

Title: Resurgence and Phase Transitions

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Series: Quantum Fields and Strings

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Abstract: There are several important conceptual and computational questions concerning path integrals in QM and QFT, which have recently been approached from new perspectives motivated by "resurgent asymptotics", a novel mathematical formalism that seeks to unify perturbative and non-perturbative physics. I will discuss the basic ideas behind the connections between resurgent asymptotics and physics, ranging from differential equations to phase transitions and QFT. I will also discuss the reconstruction problem: how to optimally reconstruct non-perturbative information from a finite amount of perturbative information.



# Resurgence and Phase Transitions

Gerald Dunne

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*QFT Seminar, Perimeter Institute  
October 20, 2020*

A.Ahmed & GD: [1710.01812](#)  
GD: [1901.02076](#)

O.Costin & GD: [1904.11593](#), [2003.07451](#), [2009.01962](#), ...

[DOE Division of High Energy Physics]



## Physical Motivation: Resurgence and Quantum Field Theory

- non-perturbative definition of QFT
  - Minkowski vs. Euclidean QFT & quantum gravity
  - "sign problem" in finite density QFT
  - dynamical & non-equilibrium physics in path integrals
  - phase transitions (Lee-Yang and Fisher zeroes)
  - common thread: analytic continuation of path integrals
- 
- question 1: does resurgence give (useful) new insight?
  - question 2: does resurgence help to decode information from a finite-order expansion?

## Physical Motivation

- what does a Minkowski path integral mean?

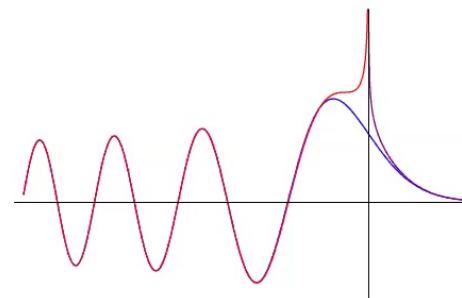
$$\int \mathcal{D}A \exp\left(\frac{i}{\hbar} S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \exp\left(-\frac{1}{\hbar} S[A]\right)$$



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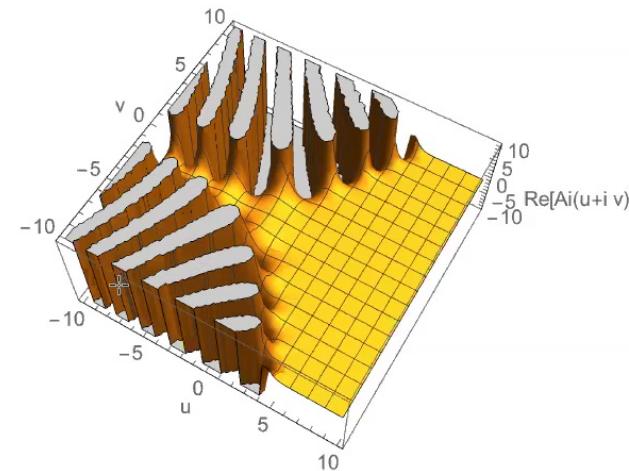
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\frac{1}{3}t^3 + xt)} dt \sim \begin{cases} \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt[4]{\pi}x^{1/4}}, & x \rightarrow +\infty \\ \frac{\sin(\frac{2}{3}(-x)^{3/2} + \frac{\pi}{4})}{\sqrt{\pi}(-x)^{1/4}}, & x \rightarrow -\infty \end{cases}$$

- real physics can be governed by complex saddles

## Physical Motivation

- behavior in complex  $x$  plane is also physically significant

- Airy Stokes sectors
- anti-Stokes lines:  $\arg(x) = \pm\frac{\pi}{3}, \pi$
- Stokes lines:  $\arg(x) = \pm\frac{2\pi}{3}, 0$



- non-perturbative connection formulas connect sectors
- encoded in a saddle decomposition (Picard-Lefschetz)
- since we need complex analysis and contour deformation to make sense of oscillatory ordinary integrals, it is natural to explore similar methods for path integrals

## The Big Question

- Can we make physical, mathematical and computational sense of a Lefschetz thimble expansion of a path integral?

$$Z(\hbar) = \int \mathcal{D}A \exp\left(\frac{i}{\hbar} S[A]\right)$$
$$" = " \sum_{\text{thimble}} \mathcal{N}_{\text{th}} e^{i\phi_{\text{th}}} \int_{\text{th}} \mathcal{D}A \times (\mathcal{J}_{\text{th}}) \times \exp\left(\mathcal{R}e\left[\frac{i}{\hbar} S[A]\right]\right)$$

- $Z(\hbar) \rightarrow Z(\hbar, N, \text{masses, couplings, } \mu, T, B, E, \dots)$
- $Z(\hbar) \rightarrow Z(\hbar, N)$ , and  $N \rightarrow \infty$  for a phase transition
- a satisfactory definition/construction of the path integral should describe Stokes sectors and transitions
- resurgence and Stokes transitions: transmutation of trans-series structures across phase transitions

# Resurgent Trans-Series

Resurgence: ‘new’ idea in mathematics

(Écalle 1980; Dingle 1960s; Stokes 1850)

resurgence = unification of perturbation theory and  
non-perturbative physics

resurgence = global complex analysis with  
asymptotic series

- perturbative series expansion  $\longrightarrow$  *trans-series* expansion

$$f(g^2) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \underbrace{c_{k,l,p} g^{2p}}_{\text{perturbative fluctuations}} \underbrace{\left( \exp \left[ -\frac{c}{g^2} \right] \right)^k}_{\text{k-instantons}} \underbrace{\left( \ln \left[ \pm \frac{1}{g^2} \right] \right)^l}_{\text{quasi-zero-modes}}$$

- trans-series ‘well-defined under analytic continuation’
- perturbative and non-perturbative physics entwined
- ODEs, PDEs, difference equations, fluid mechanics, QM, Matrix Models, QFT, Chern-Simons, String Theory,

## “Resurgence”

*resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin.*

*Loosely speaking, these functions resurrect, or **surge up** - in a slightly different guise, as it were - at their singularities*

*J. Écalle*



fluctuations about different singularities are quantitatively related

## Borel summation: extracting physics from asymptotic series

Borel transform of series, where  $c_n \sim n!$  ,  $n \rightarrow \infty$

$$f(g) \sim \sum_{n=0}^{\infty} c_n g^n \quad \longrightarrow \quad \mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

new series typically has a **finite** radius of convergence

Borel summation of original asymptotic series:

$$\mathcal{S}f(g) = \frac{1}{g} \int_0^{\infty} \mathcal{B}[f](t) e^{-t/g} dt$$

- the singularities of  $\mathcal{B}[f](t)$  provide a physical encoding of the global asymptotic behavior of  $f(g)$ , which is also much more mathematically efficient than the asymptotic series
- Borel singularities  $\leftrightarrow$  non-perturbative physical objects
- resurgence: isolated poles, algebraic & logarithmic cuts

## Airy function: recovering non-perturbative information

- “perturbation theory”:

$$y'' = x y \Rightarrow \left\{ \begin{array}{l} \text{Ai}(x) \\ \frac{1}{2} \text{Bi}(x) \end{array} \right\} \sim \frac{e^{\mp \frac{2}{3}x^{3/2}}}{2\sqrt{\pi} x^{1/4}} \sum_{n=0}^{\infty} (\mp 1)^n \frac{\Gamma(n + \frac{1}{6}) \Gamma(n + \frac{5}{6})}{2\pi n! \left(\frac{4}{3} x^{3/2}\right)^n}$$

- non-perturbative connection formula:

$$\text{Bi}(x) = 2 e^{\pm \pi i/6} \text{Ai}\left(e^{\mp \frac{2\pi i}{3}} x\right) \mp i \text{Ai}(x)$$

- Borel representation

$$\text{Ai}(x) = \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi} x^{1/4}} \left(\frac{4}{3}x^{3/2}\right) \int_0^\infty dt e^{-\frac{4}{3}x^{3/2}t} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -t\right)$$

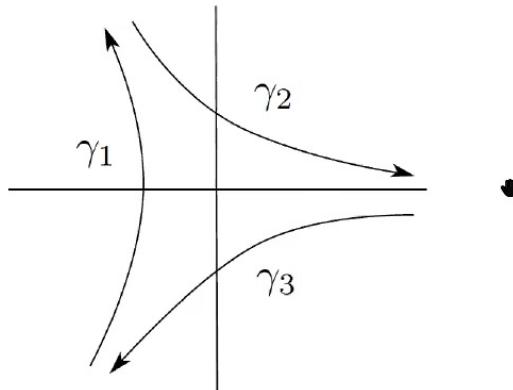
- Borel singularities  $\leftrightarrow$  non-perturbative information

## Airy function: Picard-Lefschetz decomposition

- reverse direction: “path integral”  $\rightarrow$  trans-series

$$\text{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt e^{i(xt + \frac{1}{3}t^3)} = \frac{\sqrt{r}}{2\pi i} \int_{\gamma} dz e^{r^{3/2}(e^{i\theta} z - \frac{1}{3}z^3)}$$

- saddles & steepest descent curves depend on phase  $\theta \equiv \arg(x)$



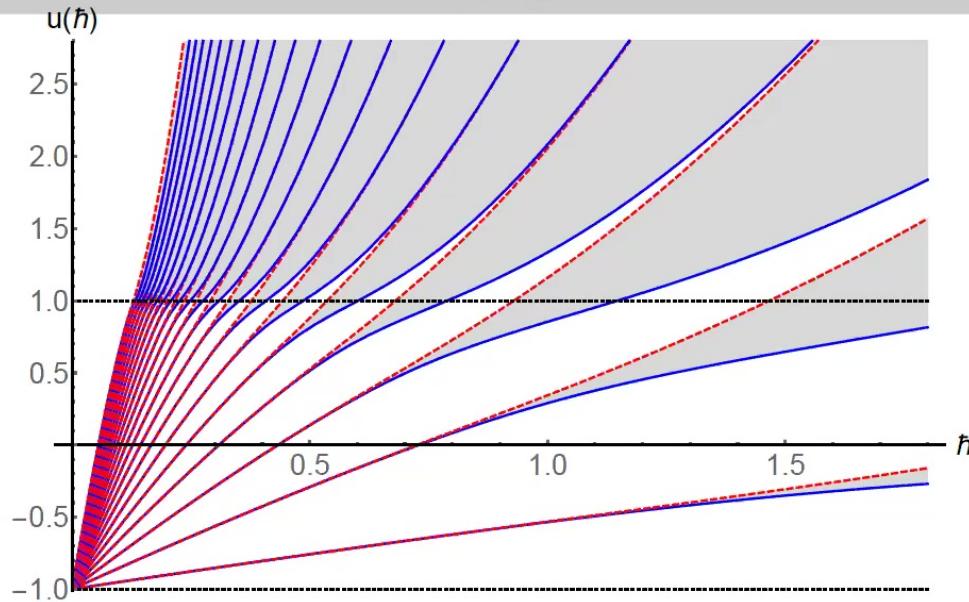
- decompose into a combination of steepest descent curves  $\Gamma_j$  deformed from the allowed curves  $\gamma_j$

to preserve analytic structure, different thimbles  $\Gamma_j$  must be related

## Towards Picard-Lefschetz theory for Path Integrals

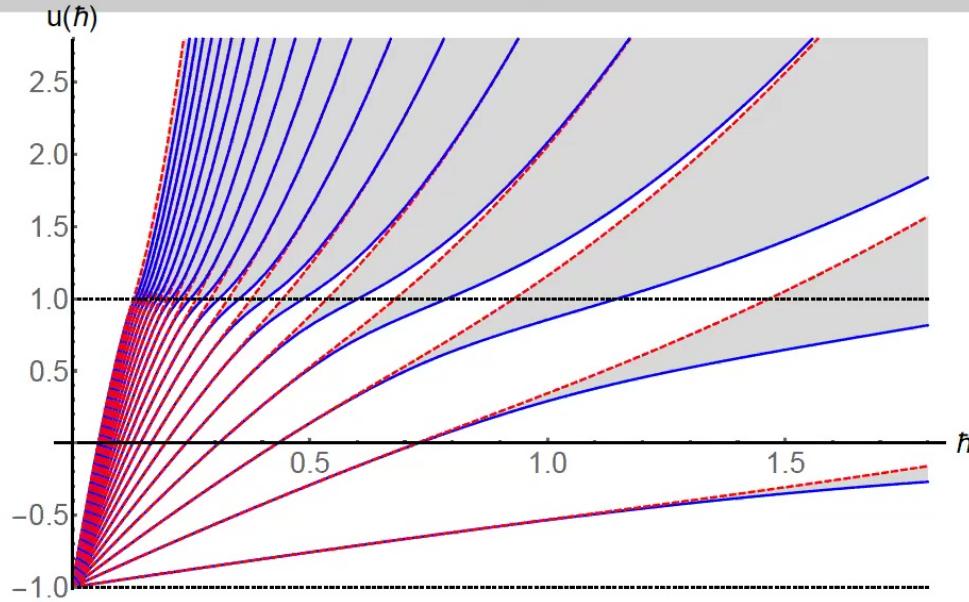
- to preserve analytic structure, different thimbles  $\Gamma_j$  must be related
- a satisfactory path integral construction should capture these physical Stokes jumps 
- infinite dimensions becomes interesting
- multiple parameters becomes interesting (phase transitions)
- renormalization and running couplings become interesting

Mathieu Equation Spectrum:  $-\frac{\hbar^2}{2} \frac{d^2\psi}{dx^2} + \cos(x) \psi = u \psi$



$$u_{\pm}(\hbar, N) = u_{\text{pert}}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left( \frac{32}{\hbar} \right)^{N+\frac{1}{2}} \exp \left[ -\frac{8}{\hbar} \right] \mathcal{P}_{\text{inst}}(\hbar, N) + \dots$$

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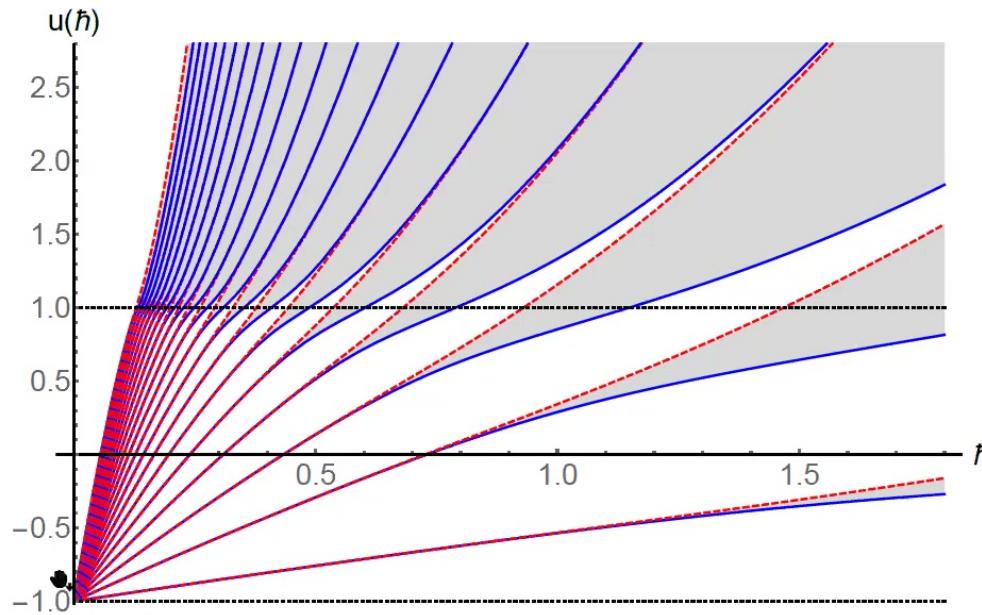


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$$\mathcal{P}_{\text{inst}}(\hbar, N) = \frac{\partial u_{\text{pert}}(\hbar, N)}{\partial N} \exp \left[ S \int_0^{\hbar} \frac{d\hbar}{\hbar^3} \left( \frac{\partial u_{\text{pert}}(\hbar, N)}{\partial N} - \hbar + \frac{(N + \frac{1}{2}) \hbar^2}{S} \right) \right]$$

all non-perturbative effects encoded in perturbative expansion

Mathieu Equation Spectrum:  $-\frac{\hbar^2}{2} \frac{d^2\psi}{dx^2} + \cos(x) \psi = u \psi$



- phase transition at  $\hbar N = \frac{8}{\pi}$ : narrow bands vs. narrow gaps
- real vs. complex instantons (Dykhne, 1961; Başar/GD)
- phase transition = "instanton condensation" (Neuberger, 1981)
- maps to  $\mathcal{N} = 2$  SUSY QFT (Nekrasov et al, Mironov et al; ...)

# Tunneling vs. Multiphoton Pair Production in QED

Keldysh (1964); Brézin/Itzykson, 1970; Popov, 1971

- "Schwinger effect" with  $E(t) = \mathcal{E} \cos(\omega t)$
- adiabaticity parameter:  $\gamma \equiv \frac{mc\omega}{e\mathcal{E}}$
- WKB  $\Rightarrow$  rate  $\Gamma_{\text{QED}} \sim \exp \left[ -\pi \frac{m^2 c^3}{e \hbar \mathcal{E}} g(\gamma) \right]$

$$\Gamma_{\text{QED}} \sim \begin{cases} \exp \left[ -\pi \frac{m^2 c^3}{e \hbar \mathcal{E}} \right] & , \quad \gamma \ll 1 \quad (\text{non-perturbative}) \\ \left( \frac{e \mathcal{E}}{\omega m c} \right)^{4mc^2/\hbar\omega} & , \quad \gamma \gg 1 \quad (\text{perturbative}) \end{cases}$$

- phase transition: **tunneling vs. multi-photon pair production**
- phase transition: real/complex instantons ([Dumlu, GD, 2011](#))
- SLAC & DESY experiments aim to probe this transition ([SLAC LoI](#); [DESY LoI](#))

## Resurgence in 2d Lattice Ising Model

- diagonal correlation function:  $\mathcal{C}(t, N) = \langle \sigma_{0,0} \sigma_{N,N} \rangle(t)$
- $\mathcal{C}(t, N)$  = tau function for Painlevé VI (Jimbo, Miwa, 1980)
- Toeplitz det representation ("linearizes")
- scaling limit:  $N \rightarrow \infty$  &  $T \rightarrow T_c$ : PVI  $\rightarrow$  PIII (McCoy et al)

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- convergent (and resurgent) conformal block expansions at high and low  $T$  (Jimbo; Lisovyy et al; Bonelli et al; GD)

$$\mathcal{C}(t, N) \sim \sum_{n=-\infty}^{\infty} s^n C(\vec{\theta}, \rho + n) \mathcal{B}(\vec{\theta}, \rho + n; t)$$

$$\mathcal{B}(\vec{\theta}, \rho; t) \propto t^{\rho^2} \sum_{\lambda, \mu \in \mathcal{Y}} \mathcal{B}_{\lambda, \mu}(\vec{\theta}, \rho) t^{|\lambda| + |\mu|}$$

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- implies resurgence for large classes of integrable models

Resurgence in Matrix Models: Mariño: 0805.3033, Ahmed & GD: 1710.01812  
Gross-Witten-Wadia Unitary Matrix Model

$$Z(g^2, N) = \int_{U(N)} DU \exp \left[ \frac{1}{g^2} \text{tr} \left( U + U^\dagger \right) \right]$$

- one-plaquette matrix model for 2d lattice Yang-Mills
- two variables:  $g^2$  and  $N$  ('t Hooft coupling:  $t \equiv g^2 N/2$ )
- 3rd order phase transition at  $N = \infty$ ,  $t = 1$  (**universal!**)
- double-scaling limit: Painlevé II
- physics of phase transition = condensation of instantons
- random matrix theory/orthogonal polynomials result:

$$Z(g^2, N) = \det (I_{j-k}(x))_{j,k=1,\dots,N} , \quad x \equiv \frac{2}{g^2}$$

## Gross-Witten-Wadia: $N = \infty$ Phase Transition

3rd order transition: kink in the specific heat

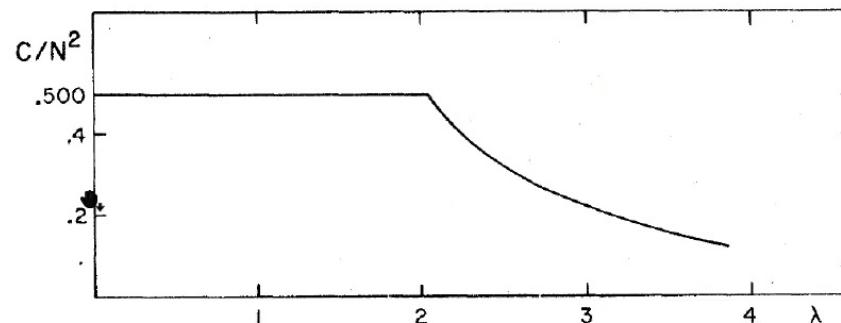


FIG. 2. The specific heat per degree of freedom,  $C/N^2$ , as a function of  $\lambda$  (temperature).

D. Gross, E. Witten, 1980

- what about non-perturbative large  $N$  effects?

$$\mathcal{F}(t, N) = \sum_n \frac{c_n^{(0)}(t)}{N^n} + e^{-NS(t)} \sum_n \frac{c_n^{(1)}(t)}{N^n} + e^{-2NS(t)} \sum_n \frac{c_n^{(2)}(t)}{N^n} + \dots$$

- Stokes transmutations between strong & weak coupling

# Resurgence in Gross-Witten-Wadia Model:

## Transmutation of the Trans-series

Ahmed & GD: 1710.01812

- “order parameter”: with ’t Hooft coupling  $t \equiv \frac{1}{2} N g^2$

$$\Delta(t, N) \equiv \langle \det U \rangle = \frac{\det [I_{j-k+1}(\frac{N}{t})]_{j,k=1,\dots,N}}{\det [I_{j-k}(\frac{N}{t})]_{j,k=1,\dots,N}}$$

- for any  $N$ ,  $\Delta(t, N)$  satisfies a Painlevé III equation:

$$t^2 \Delta'' + t \Delta' + \frac{N^2 \Delta}{t^2} (1 - \Delta^2) = \frac{\Delta}{1 - \Delta^2} \left( N^2 - t^2 (\Delta')^2 \right)$$

- $N$  is now a parameter, not necessarily integer !
- generate large  $N$  trans-series expansions

## Resurgence: Large $N$ 't Hooft limit at Weak Coupling

- large  $N$  trans-series at weak-coupling ( $t \equiv N/x < 1$ )

$$\Delta(t, N) \sim \sqrt{1-t} \sum_{n=0}^{\infty} \frac{d_n^{(0)}(t)}{N^{2n}} - \frac{\sigma_{\text{weak}}}{2\sqrt{2\pi N}} \frac{t e^{-NS_{\text{weak}}(t)}}{(1-t)^{1/4}} \sum_{n=0}^{\infty} \frac{d_n^{(1)}(t)}{N^n} + \dots$$

- weak-coupling large  $N$  instanton action

$$S_{\text{weak}}(t) = \frac{2\sqrt{1-t}}{t} - 2 \operatorname{arctanh}(\sqrt{1-t})$$

- large-order growth of perturbative coefficients ( $\forall t < 1$ ):

$$d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}} \frac{\Gamma(2n - \frac{5}{2})}{(S_{\text{weak}}(t))^{2n - \frac{5}{2}}} \left[ 1 + \frac{(3t^2 - 12t - 8)}{96(1-t)^{3/2}} \frac{S_{\text{weak}}(t)}{(2n - \frac{7}{2})} + \dots \right]$$

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- (parametric) resurgence relations, for all  $t$ :

$$\sum_{n=0}^{\infty} \frac{d_n^{(1)}(t)}{N^n} = 1 + \frac{(3t^2 - 12t - 8)}{96(1-t)^{3/2}} \frac{1}{N} + \dots$$

## Resurgence: Large $N$ 't Hooft limit at Strong Coupling

- large  $N$  transseries at strong-coupling:  $\Delta(t, N) \approx \sigma J_N\left(\frac{N}{t}\right)$

$$\Delta(t, N) = \sum_{k=1,3,5,\dots}^{\infty} (\sigma_{\text{strong}})^k \Delta_{(k)}(t, N)$$

- "Debye expansion" for Bessel function:  $J_N(N/t)$

$$\begin{aligned} \Delta(t, N) \sim & \frac{\sqrt{t} e^{-NS_{\text{strong}}(t)}}{\sqrt{2\pi N} (t^2 - 1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n(t)}{N^n} \\ & + \frac{1}{4(t^2 - 1)} \left( \frac{\sqrt{t} e^{-NS_{\text{strong}}(t)}}{\sqrt{2\pi N} (t^2 - 1)^{1/4}} \right)^3 \sum_{n=0}^{\infty} \frac{U_n^{(1)}(t)}{N^n} + \dots \end{aligned}$$

- strong-coupling large  $N$  action:  $S_{\text{st}}(t) = \text{arccosh}(t) - \sqrt{1 - \frac{1}{t^2}}$

- resurgence suggests that local analysis of perturbation theory encodes global information
- Questions:  
How much global information can be decoded from a FINITE number of perturbative coefficients ?  
How much information is needed to detect and to probe a phase transition ?

- resurgence suggests that local analysis of perturbation theory encodes global information
- Questions:  
How much global information can be decoded from a FINITE number of perturbative coefficients ?  
How much information is needed to detect and to probe a phase transition ?
- resurgent functions have orderly structure in Borel plane  
⇒ develop extrapolation and summation methods that take advantage of this
- high precision test for Painlevé I (but integrability is not important for the method)
- explicit large  $N$  estimates ( $N$  = number of terms)

## Perturbative Expansion of Painlevé I Equation

- Painlevé I equation (double-scaling limit of 2d quantum gravity)

$$y''(x) = 6y^2(x) - x$$

- large  $x$  expansion:

$$y(x) \sim -\sqrt{\frac{x}{6}} \left( 1 + \sum_{n=1}^{\infty} a_n \left( \frac{30}{(24x)^{5/4}} \right)^{2n} \right) \quad , \quad x \rightarrow +\infty$$

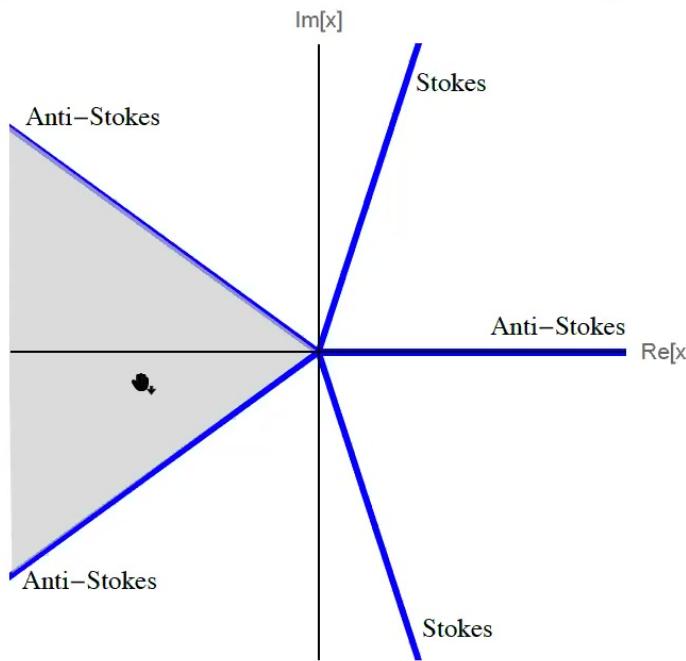
- “perturbative” input data:  $\{a_1, a_2, \dots, a_N\}$

$$\left\{ \frac{4}{25}, -\frac{392}{625}, \frac{6272}{625}, -\frac{141196832}{390625}, \frac{9039055872}{390625}, \dots, a_N \right\}$$

- this expansion defines the *tritronquée* solution to PI

Reconstruct global behavior from limited  $x \rightarrow +\infty$  data?

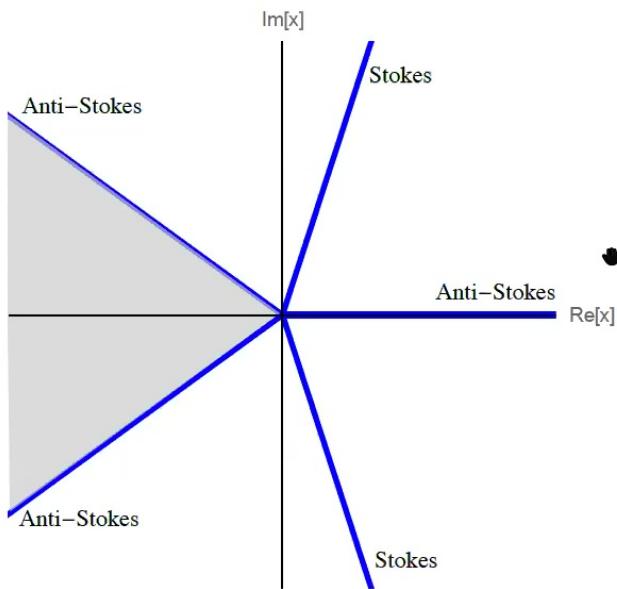
- Painlevé I equation has inherent five-fold symmetry



- do the input coefficients (from  $x = +\infty$ ) “know” this ?
- most interesting/difficult directions: phase transitions

## Nonlinear Stokes Transition: the Tritronquée Pole Region

- Boutroux (1913): asymptotically, general Painlevé I solution has poles with 5-fold symmetry
- Dubrovin conjecture (2009): this asymptotic solution to Painlevé I only has poles in a  $\frac{2\pi}{5}$  wedge

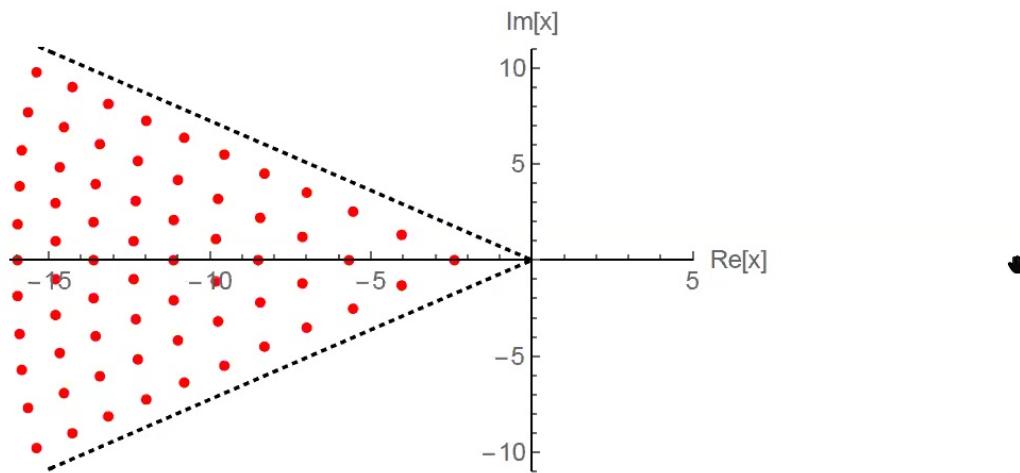


- proof: Costin-Huang-Tanveer (2012)

## Stokes Transition: Mapping the Tritronquée Pole Region

- non-linear Stokes transitions crossing  $\arg(x) = \pm \frac{4\pi}{5}$

O.Costin & GD, 1904.11593, 2009.01962



- first 66 Tritronquée poles decoded from asymptotics at  $x = +\infty$

## Metamorphosis: Asymptotic Series to Meromorphic Function

$$\begin{aligned}y(x) \approx & \frac{1}{(x - x_{\text{pole}})^2} + \frac{x_{\text{pole}}}{10}(x - x_{\text{pole}})^2 + \frac{1}{6}(x - x_{\text{pole}})^3 \\& + h_{\text{pole}}(x - x_{\text{pole}})^4 + \frac{x_{\text{pole}}^2}{300}(x - x_{\text{pole}})^6 + \dots\end{aligned}$$

- our extrapolation ( $y_N(x)$  with  $N = 50$ ) near 1st pole:

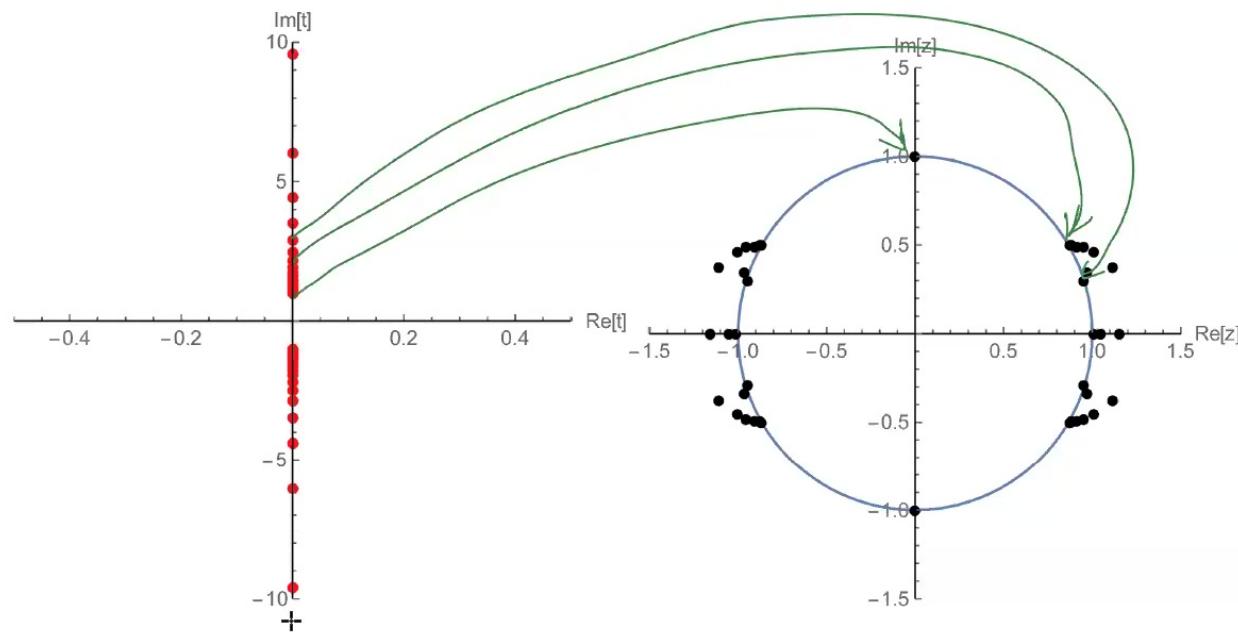
$$\begin{aligned}y(x) \approx & \frac{0.997886}{(x - x_1)^2} \\& + 3.5 \times 10^{-35} - 2.4 \times 10^{-34}(x - x_1) \\& - 0.238416876956881663929914585244923803(x - x_1)^2 \\& + 0.16666666666666666666666666666657864(x - x_1)^3 \\& - 0.06213573922617764089649014164005140(x - x_1)^4 \\& + 4 \times 10^{-31}(x - x_1)^5 \\& + 0.0189475357392909503157755851627665(x - x_1)^6 + \dots\end{aligned}$$

- estimate approx 30 digit precision for  $x_1$  and  $h_1$

- optimal extrapolation theorem:  
(Borel +) **uniformizing map** (+ Padé)
- Padé represents cuts as lines of poles  
⇒ obscures genuine higher resurgent singularities
- conformal or uniformizing maps resolve these higher singularities
- dramatic improvement of analytic continuation near singularities
- uniformizing maps extrapolate to higher Riemann sheets
- “singularity elimination” method: Borel convolutions can transform and eliminate a singularity ⇒ extreme sensitivity to the location and nature of the singularity

## Conformal Borel Analysis: resolving higher singularities

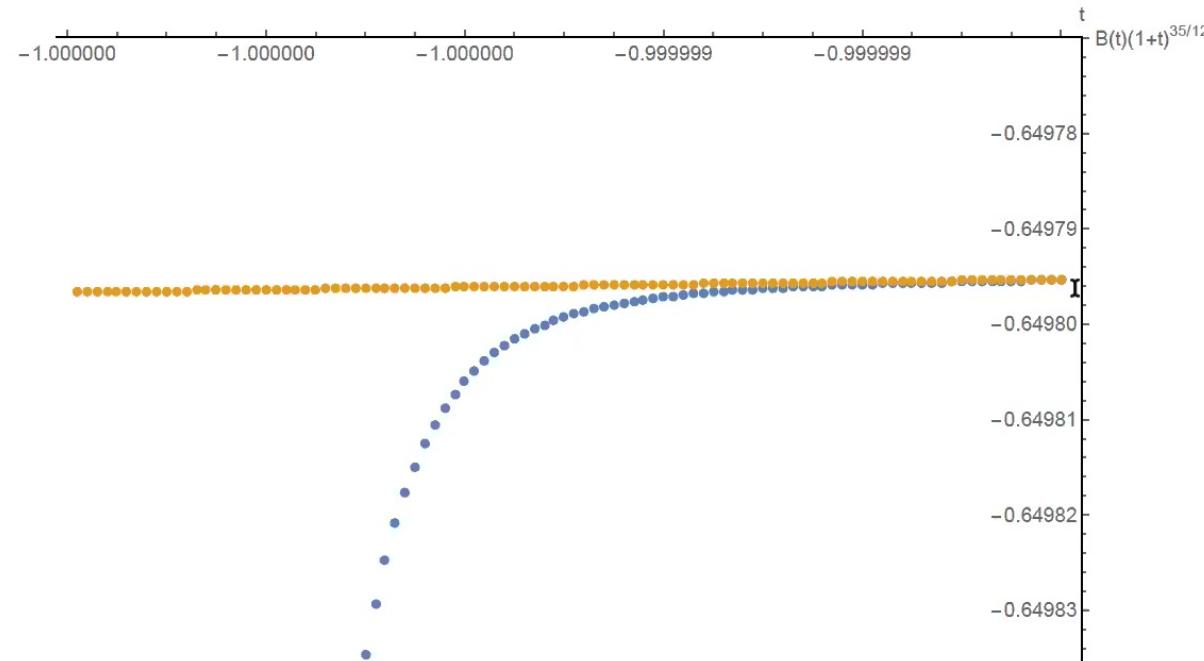
- conformal map in Borel plane resolves higher Borel singularities: ‘overlapping’ cuts are separated



$$z = \frac{t}{1 + \sqrt{1 + t^2}} \quad , \quad t = \frac{2z}{1 - z^2}$$

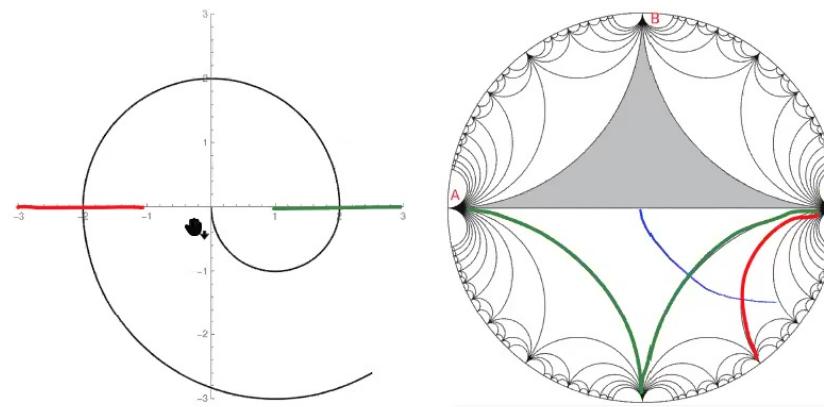
## Uniformized Borel Analysis: exponentially higher precision

- uniformization map in Borel plane enables (optimal) high precision extraction of Stokes constants:

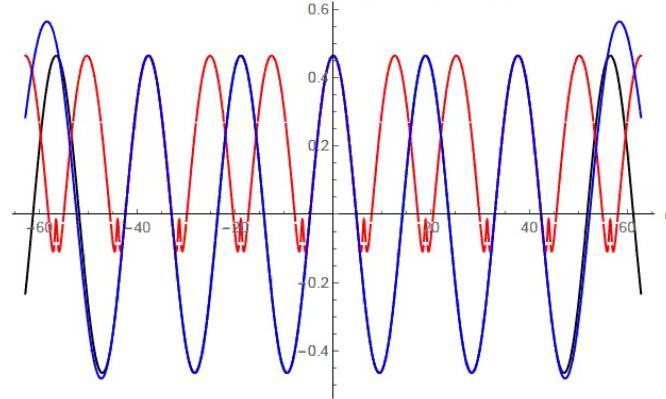


- conformal map [blue]; uniformizing map [gold]

## Beyond the horizon ... exploring higher Riemann sheets



- spiral path across cuts → blue path by Schwarz reflections



## Conclusions

- Resurgence systematically unifies perturbative and non-perturbative analysis, via trans-series, which ‘encode’ analytic continuation information
- phase transitions  $\leftrightarrow$  Stokes phenomenon
- differential/difference/integral eqs
- QM, matrix models, QFT, Chern-Simons, strings, ...
- numerical (lattice) Lefschetz thimbles
- non-perturbative resurgent effects exist even for convergent series (e.g. periodic potential; Ising model; unitary matrix model; ...)
- resurgent extrapolation: non-perturbative information can be decoded from surprisingly little perturbative data

# Resurgence in Matrix Models: Mariño: 0805.3033, Ahmed & GD: 1710.01812 I Gross-Witten-Wadia Unitary Matrix Model

$$Z(g^2, N) = \int_{U(N)} DU \exp \left[ \frac{1}{g^2} \text{tr} \left( U + U^\dagger \right) \right]$$

- one-plaquette matrix model for 2d lattice Yang-Mills
- two variables:  $g^2$  and  $N$  ('t Hooft coupling:  $t \equiv g^2 N/2$ )
- 3rd order phase transition at  $N = \infty$ ,  $t = 1$  (**universal!**)
- double-scaling limit: Painlevé II
- physics of phase transition = condensation of instantons
- random matrix theory/orthogonal polynomials result:

$$Z(g^2, N) = \det (I_{j-k}(x))_{j,k=1,\dots,N} , \quad x \equiv \frac{2}{g^2}$$