Title: Leading order correction to the QES prescription

Speakers: Geoffrey Penington

Series: Quantum Fields and Strings

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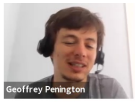
Abstract: We show that a naÃ¯ve application of the quantum extremal surface (QES) prescription can lead to paradoxical results and must be corrected at leading order. The corrections arise when there is a second QES (with strictly larger generalized entropy at leading order than the minimal QES), together with a large amount of highly incompressible bulk entropy between the two surfaces. We trace the source of the corrections to a failure of the assumptions used in the replica trick derivation of the QES prescription, and show that a more careful derivation correctly computes the corrections. Using tools from one-shot quantum Shannon theory (smooth min- and max-entropies), we generalize these results to a set of refined conditions that determine whether the QES prescription holds. We find similar refinements to the conditions needed for entanglement wedge reconstruction (EWR), and show how EWR can be reinterpreted as the task of one-shot quantum state merging (using zero-bits rather than classical bits), a task gravity is able to achieve optimally efficiently.



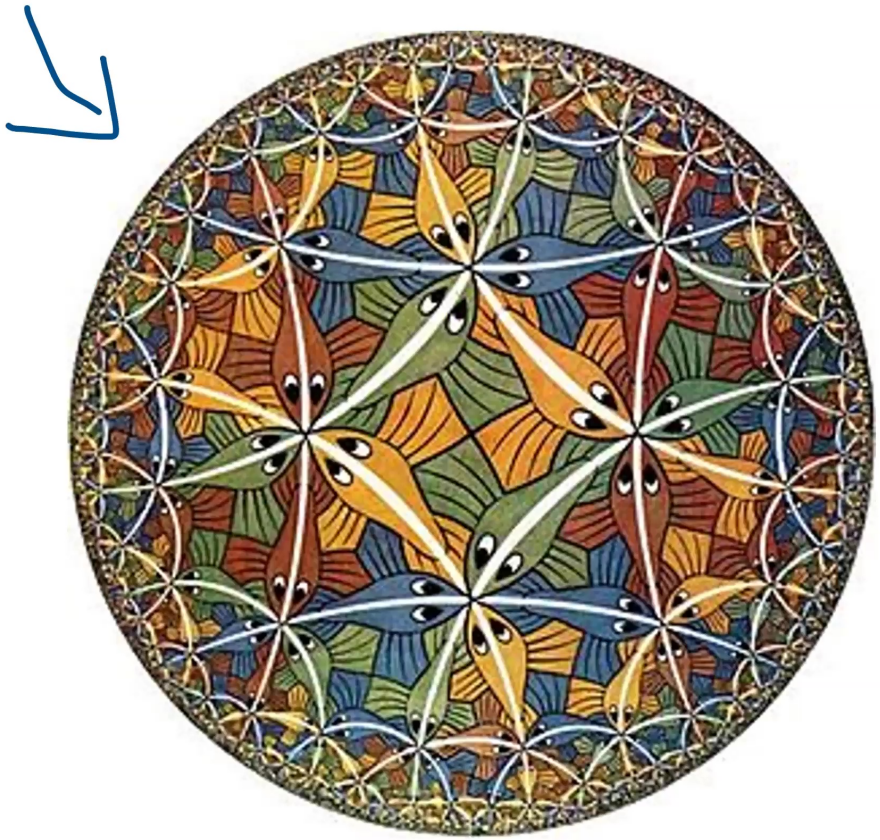
Leading Order Corrections to the QES prescription

Geoff Penington, UC Berkeley

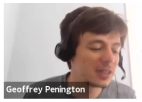
arXiv:2008.03319 (with Chris Akers)

AdS/CFT

The fish are all the same size



Quantum gravity in (d+1)-dimensional Anti-de Sitter space (the “bulk”) is **dual** to a d-dimensional conformal field theory on the boundary.



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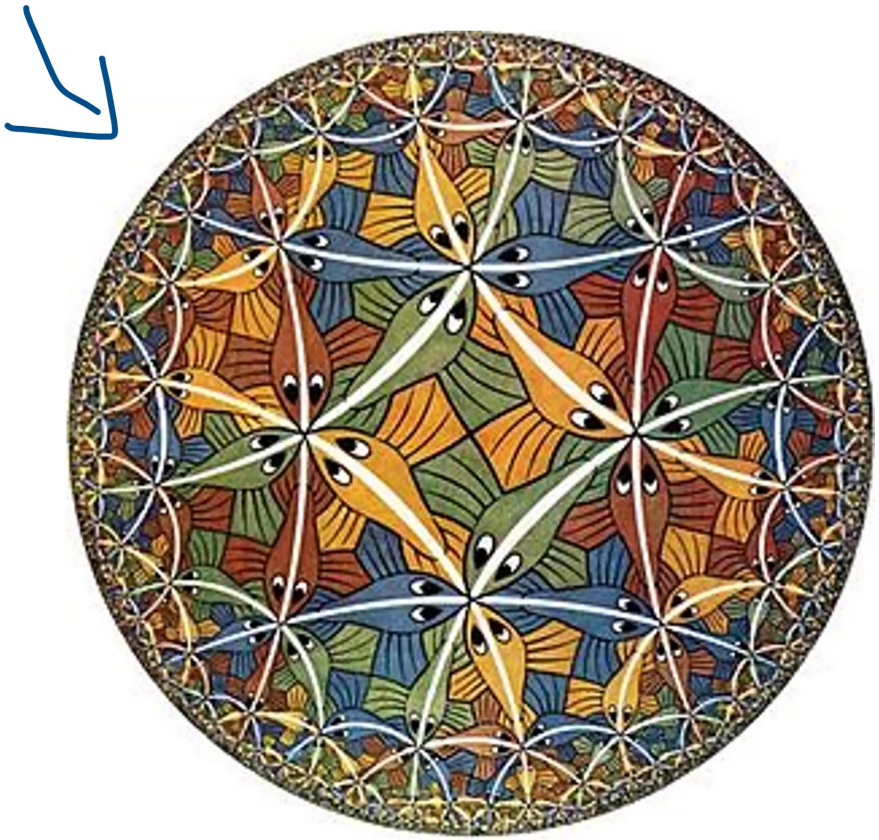
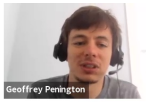


Quantum gravity in (d+1)-dimensional Anti-de Sitter space (the “bulk”) is **dual** to a d-dimensional conformal field theory on the boundary.

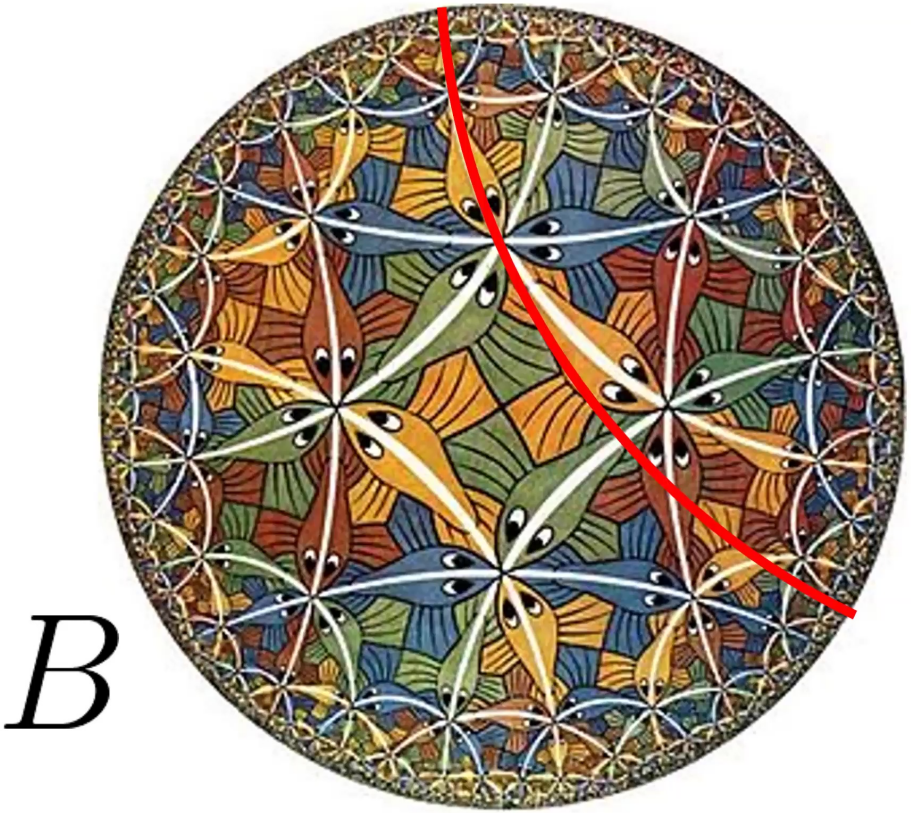
Aim: understand the **dictionary** that relates objects on each side of the duality



The Ryu-Takayanagi Formula

Entropy of the **reduced state** on some **boundary region** = **area** of minimal bulk surface anchored on that boundary region

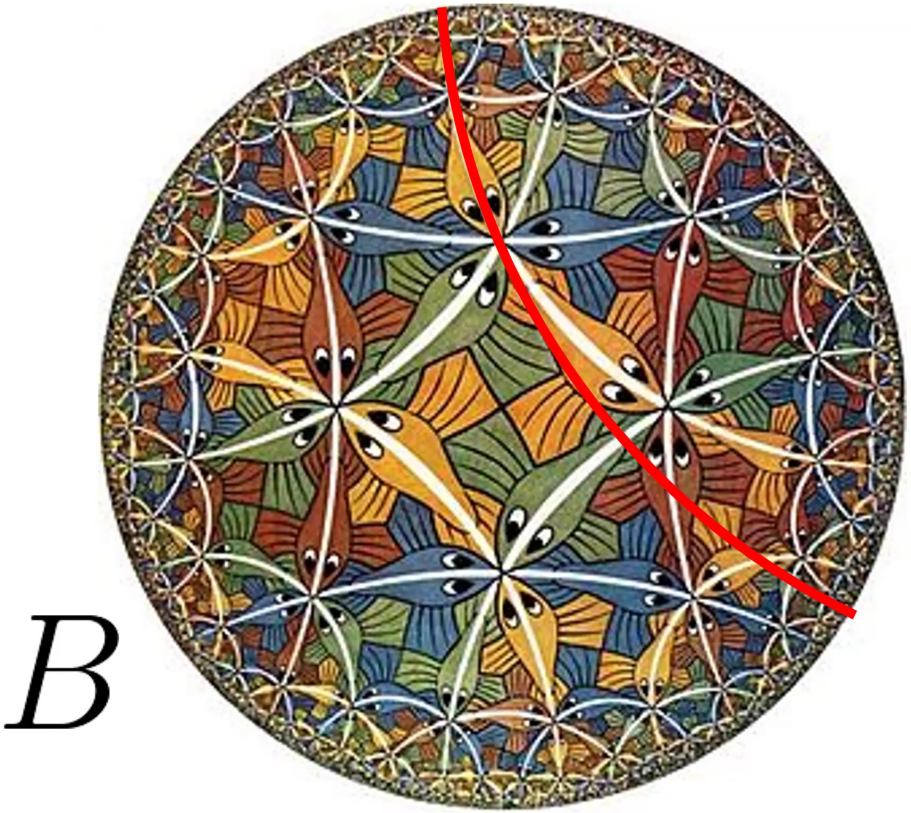
$$S(B) = \min_{\chi} \frac{A(\chi)}{4G}$$



The QES Prescription

Entropy of the **reduced state** on some **boundary region** = **generalised entropy** of minimal bulk surface anchored on that boundary region

$$S(B) = \min_{\chi} \left[\frac{A(\chi)}{4G} + S_{\text{bulk}}(\chi) \right]$$



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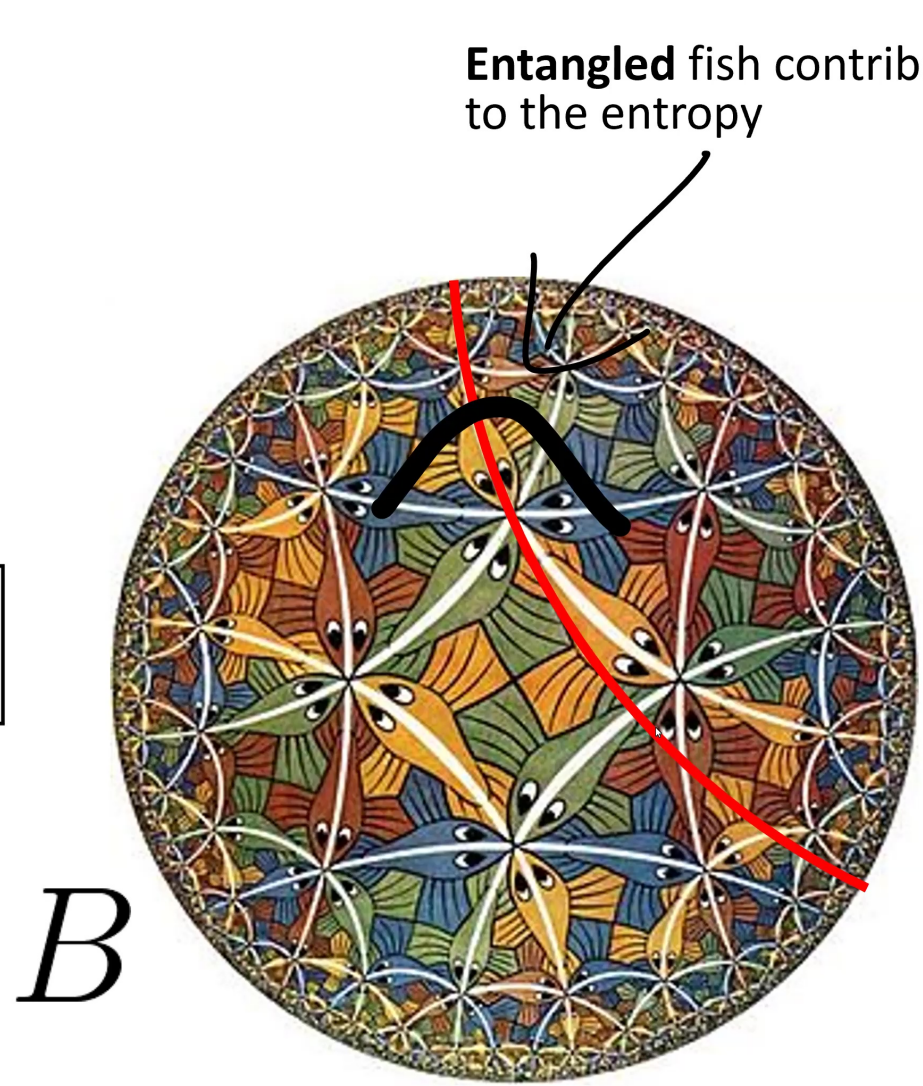

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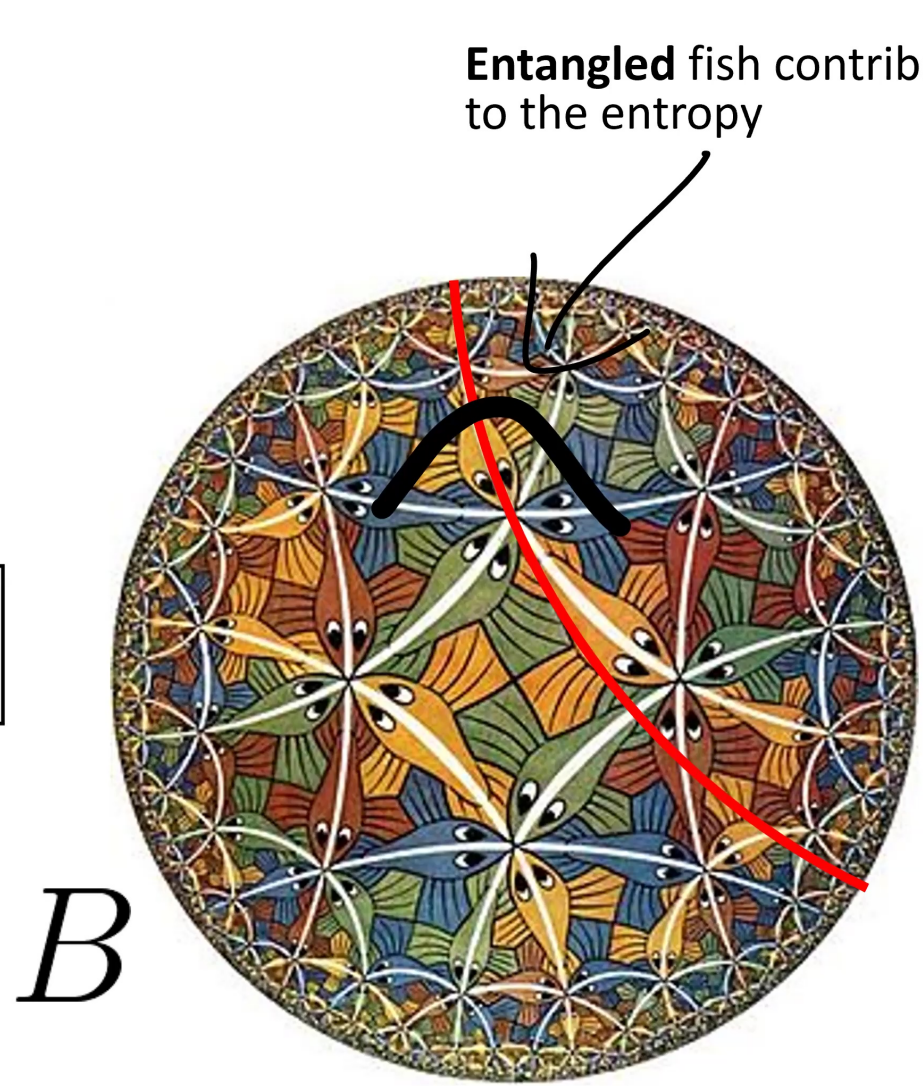
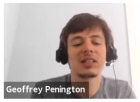
Can be derived from **gravitational path integral**

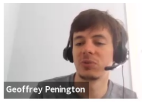
Number of successes including a derivation of the **Page curve** for an evaporating black hole



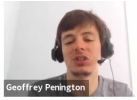


This Talk

- **Naïve application** of the QES prescription gives **paradoxical results**
- These paradoxes already exist at **leading order in G_N**, and so cannot be fixed by small corrections to the QES prescription
- The paradoxes can been seen in calculations involving **evaporating black holes**, but also in states that do not involve any black holes at all
- **More careful** entropy calculations gives **consistent results**, and show why the assumptions in the Lewkowycz-Maldacena derivation of the QES prescription **fail**, even at leading order
- Instead, the naïve QES prescription needs to be replaced by a more **refined version**, defined in terms of **smooth min- and max-entropies** (concepts from **one-shot quantum Shannon theory**)
- All the results (and specifically the consequences for **entanglement wedge reconstruction**) are close related to the task of **one-shot quantum state merging**.



Part 1: A Paradox



Dustballs in AdS3

Consider a boundary region in AdS3/CFT2 consisting of two intervals

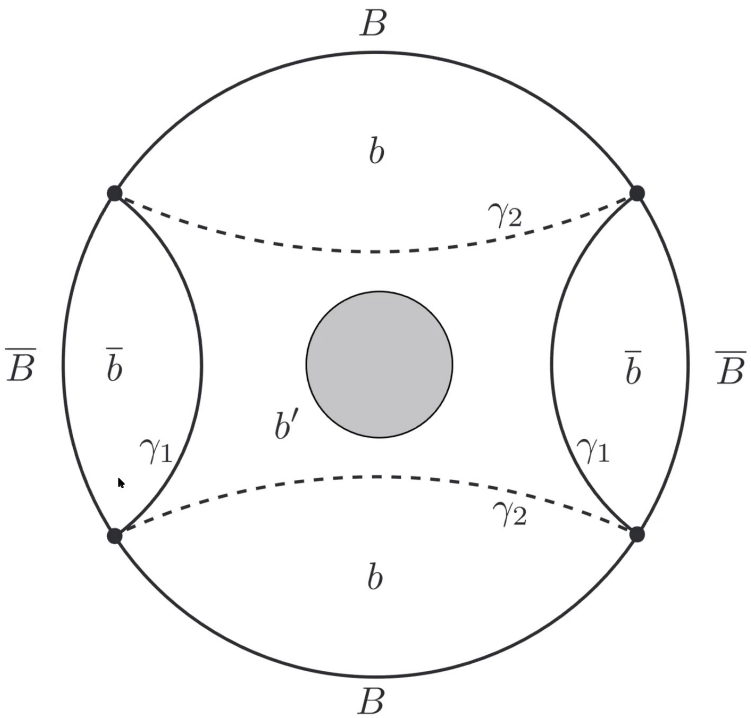
Two extremal surfaces with different topologies

We can add matter with $O(\varepsilon/G_N)$ energy while only creating a **small backreaction** on the geometry

If this matter is in a thermal state, it will have entropy $S = O(\varepsilon/G_N)$

By **tuning** the size of region B, we can ensure

$$\frac{A_1}{4G_N} + S \geq \frac{A_2}{4G_N} \geq \frac{A_1}{4G_N}$$



The Paradox

If the matter is in a **pure state**, $S(B) = \frac{A_1}{4G_N}$

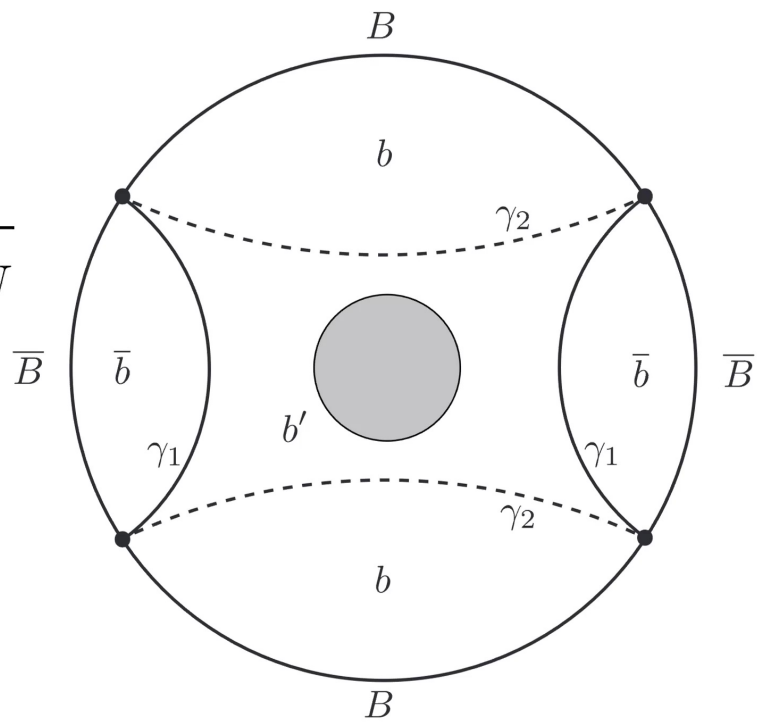
If the matter is in a **thermal state**, $S(B) = \frac{A_2}{4G_N}$

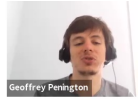
If the matter is in a **mixture** of the pure and thermal states, the **QES prescription** says that

$$S(B) = \min\left[\frac{A_1}{4G_N} + (1-p)S, \frac{A_2}{4G_N}\right]$$

However **standard bounds** on the entropy of a mixture mean

$$S(B) = p\frac{A_1}{4G_N} + (1-p)\frac{A_2}{4G_N} + O(1)$$





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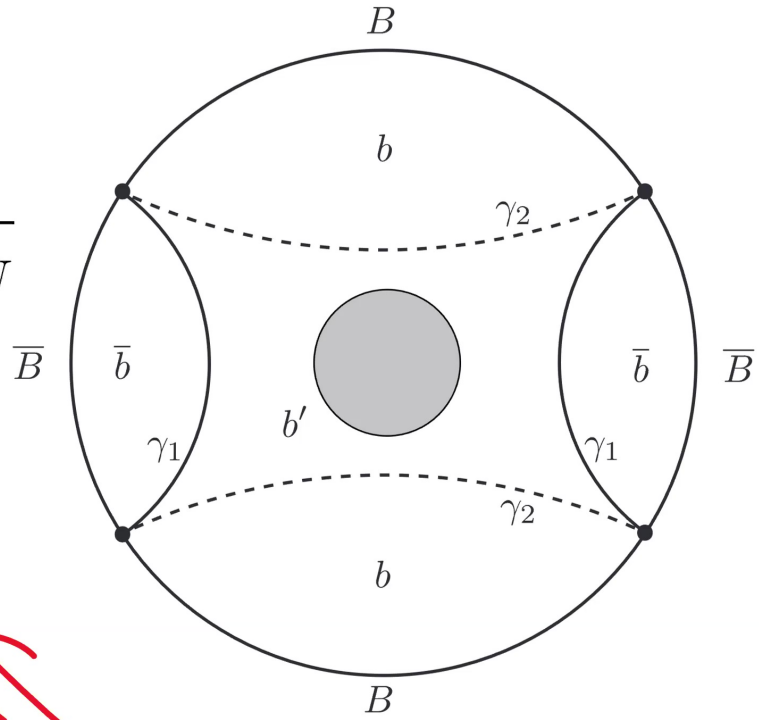
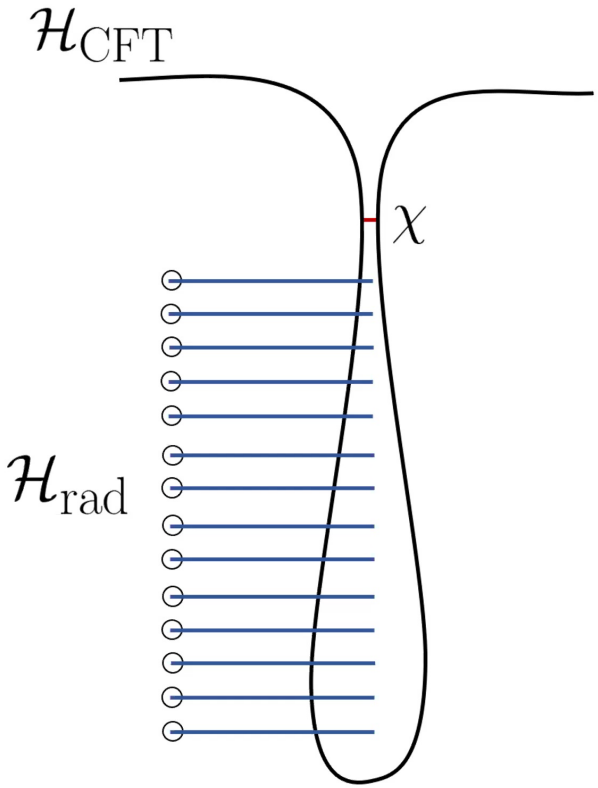
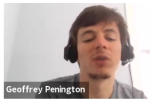


Differ at leading order

Evaporating Black Holes

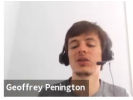
- After the Page time, the semiclassical state looks **thermal**, but the QES prescription (correctly) knows that the entropy equals the **Bekenstein-Hawking entropy**, consistent with unitarity. Great!
- Now measure all the Hawking radiation with probability p (otherwise do nothing).
- QES prescription now says $S = \min((1-p)\, S_{th}, S_{BH})$.
- But by the same arguments as the previous slide, unitarity implies $S = (1-p)\, S_{BH}$.
- QES Prescription gives the wrong answer at leading order (in fact can be just as **inaccurate** as the original semiclassical Hawking answer).





Smooth max- and min-entropies

- The **smooth max-entropy** $H^{\varepsilon}_{max}(\rho)$ is (roughly) the **log of the rank** of the smallest rank state $\tilde{\rho}$ that is ε-close to ρ: $$H^{\varepsilon}_{\max}(\rho) = \min_{\tilde{\rho}=\mathcal{B}_{\varepsilon}(\rho)} \log \mathrm{Rank}(\tilde{\rho})$$
- It characterises the minimum size Hilbert space that the state ρ can be **compressed** into.
- Equivalently, it gives a **lower confidence bound** on the size of a randomly chosen eigenvalue of ρ.
- The **smooth min-entropy** $H^{\varepsilon}_{min}(\rho)$ is minus the log of the **largest eigenvalue** (i.e. an **upper confidence bound** on the eigenvalues $$H^{\varepsilon}_{\min}(\rho) = -\min_{\tilde{\rho}=\mathcal{B}_{\varepsilon}(\rho)} \log \lambda_{\max}(\tilde{\rho})$$
- Also exist **conditional** smooth min/max-entropies but (unlike for von Neumann entropies) $H^{\varepsilon}_{min/max}(A|B) \neq H^{\varepsilon}_{min/max}(\mathrm{AB}) - H^{\varepsilon}_{min/max}(B)$.

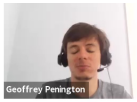


Why does the QES prescription fail?

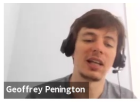
- A **thermal state**, with a large number of degrees of freedom, can be **well approximated** by a state of **rank** $\exp\big(S + o(S)\big)$ (**law of large numbers**)
- Its **smooth max-entropy** is approximately equal to its **von Neumann entropy**
- The same is true for a **pure state** (both are equal to zero)
- However the **mixture of the true** is roughly as hard to approximate as the thermal state itself. Its smooth max-entropy is related to its von Neumann entropy by

$$H_{\max}^{\varepsilon} = \frac{S}{1-p}$$

- **General rule**: large corrections can show up for these sorts of **incompressible states**, which require **more degrees of freedom** to encode than their entropy might suggest



Part 2: Resolving the Paradox



The Replica Trick

- How do you calculate **von Neumann entropies** using a path integral?
- **Answer**: we first calculate the **integer n Renyi entropies**

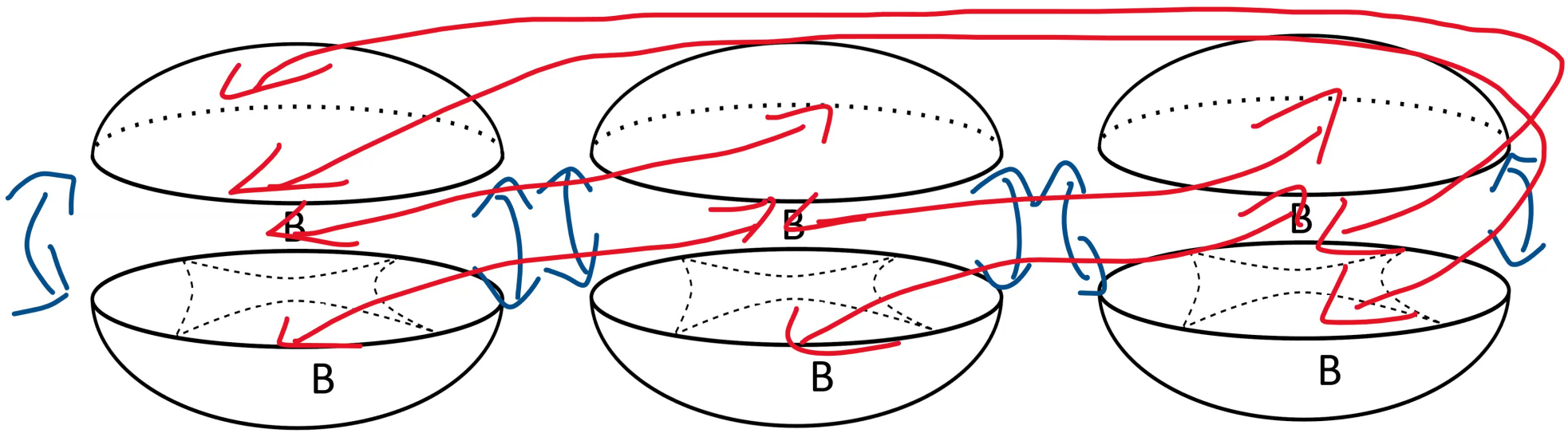
$$\frac{1}{1-n}\log \mathrm{Tr}\rho_R^n$$

- We can then calculate the von Neumann entropy by **analytically continuing the Renyi entropies** to **n=1**.

The Replica Trick in AdS/CFT

How do you evaluate $\mathrm{Tr}(\rho^n)$ in AdS/CFT?

1. States (**bras** and **kets**) are prepared **Euclidean path integrals** (2n in total). Sum over bra/ket boundary conditions for **mixed states**.
2. **Partial trace** means you **glue** the bra and ket boundaries together.
3. The B boundaries are also glued together, but with cyclic permutations



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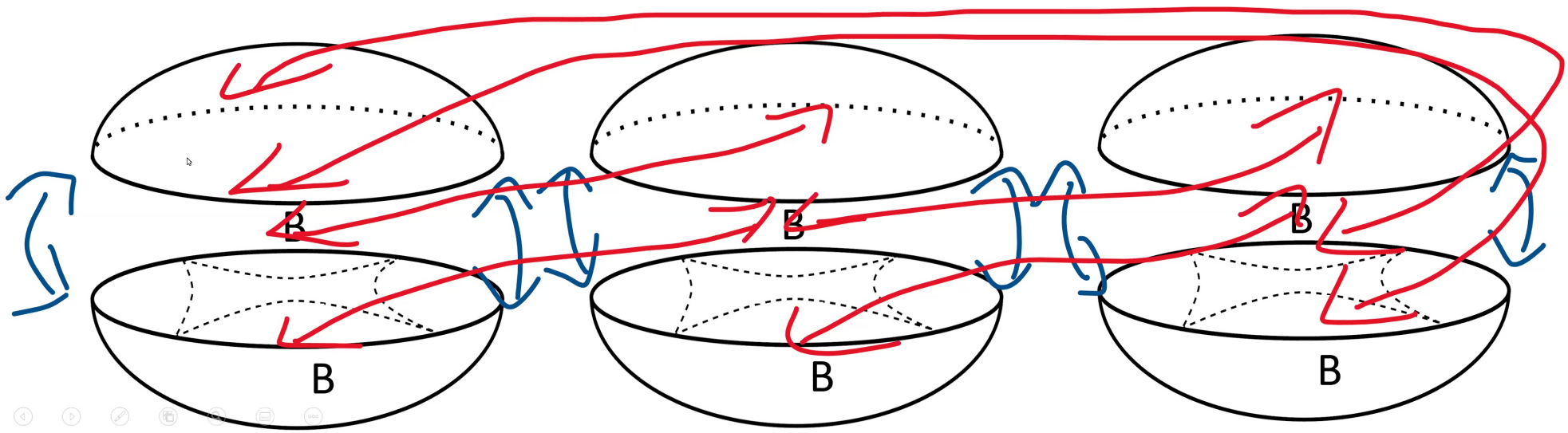
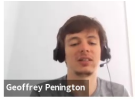
4. Integrate over all bulk geometries with those boundary conditions (dominated by saddle points)





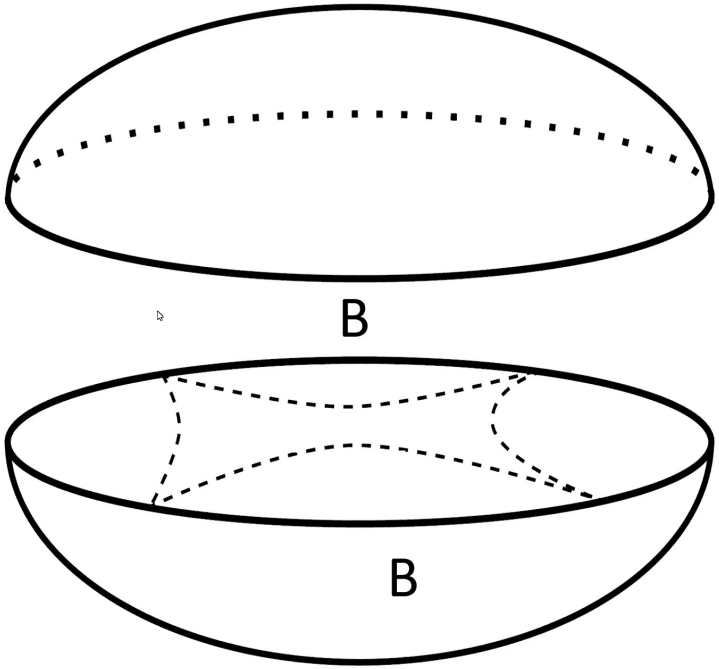
Fixed-area states

In general, finding the bulk geometries with the correct boundary conditions is very hard

However, it is easy for a particular class of states, called **fixed-area states**, where the areas of the extremal surfaces have been **measured**

This means you don't need to integrate over the area of the extremal surfaces => saddle points can have **conical singularities** at the extremal surfaces

Bulk geometry = original bulk geometry, except with **branch cuts**, permuting the different **sheets** at the extremal surfaces



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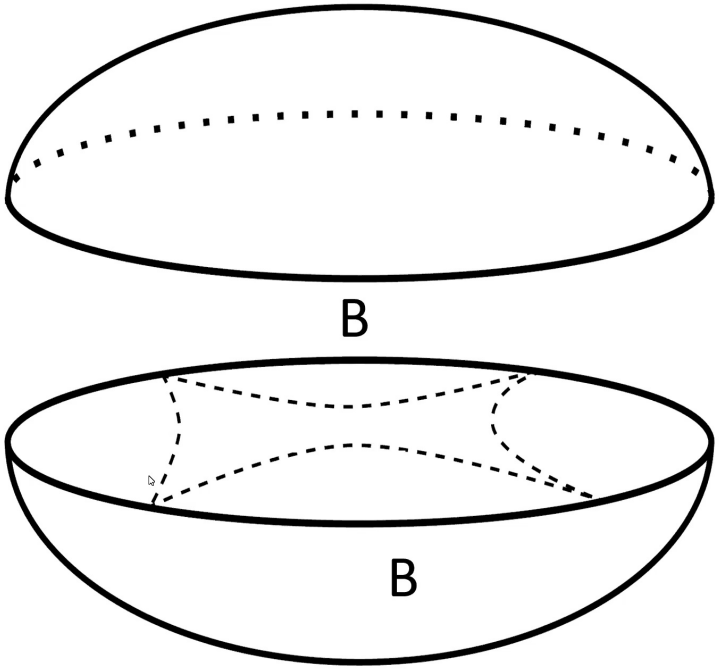
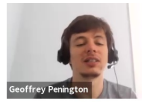
Gravitational action **cancels** with normalisation factor, except for contributions from conical singularities $((\phi - 2\,\pi)A/8\pi G)$



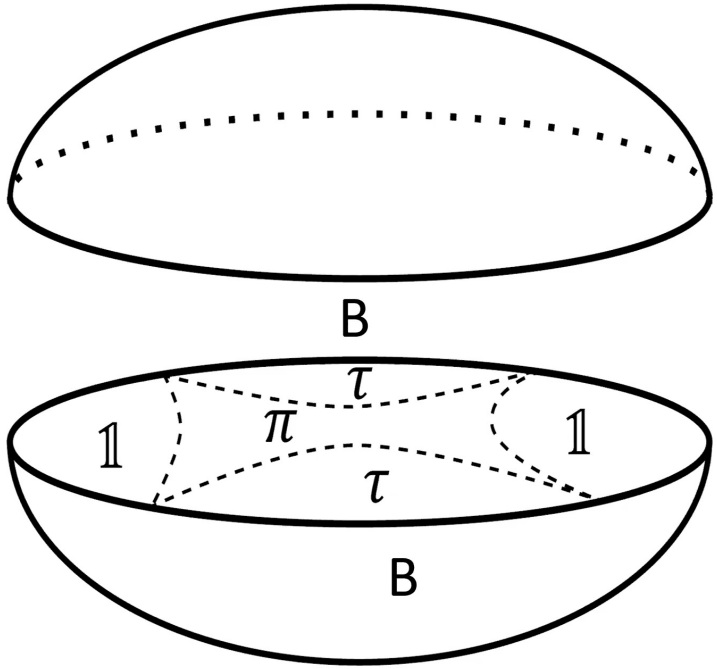


Fixed-area states

Boundary conditions **fix the permutation** in regions b and $\bar{b}$

However the permutation in the central region b' is **arbitrary**

=> sum over saddles corresponding to all permutations



$$\mathrm{tr}(\rho_B^n) = \sum_{\pi \in S_n} e^{(C(\pi)-n)A_1/4G + \left(C(\tau^{-1}\circ\pi)-n\right)A_2/4G}\, \mathrm{tr}(\rho_{bb'}^{\otimes n}\tau_b\pi_{b'}) \ .$$

If we dropped everything except the leading saddle (either τ or $\mathbb{1}$) and then analytically continued, we would get the **naïve QES** prescription

Noncrossing permutations

Number of cycles

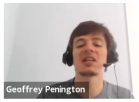
$$\mathrm{tr}(\rho_B^n) = \sum_{\pi \in S_n} e^{(C(\pi)-n)A_1/4G + \left(C(\tau^{-1}\circ\pi)-n\right)A_2/4G}\, \mathrm{tr}\left(\rho_{bb'}^{\otimes n}\tau_b \pi_{b'}\right) \ .$$

Areas formally infinite => saddles that don't maximise $C(\pi) + C(\tau^{-1} \circ \pi)$ are **infinitely suppressed**

Remaining permutations are associated with **noncrossing partitions**

Want to analytically continue this formula as a function of n. **Problem**: number of terms in the sum depends on n

Fortunately, there exists some **technology** from the theory of **free probability** that lets us do this



Resolvent

Resolvent

$$R(\lambda) = \mathrm{Tr}\left(\frac{1}{\lambda - \rho}\right) = \frac{\mathrm{rank}(\rho)}{\lambda} + \sum_n \lambda^{n-1}\,\mathrm{Tr}\,\rho^n$$

Determined by $\mathrm{Tr}(\rho^n)$

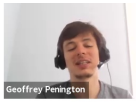
$$\mathrm{D}(\lambda) = \frac{1}{\pi}\lim_{\varepsilon\to 0}[\mathrm{Im}(R(\lambda + i\,\varepsilon))$$

Density of Eigenvalues of ρ

For the mixture of a pure and a thermal state (see *arXiv:1911.11977* or the upcoming paper for details), one can find the **recursion relation**:

$$\lambda R = e^{A_2/4G} + \frac{pR}{e^{A_2/4G} - \frac{p}{e^{A_1/4G}}R} + \frac{(1-p)R}{e^{A_2/4G} - \frac{(1-p)}{e^{A_1/4G+S}}R}\ .$$

Cubic equation => can be solved (but full solution messy)



Resolvent

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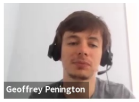
Ignorable at large R

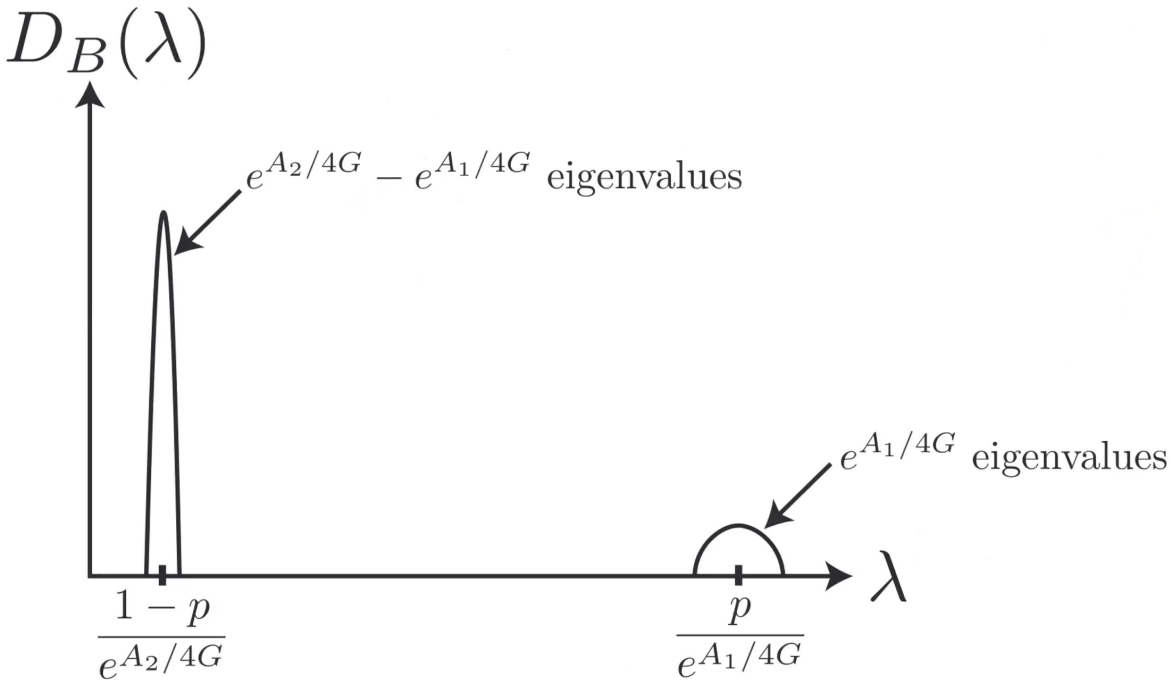
Ignorable at small R

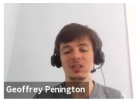
Both approximations give a **quadratic** equation, with **overlapping** regimes of validity

Nonzero eigenvalue density when the discriminant is **negative**



The Eigenvalue Density





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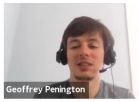
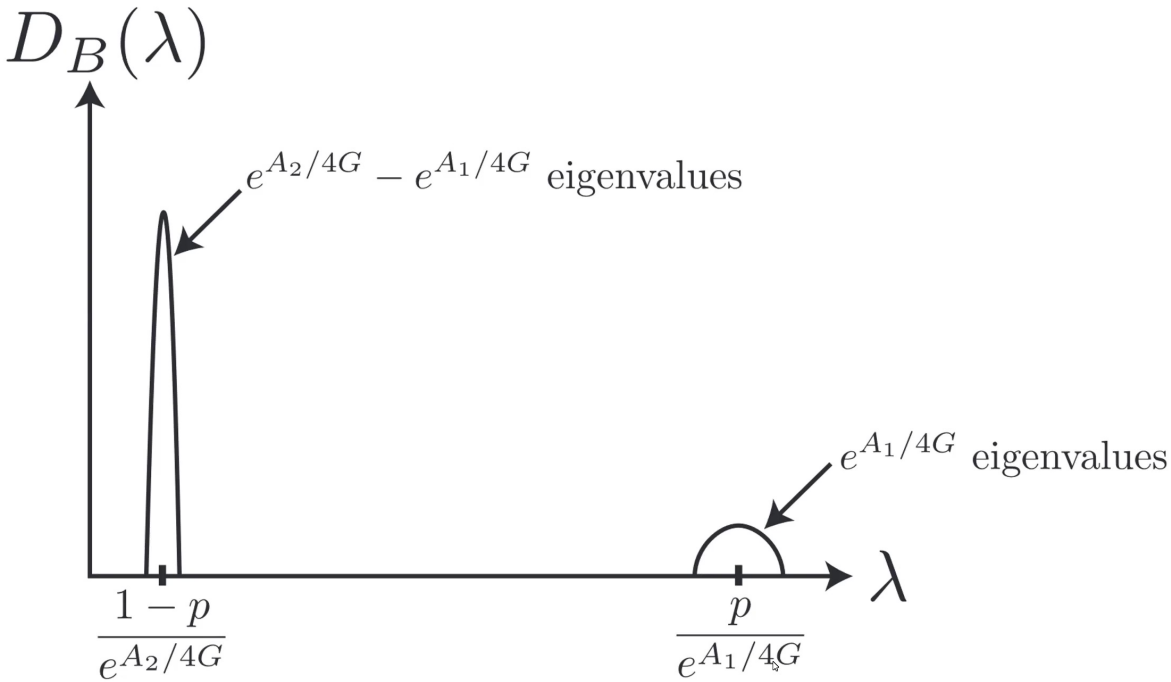
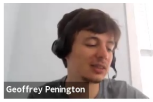


The Eigenvalue Density





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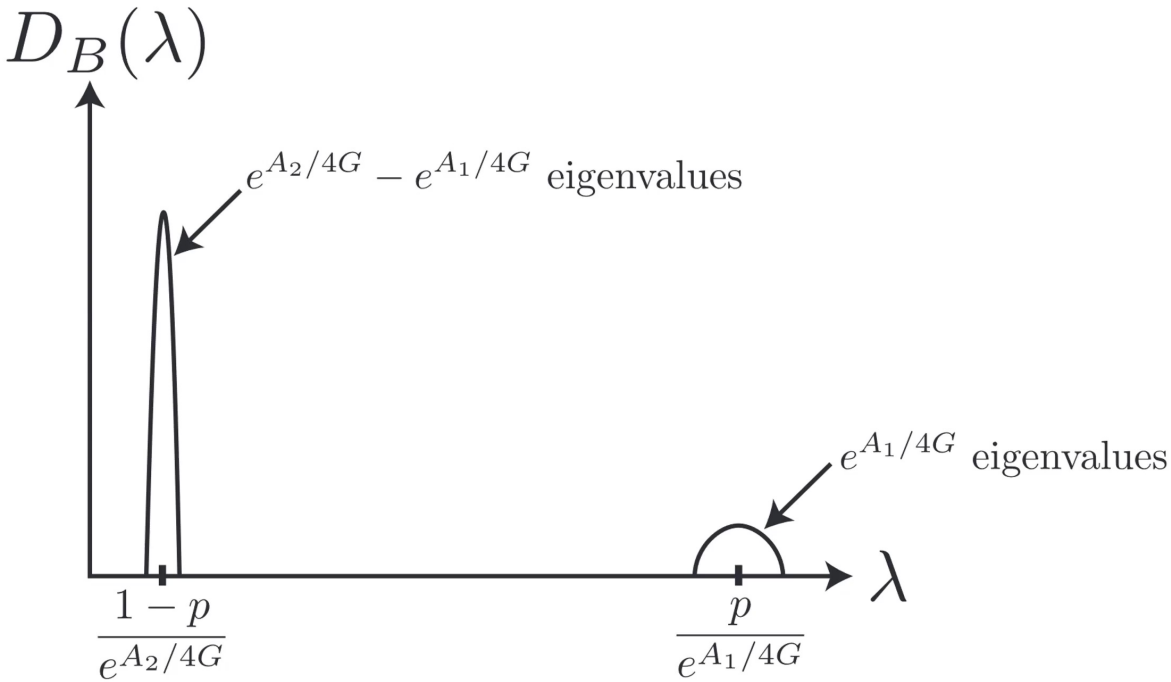
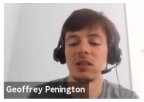


$$S(B) = p\frac{A_1}{4G_N} + (1-p)\frac{A_2}{4G_N} + O(1)$$



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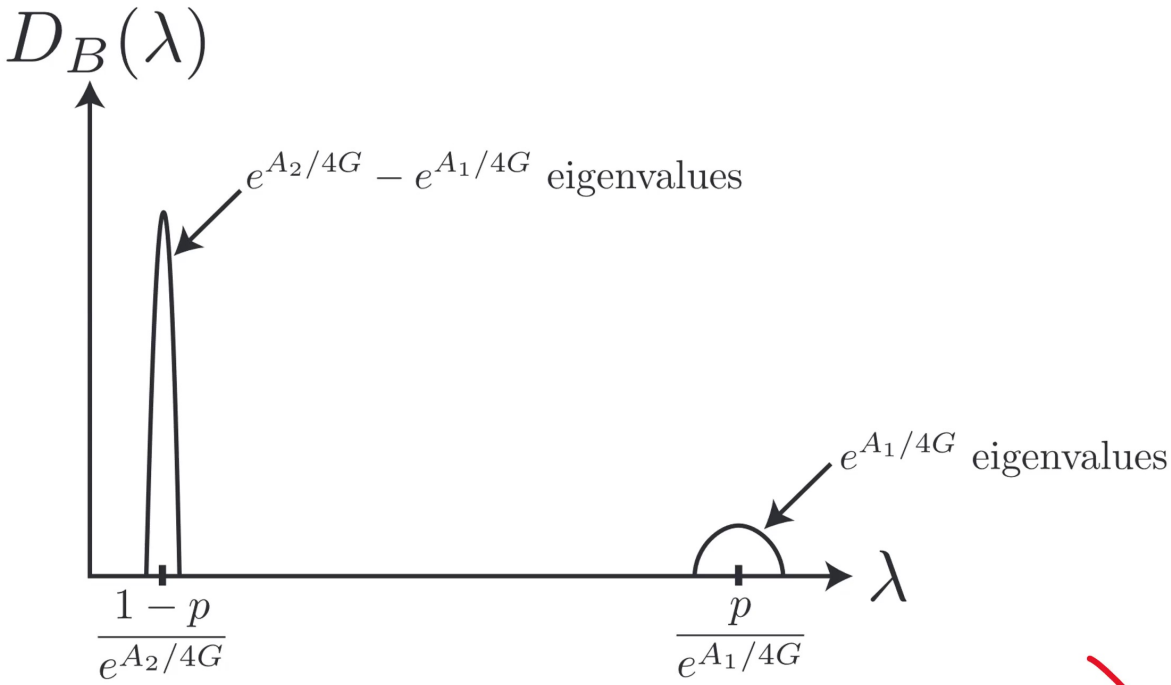

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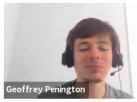
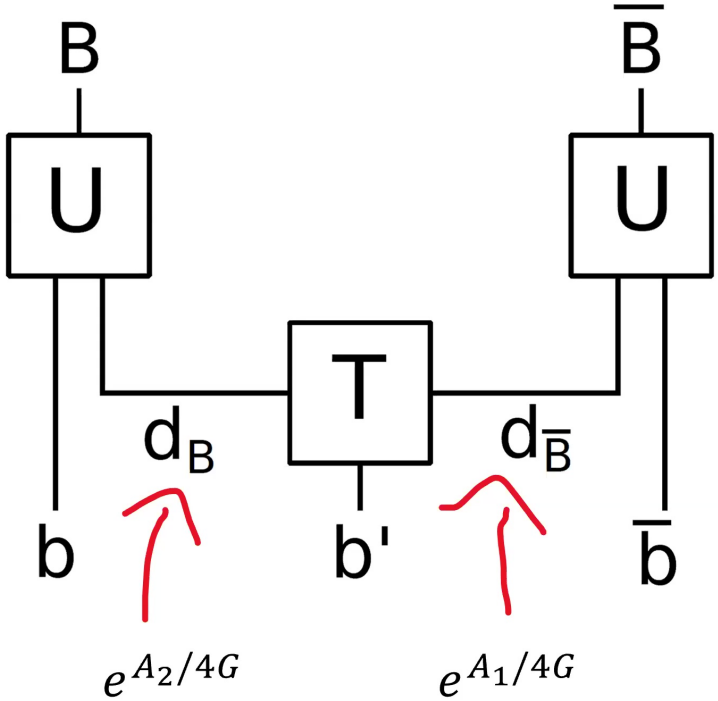
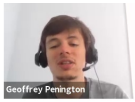
Part 3: When Do Large Corrections Exist?



General entangled bulk states

- For arbitrary bulk states, the current tools are insufficient to do the replica trick calculations **explicitly**
- However, we can make progress by noting that the formula for $\mathrm{Tr}(\rho_B^n)$ in fixed-area states is the same as the formula for a simple **random tensor network**
- We can **indirectly** do the replica trick calculation by calculating the tensor network entropies (using any technique we want)





A refined QES prescription

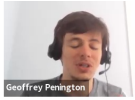
The **naïve QES prescription** says that

$$S(B)_{\text{naïve}} = \begin{cases} A_1/4G + S(bb'), & S(b'|b) \leq \frac{A_2 - A_1}{4G} \\ A_2/4G + S(b), & S(b'|b) \geq \frac{A_2 - A_1}{4G} \end{cases} .$$

A more **refined QES prescription** says that

$$S(B)_{\text{refined}} = \begin{cases} A_1/4G + S(bb'), & H^{\varepsilon}_{\max}(b'|b) \leq \frac{A_2 - A_1}{4G} \\ (\text{depends on details}), & H^{\varepsilon}_{\min}(b'|b) \leq \frac{A_2 - A_1}{4G} \leq H^{\varepsilon}_{\max}(b'|b) \\ A_2/4G + S(b), & H^{\varepsilon}_{\min}(b'|b) \geq \frac{A_2 - A_1}{4G} \end{cases} ,$$

Here $H^{\varepsilon}_{\min/\max}(b'|b)$ are the **smooth conditional min/max-entropy**



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Here $H^{\varepsilon}_{\min/\max}(b'|b)$ are the **smooth conditional min/max-entropy**

Intuition: $H^{\varepsilon}_{\max}(b'|b)$ = number of qubits needed to communicate b' to someone who already has b (**quantum state merging**).

A refined QES prescription

For pure states,

$$H^{\varepsilon}_{\min}(b'|b) = - H^{\varepsilon}_{\max}(b'|\bar{b})$$

The **naïve QES prescription** says that

$$S(B)_{\text{naïve}} = \begin{cases} A_1/4G + S(bb'), & S(b'|b) \leq \frac{A_2 - A_1}{4G} \\ A_2/4G + S(b), & S(b'|b) \geq \frac{A_2 - A_1}{4G} \end{cases} .$$

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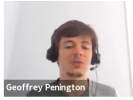
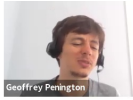
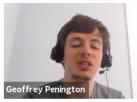


Beyond fixed-area states

- For non-fixed area states, the replica geometries involve **backreaction**, and so are hard to find **explicitly**
- However, general semiclassical holographic states can be written as a **superposition** of polynomially many fixed-area states
- By similar arguments about **entropy of mixture** $\approx$ **expectation over entropies** to the ones we saw before, this means that, at leading order, the entropy of the semiclassical state is given by the **expectation** of the entropy over the states in the superposition
- Hence semiclassical holographic states have the same **leading order entropies** as fixed-area states

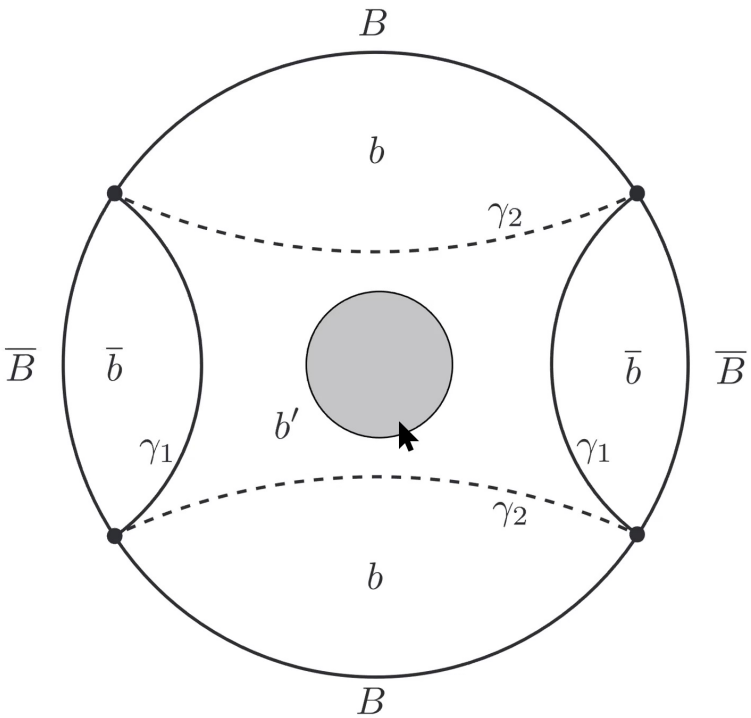


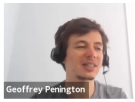
Part 4: Entanglement Wedge Reconstruction and State Merging



Entanglement Wedge Reconstruction

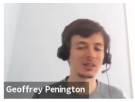
- EWR: everything between the minimal QES and the boundary region is **encoded in** said boundary region
- More precise version: given a bulk state ρ and a **bulk operator** (acting within the entanglement wedge), we can find a **boundary operator** whose action on a **purification** of ρ is the **same** as the bulk operator
- Same condition for reconstruction of b' as for the QES prescription ($H_{\max}^{\varepsilon}(b'|b) \leq (A_2 - A_1)/4G_N$)





EWR and One-shot State Merging

- One-shot state merging: given a state ρ_{AB} shared between **Alice** and **Bob**, **transfer Alice's part** to Bob by sending as few qubits as possible (so that Bob ends up with a purification of the purification of ρ_{AB})
- This is just the **Schrodinger picture** version of EWR (Bob's part of the state is the bulk state in region b, Alice's part is the bulk state in region b')
- Minimum number of qubits required for one-shot state merging is exactly $H_{\max}^{\varepsilon}(A|B)$
- Maximum number of qubits from region b' to region B is $(A_2 - A_1)/4G_N$, so gravity saturates this bound!
- For one-shot quantum state merging, it is important that you have a **classical side-channel** sending information from Alice to Bob
- In gravity, there is no classical side-channel, but there is a 'zero-bit side channel' (Hayden, GP arXiv:1807.06041). This is less useful but is still sufficient for state merging.



Beyond two extremal surfaces: min- and max-entanglement wedges

Can exist d.o.f. outside the max-EW which are fully reconstructable (or inside the min-EW that do not influence the state on B)

Max-EW = largest region b such that

$$\forall b' \subset b,\ H^{\varepsilon}_{\max}(b - b'|b') < \frac{A(b') - A(b)}{4G}$$

Min-EW = smallest region b such that

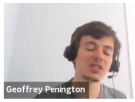
$$\forall \bar{b}' \subset \bar{b},\ H^{\varepsilon}_{\min}(\bar{b}'|b) > \frac{A(b) - A(b\bar{b}')}{4G}$$

EW = min-EW = max-EW (otherwise undefined)

Fully encoded in B

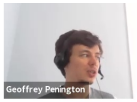
All only defined within a static slice/moment of time symmetry

D.o.f. outside this region **cannot influence** the state on B



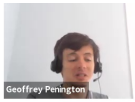
Some properties of min- and max-Ews (see paper for proofs)

1. Min- and max-EWs are **well-defined** at leading order (i.e. up to $O(\ln \varepsilon)$ variations in entropies)
2. Max-EW = Min-EW of **complementary region** (for pure states)
3. Max-EW ⊆ Min-EW
4. If $\mathrm{B}_1 \subseteq B_2$, then the min/max-EW for $\mathrm{B}_1 \subseteq$ min/max-EW for B_2 (**min/max-EW nesting)**
5. If min-EW = max-EW, then **generalised entropy is minimised** (new definition of EW is consistent with the old definition)



Some Final Comments

- Replica trick knows about more than the **QES prescription**
- **Weird oddity** (from QI point of view): holography only involves **a single state**, yet **von Neumann entropies** seemed to play a crucial role (in determining whether EWR is possible etc.)
- **Reality**: it was always **smooth min/max-entropies** that were important (as in ordinary one-shot quantum Shannon theory)
- It was just that the states we were considering happened to have smooth min/max-entropies that were close to the von Neumann entropy
- **Open questions**: time-dependent spacetimes with >2 extremal surfaces (maximin?), general derivation of refined QES directly from replica trick, many others



Thank you!