Title: Contextuality-by-default for behaviours in compatibility scenarios

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Series: Quantum Foundations

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Abstract: The compatibility-hypergraph approach to contextuality (CA) and the contextuality-by-default approach (CbD) are usually presented as products of entirely different views on how physical measurements and measurement contexts should be understood: the latter is based on the idea that a physical measurement has to be seen by a collection of random variables, one for each context containing that measurement, while the imposition of the non-disturbance condition as a physical requirement in the former precludes such interpretation of measurements. The aim of our work is to present both approaches as entirely compatible ones and to introduce in the compatibility-hypergraph approach ideas which arises from contextuality-by-default. We

introduce in CA the non-degeneracy condition, which is the analogous of consistent connectedness (an important concept from CbD), and prove that this condition is, in general, weaker than non-disturbance. The set of non-degenerate behaviours defines a polytope, therefore one can characterize non-degeneracy using a finite set of linear inequalities. We introduce extended contextuality for behaviours and prove that a behaviour is non-contextual in the standard sense if and only if it is non-degenerate and non-contextual in the extended sense. Finally, we use extended scenarios and behaviours to shed new light on our results.

Contextuality-by-default for behaviours in compatibility scenarios

Alisson Tezzin, Rafael Wagner, Barbara Amaral

October 30, 2020

Sketch of a physical theory

physical theory

$\mathscr{F}$

Postulate 1

$\mathcal{P}_1$

Postulate 2

$\mathcal{P}_2$

State

ρ

Observable

b

Outcome

β

Sketch of a physical theory

physical theory

$\mathscr{F}$

Postulate 1

$\mathcal{P}_1$

Postulate 2

$\mathcal{P}_2$

State ρ

Observable b

Outcome β

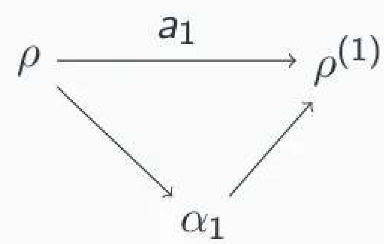
(a) Postulate 1:

$$p_b^{\rho}(\beta)$$

(b) Postulate 2:

$$\rho_{\beta}^{(b)}$$

Sequence of measurements



$$p^{\rho}_{a_1}(\alpha_1)$$

Sequences of measurements

$$\rho \xrightarrow{a_1} \rho^{(1)} \xrightarrow{a_2} \rho^{(1,2)} \xrightarrow{a_3} \rho^{(1,2,3)} \quad \cdots \quad \rho^{(,1\ldots,n-1)} \xrightarrow{a_n} \rho^{(1,\ldots,n)}$$

$$\searrow \alpha_1 \nearrow \qquad \searrow \alpha_2 \nearrow \qquad \searrow \alpha_3 \nearrow \qquad \qquad \searrow \alpha_n \nearrow$$

$$p_{a_1}^{\rho}(\alpha_1) \quad p_{a_2}^{\rho^{(1)}}(\alpha_2) \quad p_{a_3}^{\rho^{(1,2)}}(\alpha_3) \qquad \qquad p_{a_n}^{\rho^{(1,\ldots,n-1)}}(\alpha_n)$$

Sequences of measurements

$$
\begin{array}{ccccccccccc}
\rho & \xrightarrow{a_1} & \rho^{(1)} & \xrightarrow{a_2} & \rho^{(1,2)} & \xrightarrow{a_3} & \rho^{(1,2,3)} & \cdots & \rho^{(,1\ldots,n-1)} & \xrightarrow{a_n} & \rho^{(1,\ldots,n)} \\
 & \searrow \; \nearrow & & \searrow \; \nearrow & & \searrow \; \nearrow & & & & \searrow \; \nearrow & \\
 & \alpha_1 & & \alpha_2 & & \alpha_3 & & & & \alpha_n &
\end{array}
$$

$$
p^{\rho}_{a_1}(\alpha_1) \quad p^{\rho^{(1)}}_{a_2}(\alpha_2) \quad p^{\rho^{(1,2)}}_{a_3}(\alpha_3) \qquad \qquad p^{\rho^{(1,\ldots,n-1)}}_{a_n}(\alpha_n)
$$

$$
p^{(\rho)}_{a_1,a_2,\ldots,a_n}(\alpha_1, \alpha_2, \ldots, \alpha_n) \doteq \prod_{i=1}^{n} p^{\rho(1,\ldots,k-1)}_{a_k}(\alpha_k)
$$

Does it represent a "joint measurement"?

Compatibility contexts

A **compatibility context** in $\mathscr{F}$ is a *set* C of observables such that, for any finite subset $a \equiv \{a_1, ..., a_n\}$ of C,

$$p^{\rho}_{\pi(a)}(\pi(\alpha)) = p^{\rho}_{\sigma(a)}(\sigma(\alpha))$$

for any state ρ, where $\alpha \equiv \{\alpha_1, ..., \alpha_n\}$

Noncontextuality

$\mathscr{F}$ is said to be noncontextual if the set of all its observables is a compatibility context

Compatibility contexts in classical mechanics

States
Probability measures over (Λ, Σ)

Observables
Real measurable function over (Λ, Σ)

Compatibility contexts in classical mechanics

State	**Observable**	**Outcome**
μ	f	α

- Postulate $\mathcal{P}_1$:

$$p_f^{\mu}(\alpha) \doteq \mu(f^{-1}(\{\alpha\}))$$

Compatibility contexts in classical mechanics

State	**Observable**	**Outcome**
μ	f	α

- Postulate $\mathcal{P}_1$:

$$p_f^{\mu}(\alpha) \doteq \mu(f^{-1}(\{\alpha\}))$$

- Postulate $\mathcal{P}_2$:

$$\mu_\alpha^{(f)}(\cdot) \doteq \frac{\mu(\,\cdot\, \cap f^{-1}(\{\alpha\}))}{\mu(f^{-1}(\{\alpha\}))}$$

Compatibility contexts in classical mechanics

For a sequence $f_1, ..., f_n$ of measurements,

$$p^{\mu}_{f_1,...,f_n}(\alpha_1, ..., \alpha_n) = \mu(\cap_{i=1}^{n} f_i^{-1}(\{\alpha_i\}))$$

Compatibility contexts in quantum theory

For a sequence $T_1, \ldots, T_n$ of measurements,

$$p^{\psi}_{T_1,\ldots,T_n}(\alpha_1, \ldots, \alpha_n) \doteq \prod_{k=1}^{n} p^{\psi(1,\ldots,k-1)}_{T_k}(\alpha_k)$$

$$= \left\langle \psi \middle| \left(\prod_{j=1}^{n-1} P_{\alpha_j} \right) P_{\alpha_n} \left(\prod_{i=1}^{n-1} P_{\alpha_{n-i}} \right) \psi \right\rangle$$

Compatibility contexts in quantum theory

If $T_1, ..., T_n$ commutes,

$$p^{\psi}_{T_1,...,T_n}(\alpha_1, ..., \alpha_n) = \left\langle \psi \middle| \prod_{i=1}^{n} P_{\alpha_i} \psi \right\rangle,$$

Compatibility contexts in quantum theory

Standard approach
A collection of commuting bounded self-adjoint operators

Operational approach
Collection of component-wise commuting PVM's

Compatibility scenarios

A **compatibility scenario** is a triple $\mathcal{S} \equiv (\mathcal{X}, \mathcal{C}, O)$ where

- $\mathcal{X}$ is a finite set (of **measurements**)
- O is a finite set (of **outcomes**)
- $\mathcal{C} \subset \mathcal{P}(X)$ satisfies

Behaviour over $\mathcal{S}$

O^C is the set of all functions $C \to O$

Marginal distributions

Context C

Subset $E \subset C$

$$p_E^C : O^E \to [0, 1]$$

Marginal distributions

Context C

Subset $E \subset C$

$$p_E^C : O^E \to [0,1]$$

$$p_E^C(s) \doteq \sum_{\substack{u \in O^C \\ u|_E = s}} p^C(u) \; \forall s \in O^E$$

Noncontextuality

p is noncontextual if exists a distribution $\overline{p} : O^{\mathcal{X}} \to [0,1]$ such that, for any context C,

$$\overline{p}_C = p^C$$

Classical behaviours

p in $(\mathcal{X}, \mathcal{C}, O)$ is classical iff exists:

(a) A probability space (Λ, Σ, μ)

(b) A function

$$\mathcal{X} \ni x$$

Classical behaviours

p in $(\mathcal{X}, \mathcal{C}, O)$ is classical iff exists:

(a) A probability space (Λ, Σ, μ)

(b) A function

$$\mathcal{X} \ni x \qquad \mapsto \qquad R_x : \Lambda \to O$$

satisfying

- For any $C \in \mathcal{C}$ and any $s \in O^C$,

$$p^C(s) = \mu\Big(\bigcap_{x \in C} R_x^{-1}(s_x)\Big)$$

Quantum behaviours

p in $(\mathcal{X}, \mathcal{C}, O)$ is a quantum behaviour iff exists

(a) A finite dimensional Hilbert space H

(b) A function

$$\mathcal{X} \ni x \mapsto T_x \in \mathcal{B}(H)^{\mathbb{R}}$$

(c) A density operator $\rho \in \mathcal{B}(H)$

satisfying

- For any $C \in \mathcal{C}$,

$$[\theta(x), \theta(y)] = 0 \ \forall x, y \in C$$

Quantum behaviours

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satisfying

- For any $C \in \mathcal{C}$,

$$[\theta(x), \theta(y)] = 0 \ \forall x, y \in C$$

- For any $C \in \mathcal{C}$ and $s \in O^C$

$$p^C(s) = \mathrm{tr}\left(\rho \prod_{x \in C} P_{s_s}^{(x)}\right)$$

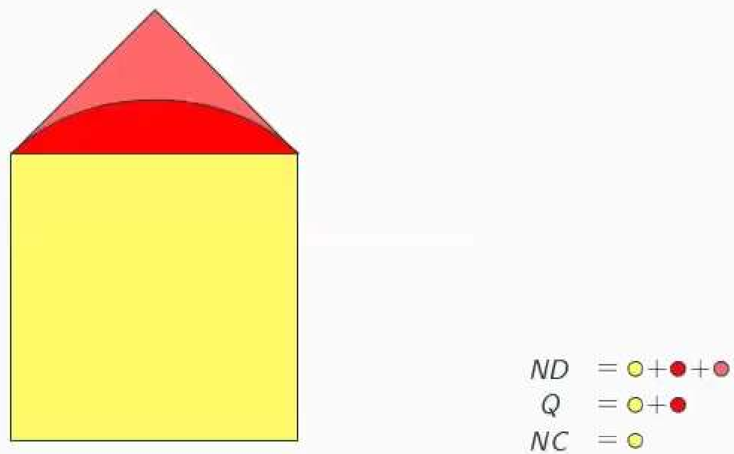
Nondisturbance

p in nondisturbing if

$$p^C_{C \cap D} = p^D_{C \cap D}$$

for any pair of intersecting contexts C, D.

Polytopes and sets



"We label all measurements contextually: this means that a property q is represented by different random variables R_q^C depending on the context C."[1]

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[1] [1] J. V. Kujala, E. N. Dzhafarov, and J.-A. Larsson, "Necessary and sufficient conditions for an extended noncontextuality in abroad class of quantum mechanical systems," Phys. Rev. Lett.,vol. 115, p. 150401, Oct 2015

Systems

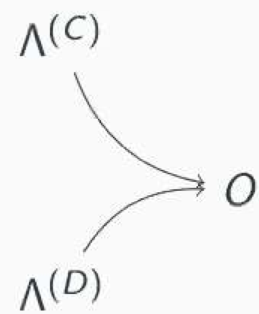
Contexts

$C \neq D$

Random variables

R_x^C, R_y^D

Stochastically unrelated random variables



Systems

Set o measurements

$\mathscr{X}$

Set of contexts

$\mathscr{C}$

Systems

Set o measurements

$\mathscr{X}$

Set of contexts

$\mathscr{C}$

Measurement

x

Context

C

R_x^C

System

$(\mathscr{X}, \mathscr{C}, \mathscr{R})$

Systems

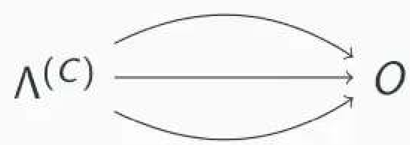
Context

C

Bunch

$\mathscr{R}^C \equiv \{R_x^C; x \in C\}$

Jointly distributed random variables



Systems

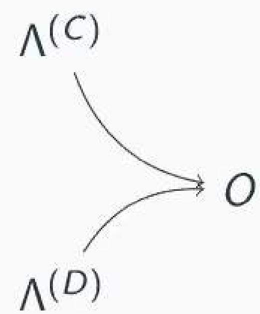
Contexts

$C \neq D$

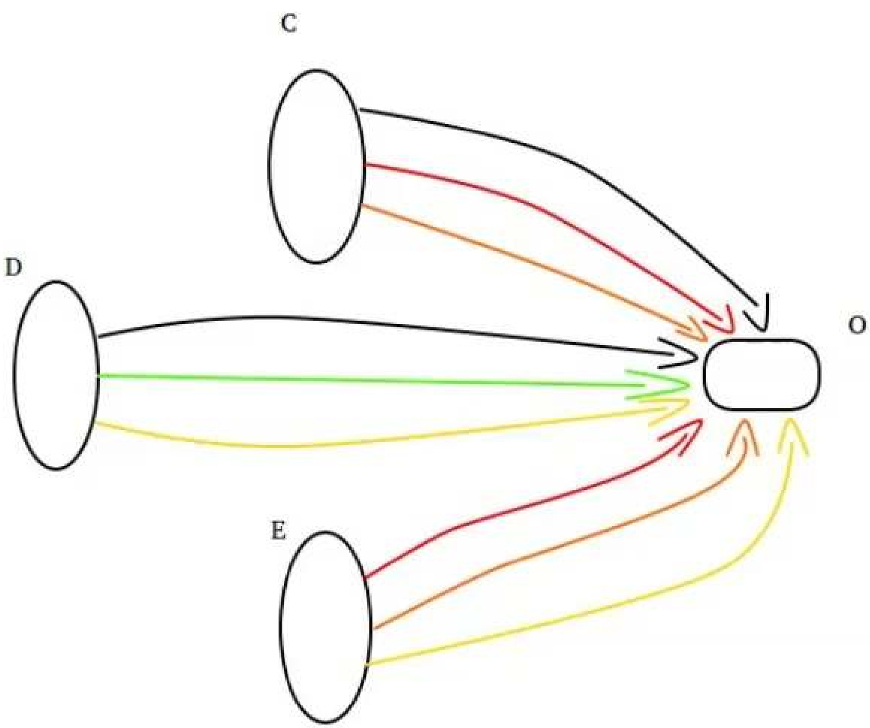
Random variables

R_x^C, R_y^D

Stochastically unrelated random variables



From behaviours to random variables



Section 4

From behaviours to random variables

Measurement	**Context**
x	C

$$p_x^C : O \to [0, 1]$$

From behaviours to random variables

Measurement x

Context C

$$p_x^C : O \to [0,1]$$

$$p_x^C \doteq p_{\{x\}}^C$$

From behaviours to random variables

Probability distribution	**Random variable**
p_x^C	R_x^C
Context	**probability space**
C	$(\Lambda^{(C)}, \Sigma^{(C)}, \mu^{(C)})$

$$\forall x \in C, \quad R_x^C : \Lambda^{(C)} \to O$$

$$p_{R_x^C} = p_x^C$$

$$p^C(s) = \mu^{(C)}(\bigcap_{x \in C} x_C^{-1}(s_x))$$

What about quantum behaviours?

Gelfand transform and Riesz Markov theorem

$$C \mapsto \Lambda^{\mathcal{V}(C)}$$

$$a \in C^{\mathbb{R}} \mapsto \widehat{a} : \Lambda^{\mathcal{V}(C)} \to \sigma(a)$$

$$\rho \mapsto \mu_\rho \in \mathcal{M}_R(\Lambda^{\mathcal{V}(C)})$$

What about quantum behaviours?

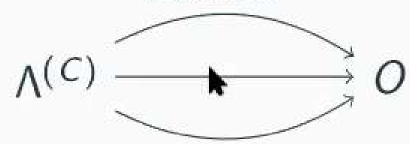
Gelfand transform and Riesz Markov theorem

C	$\mapsto$	$\Lambda^{\mathcal{V}(C)}$
$a \in C^{\mathbb{R}}$	$\mapsto$	$\widehat{a} : \Lambda^{\mathcal{V}(C)} \to \sigma(a)$
ρ	$\mapsto$	$\mu_\rho \in \mathcal{M}_R(\Lambda^{\mathcal{V}(C)})$

Context

C

Bunch



Section 5

Consistent connectedness

A system is said to be consistently connected if all the random variables in a given connection $\mathscr{R}_x$ have the same distribution

Consistent connectedness

A system is said to be consistently connected if all the random variables in a given connection $\mathscr{R}_x$ have the same distribution

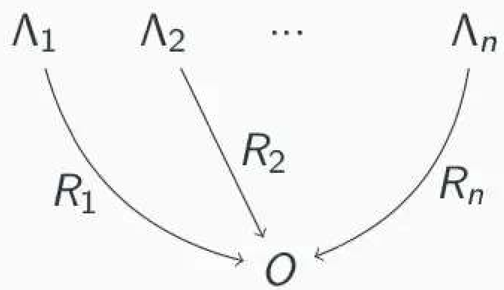
$$p_x^C = p_x^D$$

$$p_x \equiv p_x^C$$

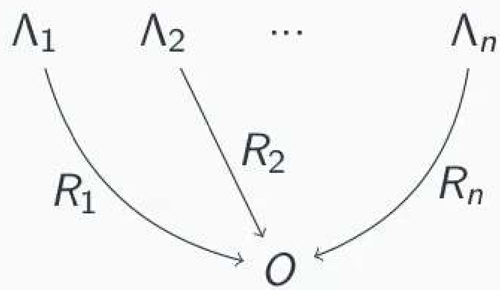
Nondegeneracy

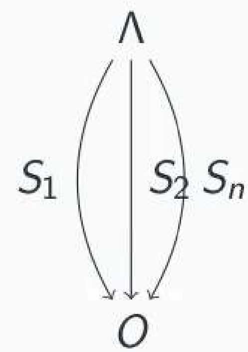
System	**Behaviour**
consistently connected	Nondegenerate

Coupling



Coupling





$$R_i \sim S_i$$

Maximal coupling

Joint distribution

$$p_S : O^n \to [0,1]$$

$$p_S(o_1, ..., o_n) = \mu(\cap_{i=1}^{n} S_i^{-1}(\{o_i\}))$$

Maximal coupling

Joint distribution

$$p_S : O^n \to [0,1]$$

$$p_S(o_1, ..., o_n) = \mu(\cap_{i=1}^n S_i^{-1}(\{o_i\}))$$

“Probability of being equal”

$$p_S(R_1 = ... = R_n) \doteq \sum_{o \in O} p_S(o, o, ..., o)$$

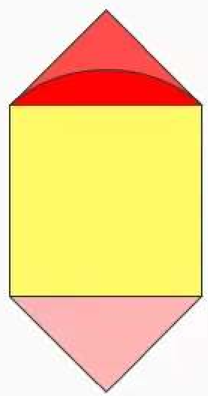
Maximally noncontextual description

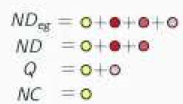
A system $(\mathscr{X}, \mathscr{C}, \mathscr{R})$ is said to have a maximally Noncontextual description if there is a coupling of $\mathscr{R}$ which provides a maximal coupling for each connection $\mathscr{R}_x$

Section 6

Nondegeneracy

Nondisturbance $\Rightarrow$ nondegeneracy
Nondegeneracy $\nRightarrow$ nondisturbance





Polytope of nondegenerate behaviours

"nondegeneracy inequalities"

Extended noncontextuality

[2][2] B.Amaral,C.Duarte, "Characterizing and quantifying extended contextuality", Phys. Rev. A.,vol. 100, p.062103 , Dec 2019

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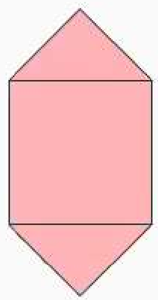
Extended noncontextuality



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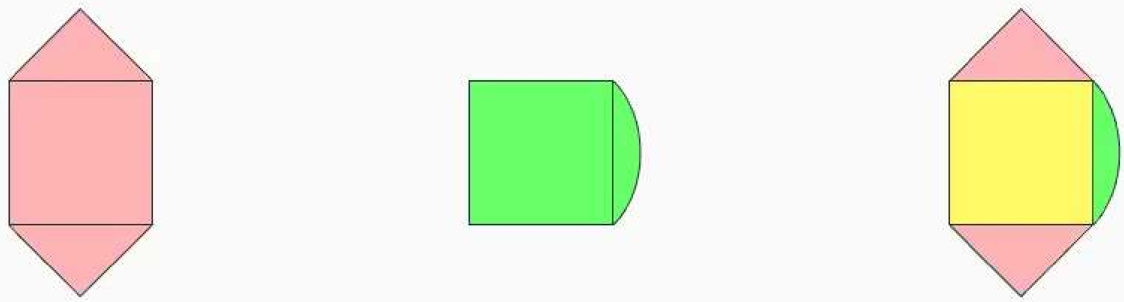
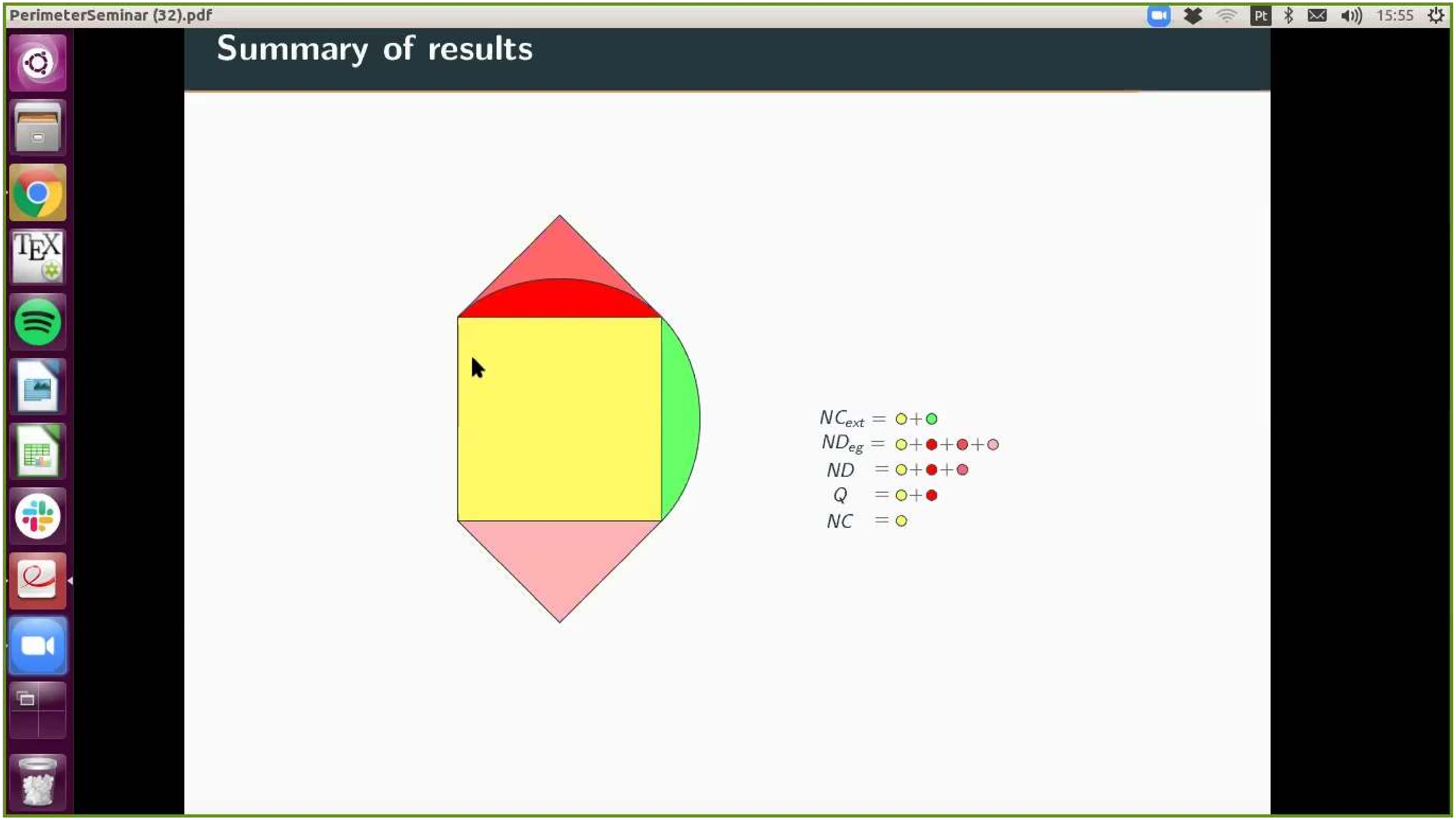
Extended noncontextuality



noncontextuality $\Leftrightarrow$ extended noncontextuailty $\wedge$ nondegeneracy

p is noncontextual in the extended sense iff its extension is noncontextual[2]

[2] [2] B.Amaral,C.Duarte, “Characterizing and quantifying extended contextuality”, Phys. Rev. A.,vol. 100, p.062103 , Dec 2019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Summary of results
NC_ext =
ND_eg =
ND =
Q =
NC =

Summary of results

- The idea behind contextuality-by-default is implicit in the compatibility-hypergraph approach to contextuality

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Summary of results

- The idea behind contextuality-by-default is implicit in the compatibility-hypergraph approach to contextuality
- We can relax the nondisturbance condition as a physical requirement

Thank you