Title: Nilpotent Slodowy slices and W-algebras

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Abstract: To any vertex algebra one can attach in a canonical way a certain Poisson variety, called the associated variety. Nilpotent Slodowy slices appear as associated varieties of admissible (simple) W-algebras. They also appear as Higgs branches of the Argyres-Douglas theories in 4d N=2 SCFTâ€™s. These two facts are linked by the so-called Higgs branch conjecture. In this talk I will explain how to exploit the geometry of nilpotent Slodowy slices to study some properties of W-algebras whose motivation stems from physics. This is a joint work with Tomoyuki Arakawa and Jethro van Ekeren (still in preparation).

Nilpotent Slodowy slices and W-algebras

(joint work with Tomoyuki Arakawa and Jethro van Ekeren)

Mathematical Physics seminar – Perimeter Institute

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October 29, 2020

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Vertex operator algebras

A *vertex operator algebras (VOA)* is a vector space V equipped with a linear map

$$V \to (\mathrm{End}(V))[[z, z^{-1}]], \quad a \mapsto Y(a,z) = a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$$

that satisfies certains properties such as the state-field correspondence

$$a = \lim_{z \to 0} a(z)|0\rangle$$

and the locality :

$$(z-w)^N [a(z), b(w)] = 0 \quad (N \gg 0),$$

or, equivalently, OPEs :

$$a(z)b(w) \sim \sum_{j \geqslant 0} \frac{1}{(z-w)^{j+1}} (a_{(j)}b)(w).$$

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Associated variety of VOAs

There is a contravariant functor

$$\begin{array}{ccc} \{\text{VOAs}\} & \longrightarrow & \{\text{affine Poisson varieties}\} \\ V & \longmapsto & X_V, \end{array}$$

where X_V is the *associated variety* of V :

$$X_V := \text{Specm } R_V.$$

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Zhu's C_2 algebra

By the state-field correspondence we can write

$$V = \mathrm{span}_{\mathbb{C}}\{ \mathopen{:} (\partial^{n_1} a_1(z) \dots \partial^{n_r} a_r(z)) \mathclose{:} \}.$$

Let $C_2(V)$ be the the subspace of V generated by the elements of the above form with $n_1 + \cdots + n_r \geqslant 1$.

Then $R_V := V/C_2(V)$. It is a Poisson algebra by

$$\overline{f(z)}.\overline{g(z)} = \overline{\mathopen{:} f(z)g(z) \mathclose{:}},$$

$$\{\overline{f(z)}, \overline{g(z)}\} = \overline{\mathrm{Res}_{z=w}\, f(w)g(z)}.$$

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Examples of associated varieties

Let $\widehat{\mathfrak{g}} := \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ be the affine Kac-Moody algebra associated with a simple Lie algebra $\mathfrak{g}$, and

$$V^k(\mathfrak{g}) := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus \mathbb{C}K)} \mathbb{C}_k$$

the universal affine vertex algebra associated with $\mathfrak{g}$ at level $k \in \mathbb{C}$.

$V^k(\mathfrak{g})$ is generated by $x(z)$, $x \in \mathfrak{g}$, with OPEs

$$x(z)y(w) \sim [x, y](w)/(z-w) + k(x|y)/(z-w)^2$$

(a $V^k(\mathfrak{g})$-module = a smooth $\widehat{\mathfrak{g}}$-module of level k).

We have

$$X_{V^k(\mathfrak{g})} = \mathfrak{g}^*.$$

Let $L_k(\mathfrak{g})$ be the simple quotient of $V^k(\mathfrak{g})$,

$$X_{L_k(\mathfrak{g})} \subset \mathfrak{g}^*, \quad G\text{-invariant and conic.}$$

The associated variety $X_{L_k(\mathfrak{g})}$ is difficult to compute in general !

Examples

- $L_k(\mathfrak{g})$ is integrable ($k \in \mathbb{Z}_{\geqslant 0}$) $\Rightarrow X_{L_k(\mathfrak{g})} = \{0\}$. (The converse is true.)

More generally, a VOA V is called *lisse* if $\dim X_V = 0$. A lisse VOA has very nice properties such as the finiteness of simple modules and the modularity of characters ([Abe-Buhl-Dong, Zhu, Miyamoto]).

- $L_k(\mathfrak{g})$ is admissible $\Rightarrow X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_k}$, for some nilpotent orbit $\mathbb{O}_k$ of $\mathfrak{g}$ [Arakawa '15].

$$\{\text{integrable } \widehat{\mathfrak{g}}\text{-repr.}\} \quad \subsetneq \quad \{\text{admissible } \widehat{\mathfrak{g}}\text{-repr.}\} \quad \subseteq \quad \{\text{modular inv. repr.}\}.$$

Remark : for $\mathfrak{g}$ simply-laced, $L_k(\mathfrak{g}) \cong L(k\Lambda_0)$ is admissible if and only if $k = -h^\vee + p/q$, with $(p,q) = 1$ and $p \geqslant h^\vee$.

Compatibility with Drinfeld-Sokolov reduction

Let f be a nilpotent element of $\mathfrak{g}$, and $\mathcal{W}^k(\mathfrak{g}, f)$ the *W-algebra* associated with g, f at the level k obtained the quantized Drinfeld-Sokolov reduction :

$$\mathcal{W}^k(\mathfrak{g}, f) = H^0_{DS,f}(V^k(\mathfrak{g}))$$

[Feigin-Frenkel, Kac-Roan-Wakimoto]. Then

$$X_{\mathcal{W}^k(\mathfrak{g},f)} \cong \mathscr{S}_f := f + \mathfrak{g}^e,$$

the *Slodowy slice* at f [De Sole-Kac]. (e,h,f) $\mathfrak{sl}_2$-triple

More generally, if V is a VOA equipped with a vertex algebra homomorphism $V^k(\mathfrak{g}) \to V$, the VOA $H^0_{DS,f}(V)$ is well-defined.

Theorem (Arakawa '15)

We have $X_{H^0_{DS,f}(V)} \cong X_V \times_{\mathfrak{g}^*} \mathscr{S}_f$. In particular, $X_{H^0_{DS,f}(L_k(\mathfrak{g}))} \cong X_{L_k(\mathfrak{g})} \cap \mathscr{S}_f$, and is isomorphic to $\overline{\mathbb{O}_k} \cap \mathscr{S}_f$ if k is admissible.

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$\rightsquigarrow$ Very useful in the study of representations of W-algebras.

Nilpotent Slodowy slices

If $\mathbb{O} = G.x$ is a nilpotent orbit of $\mathfrak{g}$, the intersection

$$\mathscr{S}_{\mathbb{O},f} := \overline{\mathbb{O}} \cap \mathscr{S}_f$$

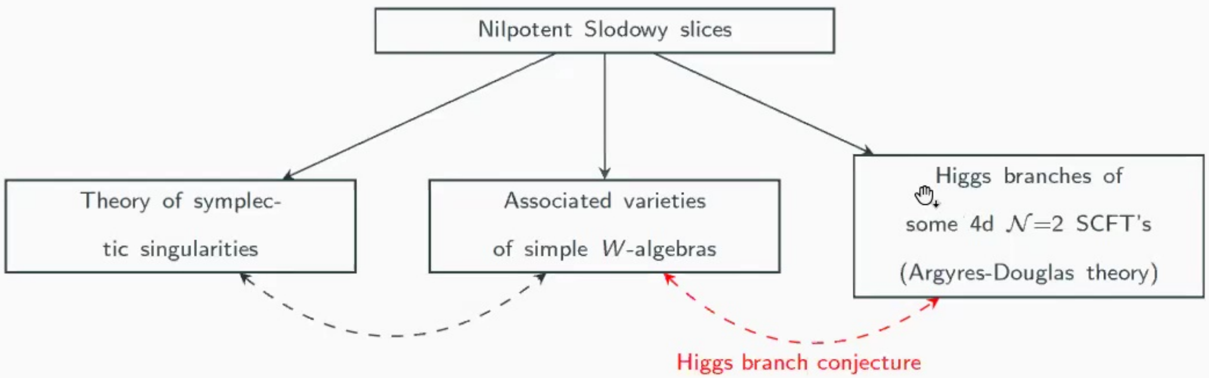
is a transverse slice to $\overline{\mathbb{O}}$ at the point f, called a *nilpotent Slodowy slice*.

$\rightsquigarrow$ *the local geometry of* $\overline{\mathbb{O}}$ *at* $f \in \overline{\mathbb{O}}$ *is encoded in* $\mathscr{S}_{\mathbb{O},f}$.

The geometry of $\mathscr{S}_{\mathbb{O},f}$ has been mainly studied in the case where $G.f$ is a *minimal degeneration* of $\overline{\mathbb{O}}$, that is, $G.f$ is a maximal orbit in the boundary $\overline{\mathbb{O}} \setminus \mathbb{O} = \operatorname{Sing} \overline{\mathbb{O}}$.

- When $\mathbb{O} = \mathbb{O}_{reg}$ ($\overline{\mathbb{O}_{reg}} = \mathcal{N}$, the nilpotent cone of $\mathfrak{g}$) and $f = f_{subreg}$, it is well-known that $\mathscr{S}_{\mathbb{O},f} = \mathcal{N} \cap \mathscr{S}_{f_{subreg}}$ has a simple surface singularity at f of the same type as $\mathfrak{g}$, provided $\mathfrak{g}$ is has type A, D, E [Brieskorn-Slodowy].
- When $\mathbb{O} = \mathbb{O}_{min}$ and $f = 0$, then $\mathscr{S}_{\mathbb{O},f} = \overline{\mathbb{O}_{min}}$ has a *minimal* symplectic singularity at 0.
- More generally, the generic singularities has been determined ([Kraft and Procesi '81-82] in the classical types, [Fu-Juteau-Levy-Sommers ' 17] in the exceptional types).

Nilpotent Slodowy slices appear in various areas



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Beem-Rastelli conjecture

Conjecture (Beem-Rastelli '18)

For any 4D $\mathcal{N} = 2$ SCFT $\mathcal{T}$, we have

$$\mathrm{Higgs}(\mathcal{T}) \cong X_{\mathbb{V}(\mathcal{T})},$$

where X_V is the associated variety of a VOA V.

Remarks :

1. the Higgs branch $\mathrm{Higgs}(\mathcal{T})$ is a hyperkähler cone, while the associated variety X_V of a VOA V is only a Poisson variety in general.
2. The VOA $\mathbb{V}(\mathcal{T})$ is not lisse unless $\mathrm{Higgs}(\mathcal{T}) = \{\text{point}\}$.
3. The conjecture has been proved in the special case of class $\mathcal{S}$ theory by Arakawa '18 (in this case, $\mathrm{Higgs}(\mathcal{T})$ is mathematically defined by Brevermann-Finkelberg-Nakajima '17).

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Quasi-lisse vertex algebras

The VOA $\mathbb{V}(\mathcal{T})$ is expected to be quasi-lisse.

Definition

V is called *quasi-lisse* if X_V has finitely many symplectic leaves.

Examples :

- $L_k(\mathfrak{g})$ is quasi-lisse $\iff$ $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$, the nilpotent cone of $\mathfrak{g}$.
  In particular, a simple admissible affine vertex algebra $L_k(\mathfrak{g})$ is quasi-lisse.
- If $\mathfrak{g}$ belongs to the *Deligne exceptional series*,

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8,$$

and $k = -h^\vee/6 - 1$, then $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{min}}$ so that $L_k(\mathfrak{g})$ is quasi-lisse [Arakawa-M. '18].

These are the VOAs (in red) that appeared in [BL^{2}PRvR] as the main examples of $\mathbb{V}(\mathcal{T})$!

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Quasi-lisse W-algebras

Let $\mathcal{W}_k(\mathfrak{g}, f)$ be the simple quotient of $\mathcal{W}^k(\mathfrak{g}, f)$.

Conjecturally [Kac-Wakimoto], $\mathcal{W}_k(\mathfrak{g}, f) = H^0_{DS,f}(L_k(\mathfrak{g}))$, provided that $H^0_{DS,f}(L_k(\mathfrak{g})) \neq 0$ (proven in many cases).

Note that $H^0_{DS,f}(L_k(\mathfrak{g})) \neq 0$ if and only if $f \in X_{L_k(\mathfrak{g})}$.

Assume that $L_k(\mathfrak{g})$ is admissible, so that $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_k}$.

Then

$$X_{\mathcal{W}_k(\mathfrak{g},f)} = \overline{\mathbb{O}_k} \cap \mathscr{S}_f = \mathscr{S}_{\mathbb{O}_k,f}.$$

In particular, $\mathcal{W}_k(\mathfrak{g}, f)$ is quasi-lisse if $f \in \overline{\mathbb{O}_k}$.

$\rightsquigarrow$ *We get many examples of lisse and quasi-lisse W-algebras...*

W-algebras and Argyres-Douglas theory

Fact (Xie-Yan-Yau '16, Song-Xie-Yan '17, Wang-Xie '19)

$\mathcal{W}_k(\mathfrak{g}, f)$ appears as $\mathbb{V}(\mathcal{T})$ for some Argyres-Douglas theory $\mathcal{T}$ if k is *boundary* admissible, that is, $k = -h^\vee + h^\vee/q$.

There are cases when $\mathcal{W}_k(\mathfrak{g}, f) \cong \mathcal{W}_{k'}(\mathfrak{g}', f')$. In particular, if

$$\mathcal{W}_k(\mathfrak{g}, f) \cong L_{k'}(\mathfrak{g}'),$$

the level k is called *collapsing* [Adamović-Kac-Möseneder-Papi-Perše '18].

- If $\mathcal{T} \cong \mathcal{T}'$ as physical theories then $\mathbb{V}(\mathcal{T}) \cong \mathbb{V}(\mathcal{T}')$, and so one can predict many isomorphisms.
- If $\mathcal{T} \cong \mathcal{T}'$ as physical theories then $\mathrm{Higgs}(\mathcal{T}) \cong \mathrm{Higgs}(\mathcal{T}')$.

  Conversely, from the coincidence of the singularities of different nilpotent Slodowy slices, we can guess many isomorphisms.

Collapsing levels for W-algebras

Let $\mathfrak{g}^\natural$ be the centralizer of the $\mathfrak{sl}_2$-triple (e, h, f). It is a reductive algebra : $\mathfrak{g}^\natural = \mathfrak{g}_0^\natural \oplus \left(\bigoplus_{i=1}^{s} \mathfrak{g}_i^\natural\right)$, where $\mathfrak{g}_0 := \mathfrak{z}(\mathfrak{g}^\natural)$ and $\mathfrak{g}_i^\natural$ are the simple factors of $[\mathfrak{g}^\natural, \mathfrak{g}^\natural]$.

[Kac-Wakimoto '04] : there is a vertex algebra morphism

$$\bigotimes_{i=0}^{s} V^{k_i^\natural}(\mathfrak{g}_i^\natural) =: V^{k^\natural}(\mathfrak{g}^\natural) \hookrightarrow \mathcal{W}^k(\mathfrak{g}, f),$$

where the $k_i^\natural$'s are some complex numbers determined by f and k.

Definition (Adamović-Kac-Möseneder-Papi-Perše '18)

We say that k is *collapsing for $\mathcal{W}_k(\mathfrak{g}, f)$* if the image of the composition map

$$V^{k^\natural}(\mathfrak{g}^\natural) \hookrightarrow \mathcal{W}^k(\mathfrak{g}, f) \twoheadrightarrow \mathcal{W}_k(\mathfrak{g}, f)$$

is surjective, that is,

$$\mathcal{W}_k(\mathfrak{g}, f) \cong L_{k^\natural}(\mathfrak{g}^\natural).$$

For example, if $\mathcal{W}_k(\mathfrak{g}, f) \cong \mathbb{C}$, then k is collapsing.

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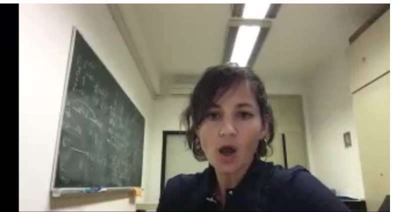
Motivations



- If k is collapsing, the vertex algebra homomorphism $\mathcal{W}^k(\mathfrak{g}, f) \longrightarrow\!\!\!\!\rightarrow \mathcal{W}_k(\mathfrak{g}, f) \cong L_{k^\natural}(\mathfrak{g}^\natural)$ induces an algebra homomorphism,

$$\mathrm{Zhu}(\mathcal{W}^k(\mathfrak{g}, \mathrm{f})) \cong \mathrm{U}(\mathfrak{g}, \mathrm{f}) \longrightarrow \mathrm{Zhu}(\mathrm{L}_{k^\natural}(\mathfrak{g}^\natural)) \cong \mathrm{U}(\mathfrak{g}^\natural)/\mathrm{I}.$$

which gives to the representation theory of $L_{k^\natural}(\mathfrak{g}^\natural)$ a richer structure.

- [AKMPP] Semisimplicity of some categories of $L_k(\mathfrak{g})$-modules for $f = f_{min}$.

- As already mentioned, collapsing levels are important in the Argyres-Douglas theory.

What is known about collapsing levels?

- [AKMPP '18] There is a full classification of collapsing levels for $\mathcal{W}_k(\mathfrak{g}, f_{min})$, including simple affine Lie superalgebras.

Furthermore, there is a full classification of pairs $(\mathfrak{g}, k)$ such that $\mathcal{W}_k(\mathfrak{g}, f_{min}) \cong \mathbb{C}$ [Arakawa-M. '18, AKMPP for the super case].

- However, little or almost nothing was known for collapsing levels for non minimal nilpotent elements.

The main reason is that for an arbitrary nilpotent element f, the commutation relations in $\mathcal{W}_k(\mathfrak{g}, f)$ are unknown.

Idea to find appropriate candidates for f and k? If k is collapsing, then obviously

$$X_{\mathcal{W}_k(\mathfrak{g},f)} \cong X_{L_{k^\natural}(\mathfrak{g}^\natural)},$$

and this is a very restrictive condition on (k, f).

When k and $k^\natural$ are admissible, such coincidences can be understood by considering singularities of nilpotent Slodowy slices...

Illustrating example

Assume that $\mathfrak{g} = \mathfrak{sl}_n$.

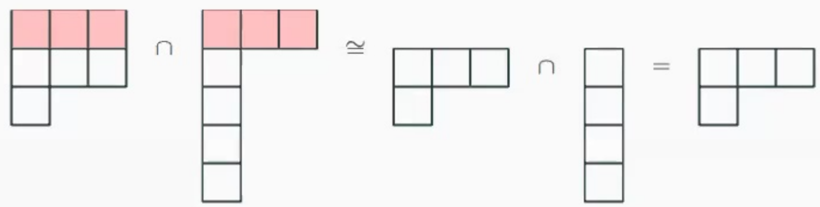
$$\begin{array}{ccc} \{\text{nilpotent orbits in } \mathfrak{sl}_n\} & \longleftrightarrow & \{\text{partitions of } n\} \\ \mathbb{O}_\lambda & \longleftarrow & \boldsymbol{\lambda} \end{array}$$

Let k be admissible, i.e., $k = -n + p/q$, $(p, q) = 1$, $p \geqslant n$.

Then $X_{L_k(\mathfrak{sl}_n)} = \overline{\mathbb{O}_k} = \overline{\mathbb{O}_{\boldsymbol{\lambda}}}$ with $\boldsymbol{\lambda} = (q^{m_0}, s_0)$, where $n = qm_0 + s_0$, $0 \leqslant s_0 < q$.

Pick $f \in \mathbb{O}_{\boldsymbol{\mu}} \subset \overline{\mathbb{O}_{\boldsymbol{\lambda}}}$. Kraft-Procesi's *removal common rows* rule (improved by Yiqiang Li, 2019) allows to describe $\mathscr{S}_{\mathbb{O}_{\boldsymbol{\lambda}}, f}$ in some cases.

Example : $n = 7$, $q = 3$ so that $\boldsymbol{\lambda} = (3^2, 1)$, $f \in \mathbb{O}_{(3,1^4)}$.



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Then $X_{L_k(\mathfrak{sl}_n)} = \overline{\mathbb{O}_k} = \overline{\mathbb{O}_{\boldsymbol{\lambda}}}$ with $\boldsymbol{\lambda} = (q^{m_0}, s_0)$, where $n = qm_0 + s_0$, $0 \leqslant s_0 < q$.

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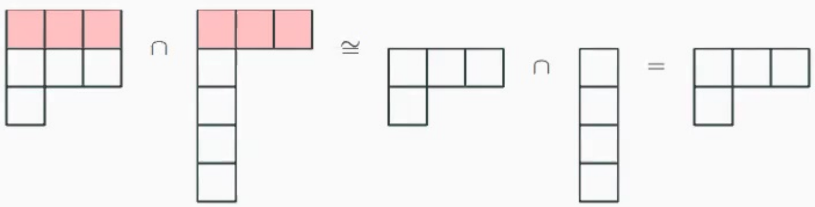


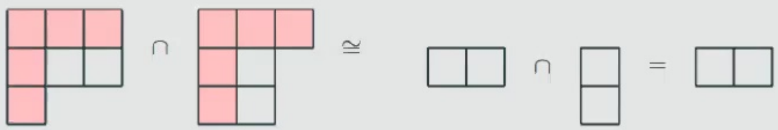
Remark : $\mathfrak{g}^\natural = \mathbb{C} \times \mathfrak{sl}_4$, $k_0^\natural = 3k + 4$, $k_1^\natural = k + 2$.

Question : $\mathcal{W}_{-7+7/3}(\mathfrak{sl}_7, f) \cong L_{-4+4/3}(\mathfrak{sl}_4)$?

Theorem (Arakawa-van Ekeren-M., 2020)

Assume that $\mathfrak{g} = \mathfrak{sl}_n$, $k = -n + p/q$ admissible.

1. Pick $f \in \mathbb{O}_k$ so that $\mathcal{W}_k(\mathfrak{sl}_n, f)$ is lisse (and even rational).
   - if $n \equiv \pm 1 \mod q$, then $\mathcal{W}_k(\mathfrak{sl}_n, f) \cong \mathbb{C}$.
   - if $n \equiv 0 \mod q$, then $\mathcal{W}_{-n+(n+1)/q}(\mathfrak{sl}_n, f) \cong L_1(\mathfrak{sl}_{m_0})$.
2. Pick $f \in \mathbb{O}_{(q^m, 1^s)} \in \overline{\mathbb{O}_k}$ with $s \neq 0$. Then $\mathcal{W}_{-n+n/q}(\mathfrak{sl}_n, f) \cong L_{-s+s/q}(\mathfrak{sl}_s)$.
3. Assume that $n = qm_0 + (q-2)$ and pick $f \in \mathbb{O}_{(q^{m_0-1},(q-1)^2)} \in \overline{\mathbb{O}_k}$. Then $\mathcal{W}_{-n+n/q}(\mathfrak{sl}_n, f) \cong L_{-2+2/q}(\mathfrak{sl}_2)$.



- We have similar results for $\mathfrak{sp}_n$, $\mathfrak{so}_n$ and the exceptional types.
- In term of associated varieties, it yields isomorphisms of Poisson varieties.
- We conjecture that our results furnish the exhaustive liste of admissible collapsing levels k.

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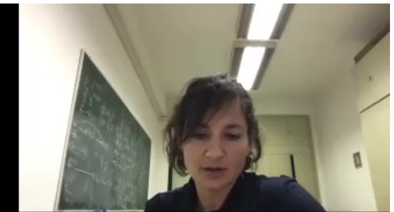
Proof : asymptotic behaviour of characters



Key point : if V is either $L_k(\mathfrak{g})$ or $H^0_{DS,f}(L_k(\mathfrak{g}))$, with k admissible, the normalized character $\chi_V(\tau)$ has a nice asymptotic behaviour.

More precisely [Arakawa-van Ekeren-M., 2020] : if V is a quasi-lisse vertex algebras, then

$$\chi_V(\tau) \sim \boldsymbol{A}_V e^{\frac{\pi i}{12\tau}\boldsymbol{g}_V} \quad \text{as} \quad \tau \downarrow 0,$$

where $\boldsymbol{A}_V$, $\boldsymbol{g}_V$ are some constants.

Theorem (Arakawa-van Ekeren-M., 2020)

Let k be an admissible level, and $f \in \overline{\mathbb{O}_k}$. If

$$\chi_{\mathcal{W}_k(\mathfrak{g},f)}(\tau) \sim \chi_{L_{k^\natural}(\mathfrak{g}^\natural)}(\tau) \quad \text{as} \quad \tau \downarrow 0,$$

then k is collapsing, that is, $\mathcal{W}_k(\mathfrak{g},f) \cong L_{k^\natural}(\mathfrak{g}^\natural)$.

Example : for $f \in \mathbb{O}_{(q^m,1^s)} \subset \mathfrak{sl}_n$,

$$\boldsymbol{A}_{\mathcal{W}_{-n+n/q}(\mathfrak{sl}_n,f)} = q^{(s^2-1)/2} = \boldsymbol{A}_{L_{-s+s/q}(\mathfrak{sl}_s)},$$

$$\boldsymbol{g}_{\mathcal{W}_{-n+n/q}(\mathfrak{sl}_n,f)} = (1-1/q)(s^2-1) = \boldsymbol{g}_{L_{-s+s/q}(\mathfrak{sl}_s)}.$$

Some examples in the exceptional cases

In the exceptional types, nilpotent orbits are classified by the Bala-Carter theory.

1. For example, in $\mathfrak{g} = E_6$, the isomorphism $\overline{\mathbb{O}_{E_6(a_3)}} \cap \mathscr{S}_{D_4} \cong \mathcal{N}_{A_2}$ admits the following liftings :

$$\mathcal{W}_{-12+12/7}(E_6, D_4) \cong L_{-3+3/7}(A_2), \quad \mathcal{W}_{-12+13/6}(E_6, D_4) \cong L_{-3+4/3}(A_2).$$

   Similarly, the isomorphism $\overline{\mathbb{O}_{A_4+A_2}} \cap \mathscr{S}_{A_4} \cong \overline{a_1}$ has the following lifting :

$$\mathcal{W}_{-12+12/5}(E_6, A_4) \cong L_{-2+2/5}(A_1).$$

2. In $\mathfrak{g} = E_8$, we have (among others) the isomorphisms

$$\mathcal{W}_{-30+30/7}(E_8, D_4) \cong L_{-9+9/7}(F_4), \quad \mathcal{W}_{-30+31/6}(E_8, D_4) \cong L_{-9+13/6}(F_4),$$

$$\mathcal{W}_{-30+31/3}(E_8, 2A_2) \cong L_{-4+7/3}(G_2).$$

   As a consequence we obtain (new ?) isomorphisms of Poisson varieties :

$$\mathscr{S}_{A_6+A_1, D_4} \cong \overline{\mathbb{O}_{F_4(a_2)}}, \quad \mathscr{S}_{E_8(a_7), D_4} \cong \overline{\mathbb{O}_{F_4(a_3)}},$$

$$\mathscr{S}_{3A_2+2A_1, 2A_2} \cong \overline{g_2} \times \overline{g_2}.$$

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A conjecture

Sometimes, $\mathcal{W}_k(\mathfrak{g}, f)$ is merely a finite extension of $L_{k^\natural}(\mathfrak{g}^\natural)$.

In most of such cases, the associated varieties of $\mathcal{W}_k(\mathfrak{g}, f)$ and $L_{k^\natural}(\mathfrak{g}^\natural)$ are isomorphic.

However, it is not always true : when it is not, we observe that they are at least birationally equivalent.

Example : $\mathcal{W}_{-12+13/2}(E_6, A_1) \cong L_{-6+7/2}(A_5) \oplus L_{-6+7/2}(A_5; \varpi_3)$. But $X_{\mathcal{W}_{-12+13/2}(E_6,A_1)} \cong X_{L_{-6+7/2}(A_5)}$ since $\mathscr{S}_{3A_1,A_1} \ncong \overline{\mathbb{O}_{(2^3)}}$. However, $\mathscr{S}_{3A_1,A_1}$ and $\overline{\mathbb{O}_{(2^3)}}$ are birationally equivalent.

We formulate a more general conjecture :

Conjecture (Arakawa-van Ekeren-M., 2020)

If a vertex algebra V is finite extension of a vertex subalgebra $W \subset V$, then the associated varieties of V and W are birationally equivalent.

Note that it is known that if W is lisse then V is lisse. Our conjecture suggests that the converse holds as well.

Some non-admissible collapsing levels

Using different methods one can obtain a few non-admissible collapsing levels.

For examples, from explicit OPEs in type B_2 and G_2 (Justine Fasquel's computations), one obtains (Adamović-Fasquel) the following isomorphisms :

- $\mathcal{W}_{-1}(B_2, f_{subreg}) \cong M(1)$,
- $\mathcal{W}_{-2}(B_2, f_{subreg}) \cong Vir_{-2}$,
- $\mathcal{W}_{-2}(G_2, f_{subreg}) \cong \mathbb{C}$.

In particular, the last one gives $X_{L_{-2}(G_2)} \cong \overline{\mathbb{O}_{subreg}}$.

We also formulate a number of conjectures :

- $\mathcal{W}_{-9}(E_6, 2A_2) \cong L_{-3}(G_2)$,
- $\mathcal{W}_{-6}(E_6, 2A_1) \cong L_{-2}(B_3)$,
- $\mathcal{W}_{-12}(E_7, A_2 + 3A_1) \cong L_{-2}(G_2)$,
- $\mathcal{W}_{-6}(F_4, \tilde{A}_2) \cong L_{-2}(G_2)$, etc.

It would be interesting to know whether these isomorphisms have a physical interpretation.

Thank you !