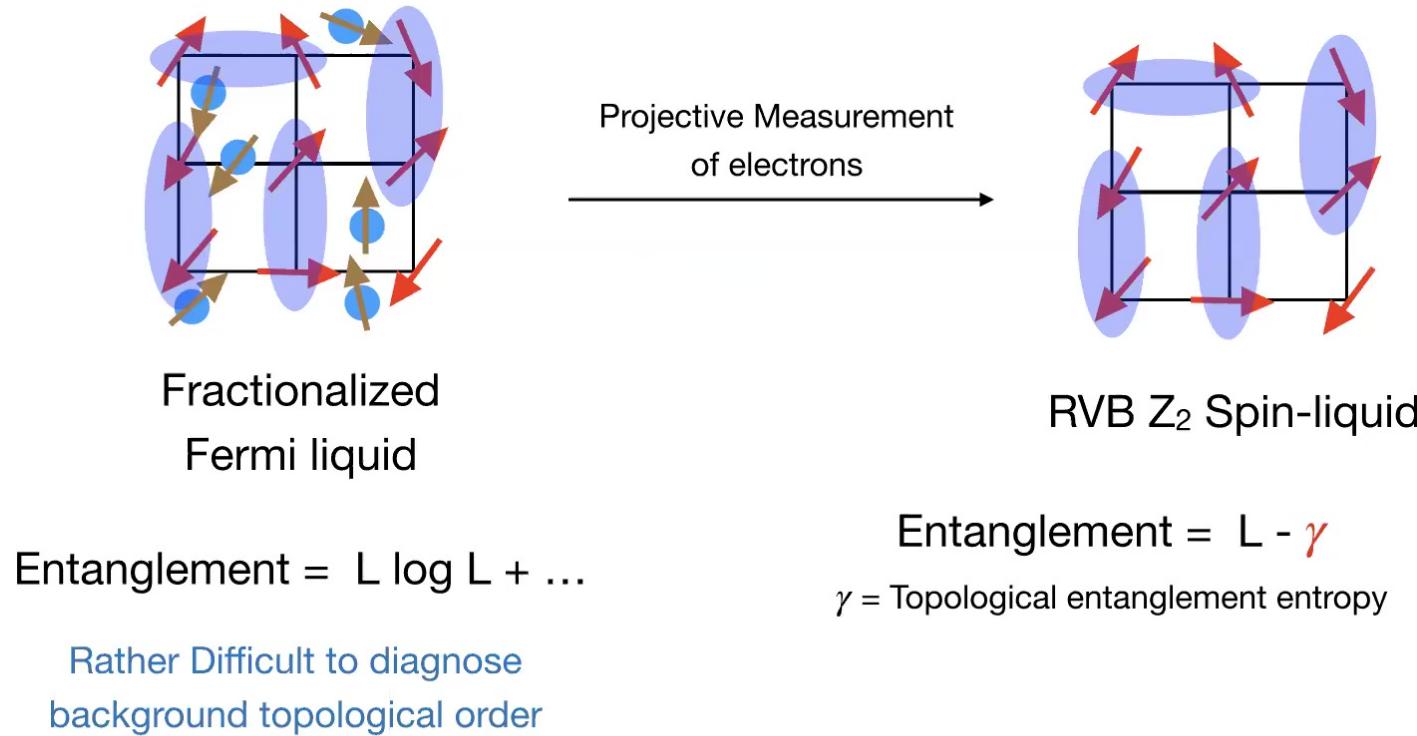
Interactions can lead to dynamical decoupling
between topological and gapless d.o.f.

Diagnosing “Gapless topological order” using entanglement



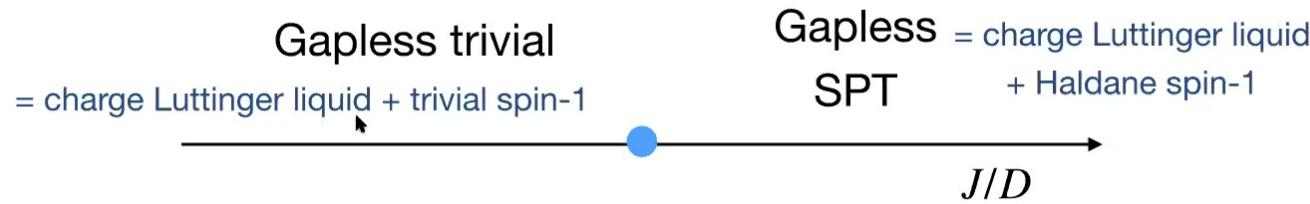
Interactions can lead to dynamical decoupling
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Entanglement Diagnostic of Fractionalized Fermi liquid



A toy model of gapless topological phase

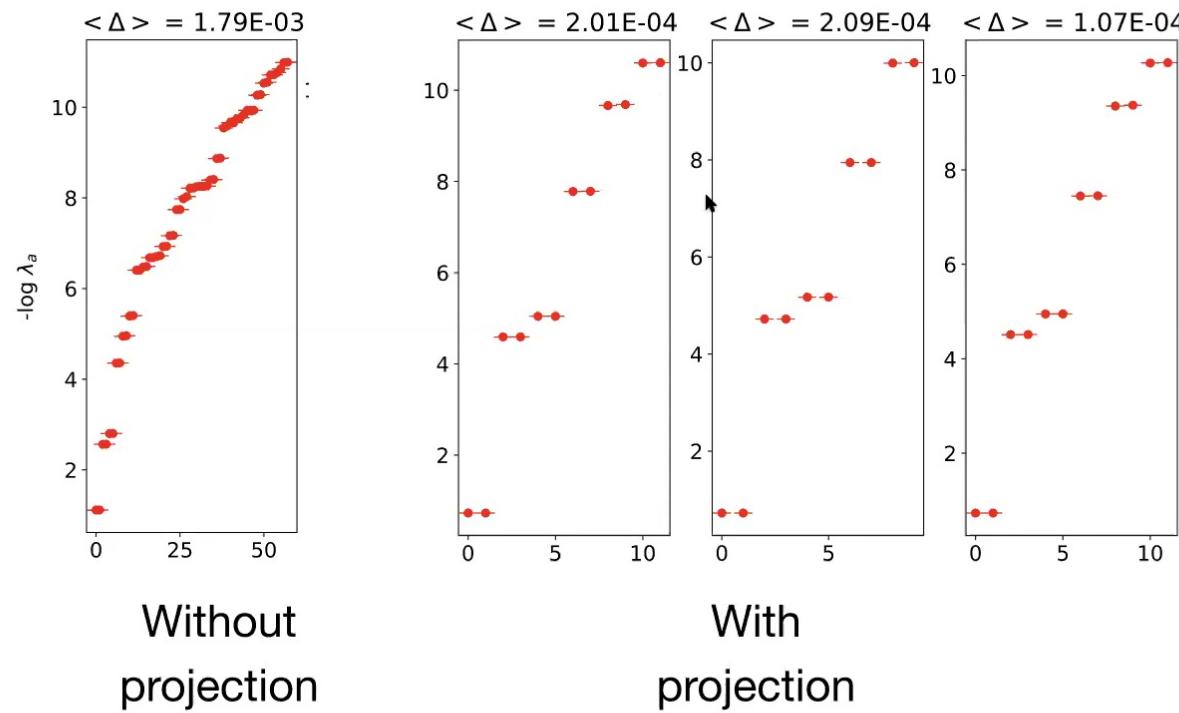
$$\mathbf{H} = -t \sum_{i\sigma} b_{i\sigma}^\dagger b_{i+i\sigma} + h.c. + J \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + D \sum_i (S_i^z)^2 \quad [\text{Jiang, Li, Seidel, Lee 2018}]$$



Protocol to detect gapless topological order:
Project out gapless d.o.f.'s to study topological d.o.f.'s

[Ben-Zion,
McGreevy, TG 2019]

Entanglement spectrum with and without projecting out charge d.o.f.



[Ben-Zion,
McGreevy, TG 2019]

Summary and a few questions...

- Topological negativity seemingly a candidate order parameter for finite-T topological order.
- Entanglement transitions in subsystem negativity can capture distinction between chaotic, MBL, integrable and a class of scar states.
- Quantum disentangled liquid type diagnostic for gapless topological orders.
- Exploiting separation of classical Vs quantum correlations for real time dynamics?
- “Industrial” applications of negativity? e.g., optimize parameters in a noisy quantum computer by maximizing negativity between qubits. Understanding “Fault tolerance” using mixed-state entanglement? c.f. Aharanov 2000.
- Loss of topological order due to thermal fluctuations reminiscent of “anyon condensation”. Accessing new topological phases by heating?
- Are there models where singularity exists *only* in quantum correlations at finite temperature? “Truly quantum”.
- Calculation of mixed-state entanglement measures other than negativity?