Title: Quantum algorithms for the Petz recovery channel, pretty-good measurements and polar decomposition

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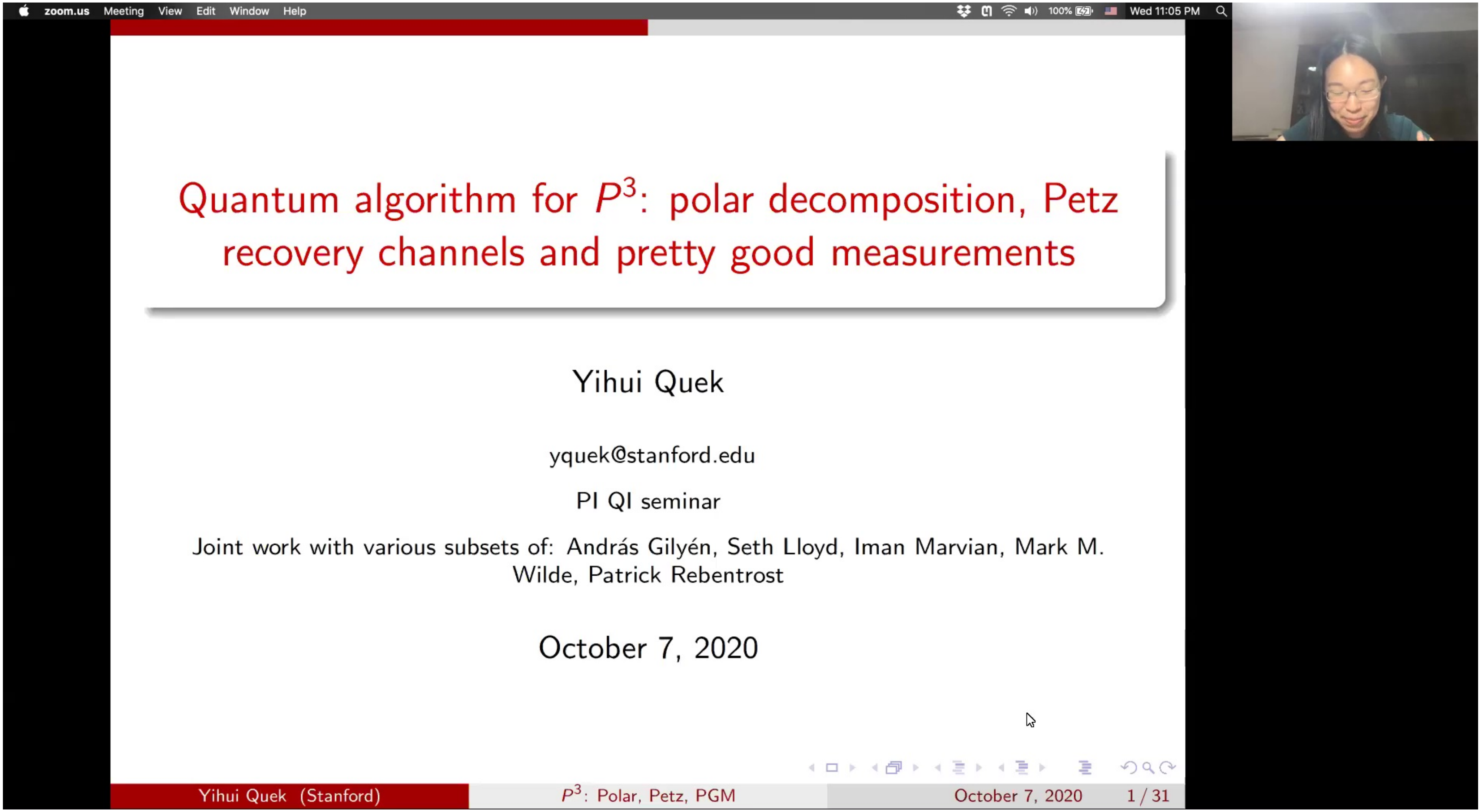
Abstract: The Petz recovery channel plays an important role in quantum information science as an operation that approximately reverses the effect of a quantum channel. The pretty good measurement is a special case of the Petz recovery channel, and it allows for near-optimal state discrimination. A hurdle to the experimental realization of these vaunted theoretical tools is the lack of a systematic and efficient method to implement them. We rectify this lack using the recently developed tools of quantum singular value transformation and oblivious amplitude amplification, providing a quantum algorithm to implement the Petz recovery channel. Our quantum algorithm also provides a procedure to perform pretty good measurements when given multiple copies of the states that one is trying to distinguish.

Using the same toolbox, we also develop a quantum algorithm for enacting the polar decomposition, a workhorse in linear algebra. This provides an alternative route to implementing a pretty-good measurements for the special case of pure states, which speeds up the general-purpose algorithm developed above.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Quantum algorithm for P^3: polar decomposition, Petz recovery channels and pretty good measurements
Yihui Quek
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PI QI seminar
Joint work with various subsets of: András Gilyén, Seth Lloyd, Iman Marvian, Mark M. Wilde, Patrick Rebentrost
October 7, 2020
Yihui Quek (Stanford)
P^3: Polar, Petz, PGM
October 7, 2020
1 / 31

Summary of both results

- Using Quantum Singular Value Transformation, we provide two quantum algorithms for two theoretical tools:

  1. From classical linear algebra: **polar decomposition**, matrix analog of $z = re^{i\theta}$.
  2. From quantum information: **Petz recovery map**, which approximately 'reverses' a quantum noise channel.

- Application: implementation of **Pretty-Good Measurements**, another ubiquitous proof tool now brought to life!

Block-encodings

Unitary U is a *block-encoding* of A if

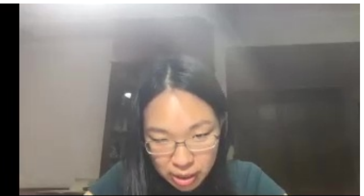
$$U = \begin{bmatrix} A/\alpha & \cdot \\ \cdot & \cdot \end{bmatrix} \iff A = \alpha(\langle 0|^{\otimes s} \otimes I)U(|0\rangle^{\otimes s} \otimes I). \qquad (1)$$

U (acts on a qubits $+s$ ancillae) can be used to realize a **probabilistic implementation of** A/α.

On a-qubit input $|\psi\rangle$,

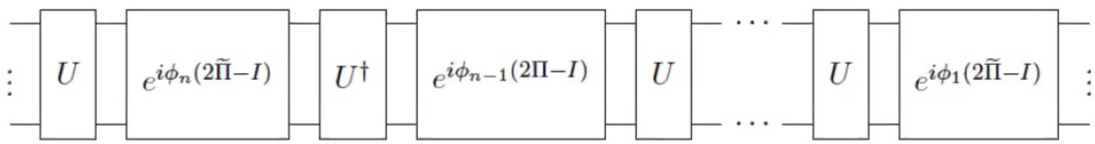
- Apply U to $|0\rangle^{\otimes s} \otimes |\psi\rangle$
- Measure ancillae; if outcome was $|0\rangle^{\otimes s}$, the first a qubits contain a state $\sim A|\psi\rangle$.

Quantum singular value transformation (I)



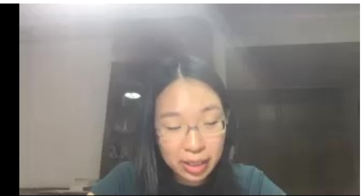
QSVT: A method to transform singular values of block-encodings

[Gilyén-Su-Low-Wiebe'18, Low-Chuang '16]



This circuit composes $U = \begin{bmatrix} \rho & \cdot \\ \cdot & \cdot \end{bmatrix}$ into $\begin{bmatrix} \tilde{f}(\rho) & \cdot \\ \cdot & \cdot \end{bmatrix}$ for your choice of polynomial $\tilde{f}$.

- $\tilde{f}(\rho)$ means 'apply $\tilde{f}$ to the singular values of ρ'.
- Usually a polynomial approximation of ideal function f.



Quantum singular value transformation (II)

$$U^\rho = \begin{bmatrix} \rho & \cdot \\ \cdot & \cdot \end{bmatrix} \xrightarrow{QSVT} \begin{bmatrix} \tilde{f}(\rho) & \cdot \\ \cdot & \cdot \end{bmatrix}$$

- Gate complexity measured in **number of uses of U^ρ**.
- Depends on **approximation's domain ($[\theta, 1]$) and error (δ)**.

e.g. Can approximate $x^{1/2}$ with error $\frac{1}{2}\left\|\tilde{f}(x) - x^{1/2}\right\|_{[\lambda_{\min},1]} \leq \delta$

- Let $\kappa := \frac{1}{\lambda_{\min}(\rho)} \sim$ "condition number", overall gate complexity

$$\boxed{\mathcal{O}\left(\kappa \log \frac{1}{\delta}\right) \text{ uses of } U^\rho.}$$

Algorithm 1: Polar decomposition

Polar decomposition

Definition (Polar decomposition)

The polar decomposition of $A \in \mathbb{C}^{M \times N}$ is the factorization

$$A = UB = \tilde{B}U$$

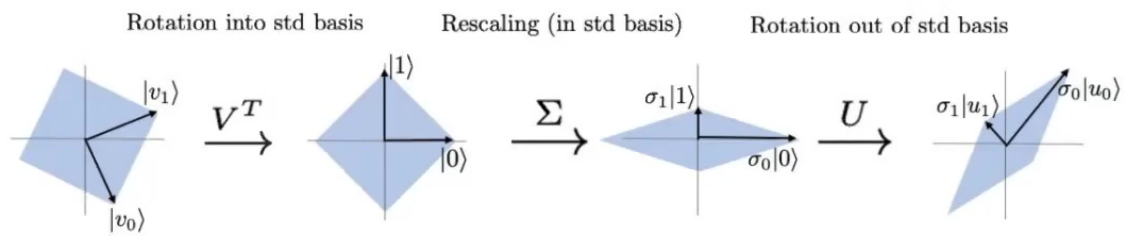
where U is a unitary/isometry and $B = \sqrt{A^\dagger A}$, $\tilde{B} = \sqrt{AA^\dagger}$ Hermitian.

Task: enact $U := U_{\mathrm{polar}}(A)$ on an input quantum state.

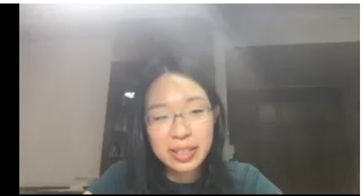
$$|\psi\rangle \rightarrow \mathcal{N} U_{\mathrm{polar}}(A)|\psi\rangle$$

Geometrical intuition for the polar decomposition

Singular value decomposition: $A = U\Sigma V^T$



But why should we have to go to the std basis?



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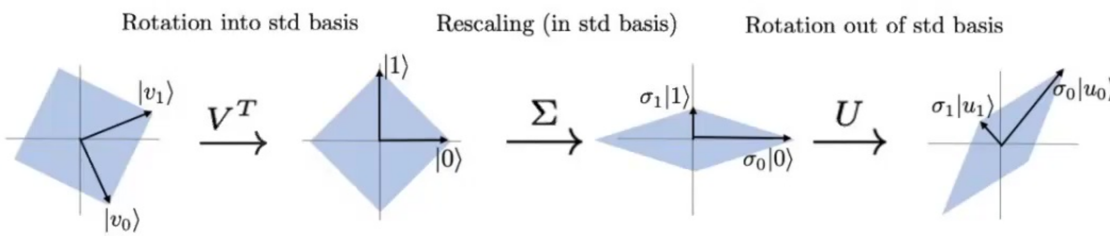

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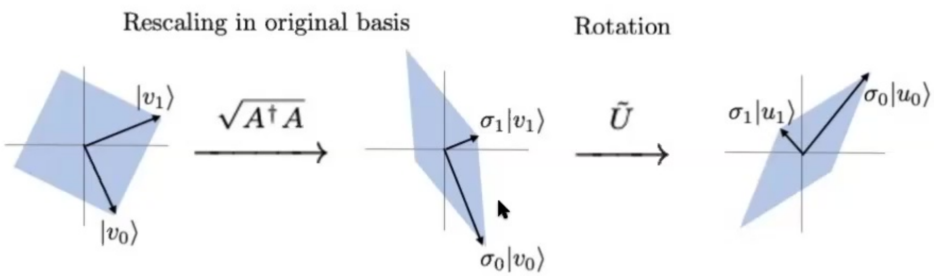
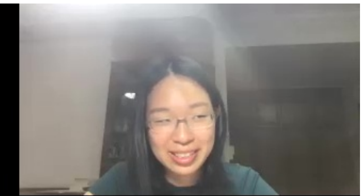
Polar decomposition: $A = \tilde{U}\sqrt{A^\dagger A}$





Application of polar decomposition: Procrustes problem

Given: r (input, output) quantum state pairs: $(|\phi_i\rangle, |\psi_i\rangle)_{i=1}^r$. Which U best transforms each input to output?

Figure 1: Taken from `http://atlasgeographica.com/the-bed-of-procrustes/`

- **Solution: U^* is the polar decomposition of** $FG^\dagger$, where F has $|\phi_i\rangle$ as columns and G has $|\psi_i\rangle$ as columns.

Our algorithm

- [Lloyd'20] provided an algorithm for enacting $U_{\mathrm{polar}}(A)$ based on density matrix exponentiation and quantum phase estimation.
- QSVT simplifies this dramatically: One realizes that

$$\mathrm{SVD}(A) = \sum_{i=1}^{r} \sigma_{ii} |p_i\rangle\langle q_i| \Rightarrow U_{\mathrm{polar}}(A) = \sum_{i=1}^{r} |p_i\rangle\langle q_i|.$$

i.e. can simply use QSVT to set all non-zero singular values to 1.

Significant speedups

vs [Lloyd '20]: exponentially faster in ε, with polynomial speedups in r, κ.

Algorithm 2: Petz recovery map

Quantum algorithm for Petz recovery channels and pretty good measurements

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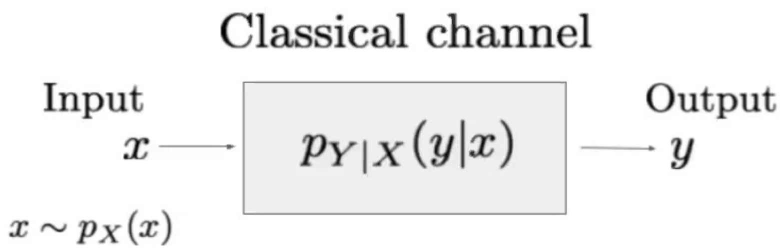
(Dated: July 28, 2020)

The Petz recovery channel plays an important role in quantum information science as an operation that approximately reverses the effect of a quantum channel. The pretty good measurement is a special case of the Petz recovery channel, and it allows for near-optimal state discrimination. A hurdle to the experimental realization of these vaunted theoretical tools is the lack of a systematic and efficient method to implement them. This paper sets out to rectify this lack: using the recently developed tools of quantum singular value transformation and oblivious amplitude amplification, we provide a quantum algorithm to implement the Petz recovery channel when given the ability to perform the channel that one wishes to reverse. Moreover, we prove that our quantum algorithm's usage of the channel implementation cannot be improved by more than a quadratic factor. Our quantum algorithm also provides a procedure to perform pretty good measurements when given multiple copies of the states that one is trying to distinguish.

arxiv:2006.16924

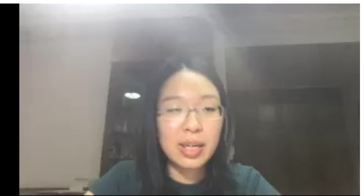
Classical 'reversal' channel from Bayes' theorem

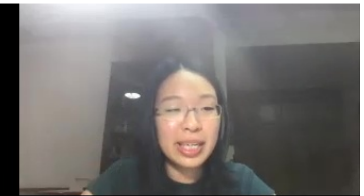


Probability distribution over outputs

$$p_Y(y) = \sum_x p_X(x) p_{Y|X}(y|x)$$

Given input $p_X(x)$ and channel $p_{Y|X}(y|x)$, what is $p_{X|Y}(x|y)$?





Petz recovery

Classically, Bayes theorem yields 'reverse channel':

$$p_{X|Y}(x|y) = \frac{p_X(x)p_{Y|X}(y|x)}{p_Y(y)}. \tag{2}$$

Quantumly: Petz recovery map!
Given a forward channel, $\mathcal{N}$ and an 'implicit' input state σ_A:

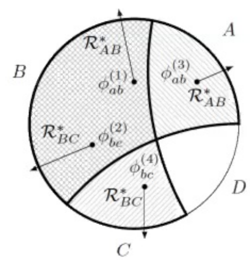
$$\mathcal{P}^{\sigma,\mathcal{N}}_{B\to A}(\cdot) := \sigma_A^{1/2}\mathcal{N}^\dagger\left(\mathcal{N}(\sigma_A)^{-1/2}(\cdot)\mathcal{N}(\sigma_A)^{-1/2}\right)\sigma_A^{1/2} \tag{3}$$

Why should you care about the Petz map?

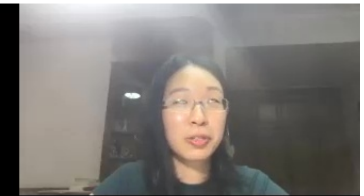
1. **Universal recovery operation** in error correction [Barnum-Knill'02, Ng-Mandayam'09]

2. **Important proof tool in QI**: [Beigi-Datta-Leditzky'16] as a decoder in quantum communication, achieves coherent information rate.

3. A wild Petz map has appeared in **quantum gravity**! [Cotler-Hayden-Penington-Salton-Swingle-Walter '18]



4. Is a type of **quantum "Bayesian inference"** [Leifer-Spekkens'13] (see: $\star$-product).



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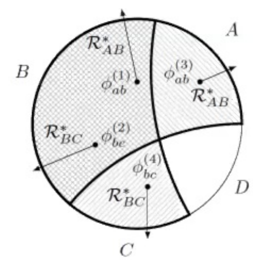

4. Is a type of **quantum "Bayesian inference"** [Leifer-Spekkens'13] (see: $\star$-product).

5. Has **pretty-good measurements** as a special case (later)

Quantum channels I

Quantum channel (informal): a physically valid map bringing one quantum state to another.

- Important use case: **model for quantum noise**, e.g. amplitude damping channels

Theorem (Choi-Kraus theorem)

Any physically valid channel $\mathcal{N}_{A\to B}(\cdot)$ can be decomposed as

$$\mathcal{N}_{A\to B}(X_A) = \sum_{l=0}^{d-1} V_l X_A V_l^\dagger$$

where V_l are linear ('Kraus') operators and $\sum_{l=0}^{d-1} V_l^\dagger V_l = I_A$.
(and vice versa!)

Quantum channels II

Definition (Channel adjoint)

Given $\mathcal{N}_{A\to B}$, the channel adjoint $\mathcal{N}^{\dagger}_{B\to A}$ satisfies

$$\langle Y, \mathcal{N}(X)\rangle = \langle \mathcal{N}^{\dagger}(Y), X\rangle \qquad \forall X \in \mathcal{H}_A, Y \in \mathcal{H}_B$$

Every channel can be replicated by an isometry acting on a larger input.

Definition (Isometric extension)

Given a channel $\mathcal{N}_{A\to B}$, an *isometric extension* $U : \mathcal{H}_A \otimes \mathcal{H}_E \to \mathcal{H}_B \otimes \mathcal{H}_{E'}$ of $\mathcal{N}$ satisfies

$$\mathrm{Tr}_{E'}(U(\rho \otimes |0\rangle\langle 0|_E)U^{\dagger}) = \mathcal{N}_{A\to B}(\rho) \tag{4}$$

Assumptions

Assume we have the following quantum circuits to start with:

1. **Block-encodings of two states**
   - One can efficiently block-encode density matrices (proof omitted).
   - The implicit state σ_A ($U^\sigma = \begin{bmatrix} \sigma_A & \cdot \\ \cdot & \cdot \end{bmatrix}$)
   - The state $\mathcal{N}(\sigma_A)$ ($U^{\mathcal{N}(\sigma)} = \begin{bmatrix} \mathcal{N}(\sigma_A) & \cdot \\ \cdot & \cdot \end{bmatrix}$).
2. $U^{\mathcal{N}}_{E'A\to EB}$, a **unitary extension of the forward channel** $\mathcal{N}$
   - Setting: we have characterized the noise and can simulate it using quantum gates.

A peek at the Petz recovery map

Given: quantum state σ_A (implicit 'input' to channel $\sim p_X$), quantum channel $\mathcal{N}_{A\to B}$, Petz map is:

$$\mathcal{P}^{\sigma,\mathcal{N}}_{B\to A}(\omega_B) := \sigma_A^{1/2}\mathcal{N}^\dagger\left(\mathcal{N}(\sigma_A)^{-1/2}\omega_B\mathcal{N}(\sigma_A)^{-1/2}\right)\sigma_A^{1/2},$$

Composition of 3 CP maps (overall trace-preserving):

$$(\cdot) \to [\mathcal{N}(\sigma_A)]^{-1/2}(\cdot)[\mathcal{N}(\sigma_A)]^{-1/2}$$
$$(\cdot) \to \mathcal{N}^\dagger(\cdot),$$
$$(\cdot) \to \sigma_A^{1/2}(\cdot)\sigma_A^{1/2}.$$

Re-writing the channel adjoint

- Second step of map: $(\cdot) \to \mathcal{N}^\dagger(\cdot)$
- Can write adjoint $\mathcal{N}^\dagger$ in terms of unitary extension $U^{\mathcal{N}}$:

$$\mathcal{N}^\dagger(\omega_B) = \langle 0|_{E'} U^{\mathcal{N}\dagger} (I_E \otimes \omega_B) U^{\mathcal{N}} |0\rangle_{E'}$$

- Problem: I_E is not a quantum state.
  Solution: act on maximally-entangled state $\frac{1}{d_E} \sum_{i=0}^{d_E-1} |i\rangle_E |i\rangle_{\tilde{E}}$, whose density matrix is $\sim$ identity.

Overview of algorithm

- Implement an isometric extension of the Petz map:

$$V^{\mathcal{P}}_{B\to\tilde{E}A} := (\langle 0|_{E'} \otimes I_{\tilde{E}A})\sigma_A^{\frac{1}{2}}(U^{\mathcal{N}}_{E'A\to EB})^{\dagger}\left[\mathcal{N}(\sigma_A)\right]^{-\frac{1}{2}}(|\Gamma\rangle_{E\tilde{E}} \otimes I_B).$$

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- Implement an isometric extension of the Petz map:

I)

$$V^{\mathcal{P}}_{B\to\tilde{E}A} := (\langle 0|_{E'} \otimes I_{\tilde{E}A})\sigma_A^{\frac{1}{2}}(U^{\mathcal{N}}_{E'A\to EB})^\dagger \left[\mathcal{N}(\sigma_A)\right]^{-\frac{1}{2}} (|\Gamma\rangle_{E\tilde{E}} \otimes I_B).$$

  - (I) $[\mathcal{N}(\sigma_A)]^{-1/2}\,(\cdot)\,[\mathcal{N}(\sigma_A)]^{-1/2}$

Overview of algorithm

- Implement an isometric extension of the Petz map:

$$V^{\mathcal{P}}_{B\to\tilde{E}A} := \underbrace{(\langle 0|_{E'} \otimes I_{\tilde{E}A})}_{\text{II)}}\sigma_A^{\frac{1}{2}}(U^{\mathcal{N}}_{E'A\to EB})^{\dagger}\underbrace{[\mathcal{N}(\sigma_A)]^{-\frac{1}{2}}}_{\text{I)}}(|\Gamma\rangle_{E\tilde{E}} \otimes I_B).$$

  - (I) $[\mathcal{N}(\sigma_A)]^{-1/2}(\cdot)[\mathcal{N}(\sigma_A)]^{-1/2}$
  - (II) $\mathcal{N}^{\dagger}(\cdot)$

Overview of algorithm

- Implement an isometric extension of the Petz map:

II) III) I)

$$V^{\mathcal{P}}_{B\to\tilde{E}A} := (\langle 0|_{E'} \otimes I_{\tilde{E}A})\sigma_A^{\frac{1}{2}}(U^{\mathcal{N}}_{E'A\to EB})^{\dagger}\,[\mathcal{N}(\sigma_A)]^{-\frac{1}{2}}(|\Gamma\rangle_{E\tilde{E}} \otimes I_B).$$

  - (I) $[\mathcal{N}(\sigma_A)]^{-1/2}(\cdot)\,[\mathcal{N}(\sigma_A)]^{-1/2}$
  - (II) $\mathcal{N}^{\dagger}(\cdot)$
  - (III) $\sigma_A^{1/2}(\cdot)\sigma_A^{1/2}$.

- Finally: tracing over environment $\tilde{E}$ implements the map.

Theorem

For a forward channel $\mathcal{N}$ and an implicit input state σ_A, we can realize an approximation $\tilde{\mathcal{P}}$ of the associated Petz recovery channel $\mathcal{P}$, such that:

$$\left\| \tilde{\mathcal{P}}^{\sigma_A, \mathcal{N}} - \mathcal{P}^{\sigma_A, \mathcal{N}} \right\|_\diamond \leq \varepsilon, \tag{5}$$

with

$$\widetilde{\mathcal{O}}\left(\sqrt{d_E \kappa_{\mathcal{N}(\sigma)}}\right) \text{ uses of } U^{\mathcal{N}}_{AE' \to BE} \ (\sim\text{optimal}) \tag{6}$$

$$\widetilde{\mathcal{O}}\left(\text{poly}(d_E, \kappa_{\mathcal{N}(\sigma)}, \kappa_{(\sigma)})\right) \text{ uses of } U^{\sigma_A} \text{ and } U^{\mathcal{N}(\sigma_A)} \tag{7}$$

d_E is the dimension of the system E, which is at least the Kraus rank of the channel $\mathcal{N}(\cdot)$.

Maps I and III

Maps I and III are a matter of transforming block-encodings:

$$V^{\mathcal{P}}_{B\to\tilde{E}A} := (\langle 0|_{E'} \otimes I_{\tilde{E}A})\underbrace{\sigma_A^{\frac{1}{2}}}_{\text{III)}}(U^{\mathcal{N}}_{E'A\to EB})^{\dagger}\underbrace{[\mathcal{N}(\sigma_A)]^{-\frac{1}{2}}}_{\text{I)}}(|\Gamma\rangle_{E\tilde{E}} \otimes I_B).$$

- Map I: $U^{\mathcal{N}(\sigma)} = \begin{bmatrix} \mathcal{N}(\sigma_A) & \cdot \\ \cdot & \cdot \end{bmatrix} \xrightarrow{QSVT} \begin{bmatrix} \mathcal{N}(\sigma_A)^{-1/2} & \cdot \\ \cdot & \cdot \end{bmatrix}$ with $\tilde{\mathcal{O}}(\kappa_{\mathcal{N}(\sigma)})$ uses of $U^{\mathcal{N}(\sigma)}$.
- Map III: $U^{\sigma} = \begin{bmatrix} \sigma_A & \cdot \\ \cdot & \cdot \end{bmatrix} \xrightarrow{QSVT} \begin{bmatrix} \sigma_A^{1/2} & \cdot \\ \cdot & \cdot \end{bmatrix}$ with $\tilde{\mathcal{O}}(\kappa_{\sigma})$ uses of U^{σ}.

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Map II: Channel adjoint $\mathcal{N}^\dagger(\cdot)$

$$V^{\mathcal{P}}_{B\to\tilde{E}A} := (\langle 0|_{E'} \otimes I_{\tilde{E}A})\sigma_A^{\frac{1}{2}}(U^{\mathcal{N}}_{E'A\to EB})^\dagger [\mathcal{N}(\sigma_A)]^{-\frac{1}{2}} (|\Gamma\rangle_{E\tilde{E}} \otimes I_B).$$

1. Tensor in the maximally entangled state $\Gamma_{E\tilde{E}}/d_E$
2. Perform $U^{\mathcal{N}\dagger}$.

- Easy peasy, no QSVT needed.

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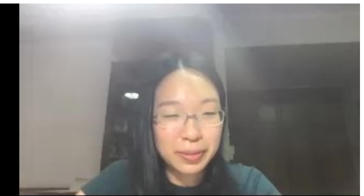
3. Measure the system E', accepting if the all-zeros outcome occurs.
4. Ignore the system $\tilde{E}$ (i.e. trace it out).



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- Not contiguous with steps 1 and 2! Do these steps *after* Map III which applies $\sigma_A^{\frac{1}{2}}$.

More on measurement

Implementing in sequence the unitaries created through SVT obtains the overall unitary

$$\tilde{W} = \begin{bmatrix} \frac{1}{4}\sqrt{\frac{1}{d_{E^{\mathsf{K}}\mathcal{N}(\sigma)}}}\tilde{V} & \cdot \\ \cdot & \cdot \end{bmatrix} \tag{8}$$

where V= ideal Petz map, $\|\tilde{V} - V\| < O(\varepsilon)$.

- This is a **probabilistic** implementation: Measuring E' system, probability $p_{\text{success}} = O(\frac{1}{d_{E^{\mathsf{K}}}})$ of getting $|0\rangle$.
- Make this **deterministic**: use Oblivious Amplitude Amplification to boost probability by repeating $\tilde{W}$ $\mathcal{O}\left(1/\sqrt{p_{\text{success}}}\right)$ times .

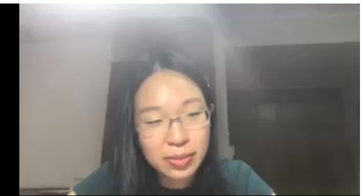
Application of our algorithms: Pretty-Good Measurements

Pretty-Good Measurements

Given an ensemble of mixed states $\{\sigma^x\}_{x \in \mathcal{X}}$, and a quantum state ρ, we are promised that ρ is in state σ^x with probability $p(x)$. What POVM maximizes Pr(correctly identify ρ)?

- No optimal strategy when $|\mathcal{X}| \geq 3$, but 'pretty-good measurement' does pretty-well on this. [Belavkin'75, Hausladen-Wootters'94]
- Special case of the Petz map with $\sigma_{XB} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \sigma_B^x$, and $\mathcal{N}$ the partial trace over X.
- Our Petz map algorithm can implement this, with $\mathcal{O}\left(\sqrt{|\mathcal{X}|}\text{poly}(\kappa)\right)$ uses of unitary preparing σ_{XB}.

Faster PGM

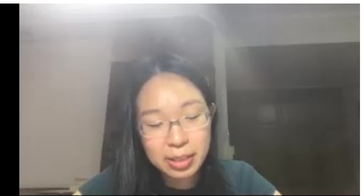


Special case: ensemble of r *pure* states over a *uniform* distribution $\{p(j) = 1/r, |\phi_j\rangle\}_{j=0}^{r-1}$.

- Not that uncommon: this setting was considered by [Holevo'79].
- Let $\kappa = 1/\sigma_{\min}(\sum_j |j\rangle\langle\phi_j|)$.
  - **Petz map algorithm:** $\tilde{\mathcal{O}}\left(r^2\kappa^3\right)$ uses of U_ϕ.
  - **Polar decomposition algorithm** can handle this, with polynomial speedup: $\tilde{\mathcal{O}}(r\kappa)$ uses of U_ϕ.

Why should you care about Pretty-Good Measurements?

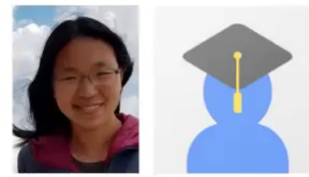
1. Used to approach the Holevo information rate [Hausladen-Jozsa-Schumacher-Westmoreland-Wootters'96]
2. Important proof technique: PGM $\sim$ optimal measurement to distinguish states
   - i. Is the optimal measurement in q. algorithm for dihedral hidden subgroup problem [Bacon-Childs-vanDam'06]
   - ii. Bounds on sample complexity for Quantum Probably Approximately Correct (PAC) learning [Arunachalam-de Wolf'16]
   - iii. Optimal for port-based teleportation [Leditzky'20]



Takeaways!

The quantum singular value transform allows us to be systematic, rigorous and fast.

Our algorithm for Polar decomposition (arxiv:20??.?????)



- provides new tools for quantum linear algebra; and
- speeds up PGM, for pure states and uniform distribution.

Our algorithm for Petz recovery maps and general-purpose PGM (arxiv:2006.16924)

- brings these theoretical tools closer to implementation; and
- is almost optimal in gate complexity.