Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 13

Speakers: Daniel Siegel

Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

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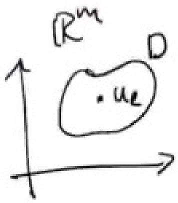
Theorem:

- strictly hyperbolic system

$$u_t + f(u)_x = 0$$

- $Df(u) \in C^1(\mathbb{R}^m, \mathbb{R}^{m\times m})$



- λ_k genuinely non-linear characteristic on $D \subset \mathbb{R}^m$

$$\nabla \lambda_k(v) \cdot r_k(v) > 0 \quad (=1) \quad \forall\, v \in D$$

- $u_\ell \in D$

Then there exists $a > 0$ and a function

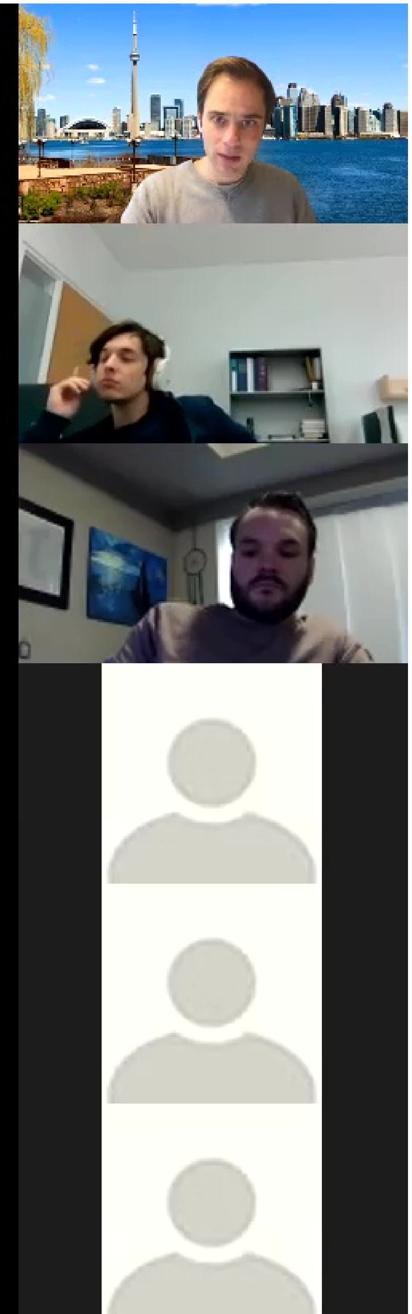
$v \in C^1([\lambda_k(u_\ell), \lambda(u_\ell) + a); \mathbb{R}^m)$ $\quad (y \mapsto v(y))$

such that RP

$$u(x,0) = u_0(x) = \begin{cases} u_\ell, & x < 0 \\ \underbrace{v(y)}_{\text{"}u_r\text{"}}, & x > 0 \end{cases}$$

for all $y \in [\lambda_k(u_\ell), \lambda_k(u_\ell) + a)$ can be solved by a continuous weak solution.

One has $\frac{d}{dy} \lambda_k(v(y)) > 0$, $\quad \lambda_k(u_\ell) < \lambda_k(u_r)$



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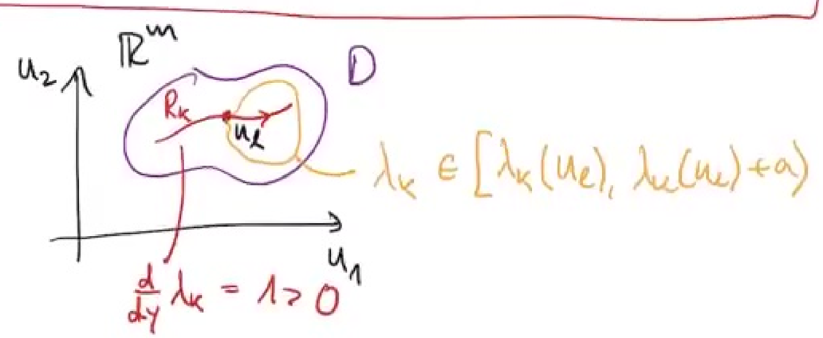
$$\underbrace{v(y)}_{\text{"}u_r\text{"}}, \quad x > 0$$

for all $y \in [\lambda_k(u_l), \lambda_k(u_l)+a)$ can be solved by a continuous weak solution.

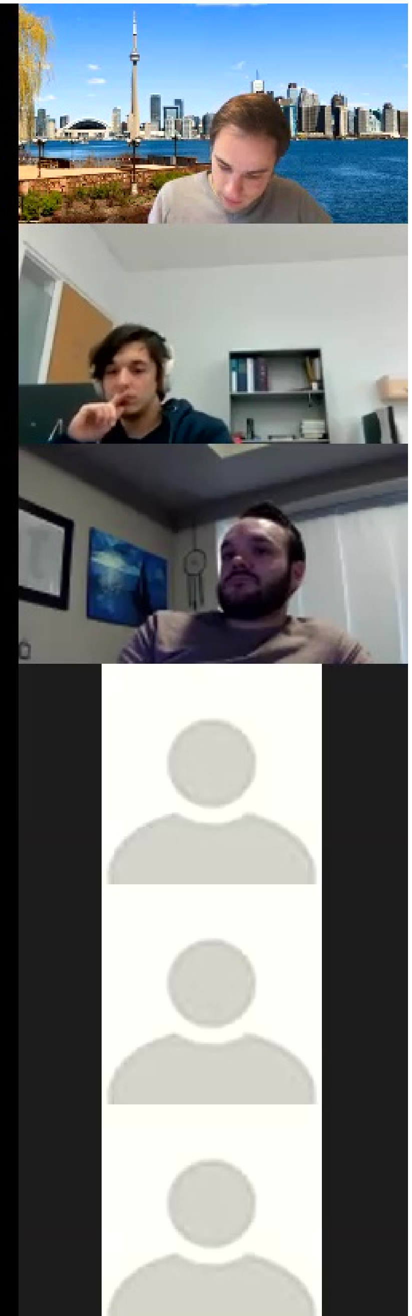
One has $\frac{d}{dy}\lambda_k(v(y)) > 0$, $\lambda_k(u_l) < \lambda_k(u_r)$

and

$$u(x,t) = \begin{cases} u_l, & \frac{x}{t} < \lambda_k(u_l) & \text{I} \\ v(\frac{x}{t}), & \lambda_k(u_l) < \frac{x}{t} < y & \text{II} \\ v(y), & y < \frac{x}{t} & \text{III} \end{cases}$$



Proof: $\exists\, a > 0$ sufficiently small such t



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Proof: $\exists\, a > 0$ sufficiently small such that the IVP

$$\begin{cases} v'(y) = r_k(v(y)) , & \lambda_k(u_\ell) < y < \lambda_k(u_\ell) + a \\ v(\lambda_k(u_\ell)) = u_\ell \end{cases}$$

has a unique solution (ODE)

Also: $$\frac{d}{dy}\lambda_k(v(y)) = \nabla\lambda_k(v(y))\, \underline{v'(y)} = \nabla\lambda_k \cdot r_k = 1$$
↖ by assumption

Since $\lambda_k(\underbrace{v(\lambda_k(u_\ell))}_{u_\ell}) = \lambda_k(u_\ell)$

$\lambda_k(v(y)) = y$ along $\lambda_k(u_\ell) < y < \lambda_k(u_\ell) + a$

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Since $\lambda_k(v(\underbrace{\lambda_k(u_\ell)}_{u_\ell})) = \lambda_k(u_\ell)$

(**) $\lambda_k(v(y)) = y$ along $\lambda_k(u_\ell) < y < \lambda_k(u_\ell) + a$

For ansatz (*)

we have $\partial_t u + f(u)_x = 0$

in the distributional sense:

In I and III: $u = \text{const.} \Rightarrow u_t + f(u)_x = 0$ ✓

In II: $v'\left(\frac{x}{t}\right)\left(-\frac{x}{t^2}\right) + Df\left(v(x/t)\right)\underbrace{v'\left(\frac{x}{t}\right)}\frac{1}{t}$

$$= r_k\left(v\left(\tfrac{x}{t}\right)\right)\left(-\frac{x}{t^2}\right) + \frac{1}{t}\underbrace{\lambda_k\left(v\left(\tfrac{x}{t}\right)\right)}_{\overset{(**)}{=}\frac{x}{t}} r_k\left(v\left(\tfrac{x}{t}\right)\right)$$

$$= 0$$ ✓

Let u

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$$= r_k\left(v\left(\tfrac{x}{t}\right)\right)\left(-\frac{x}{t^2}\right) + \frac{1}{t}\underbrace{\lambda_k\left(v\left(\tfrac{x}{t}\right)\right)}_{\overset{(**)}{=}\frac{x}{t}} r_k\left(v\left(\tfrac{x}{t}\right)\right)$$

$$= 0 \quad \checkmark$$

Let w be a k-Riemann invariant in D. Then:

$$\frac{d}{dt} w\big(\underbrace{u(\gamma(t),t)}_{=\text{const in } I, III}\big) = 0$$

$$\frac{d}{dt} w(u(\gamma(t),t)) = \nabla w\,\big(\underbrace{u_x \overset{\lambda_k}{\gamma'} + u_t}_{=0 \text{ in } II}\big) = 0$$

γ: corresponding k-characteristic with $\gamma'(t) = \lambda_k(u(\gamma(t),t))$

$\Rightarrow$ any w is constant in D

$\Rightarrow$ u is rarefaction wave $\square$

Remar

$$\frac{d}{dt} w(u(\gamma(t),t)) = \nabla w \, (u_x \underbrace{\gamma'}_{\lambda_k} + u_t) = 0$$

$\underbrace{u_x \gamma' + u_t}_{= 0 \text{ in } II}$

γ: corresponding k-characteristic with $\gamma'(t) = \lambda_k(u(\gamma(t),t))$

$\Rightarrow$ any w is constant in D

$\Rightarrow$ u is rarefaction wave $\square$

<u>Remarks:</u> 1) This rarefaction wave solution is the uniquely determined weak solution satisfying the entropy condition.

2) Note that from $\frac{d}{dy} \lambda_k(v(y)) > 0$ it follows that

$$\lambda_k(u_l) < \lambda_k(u_r)$$

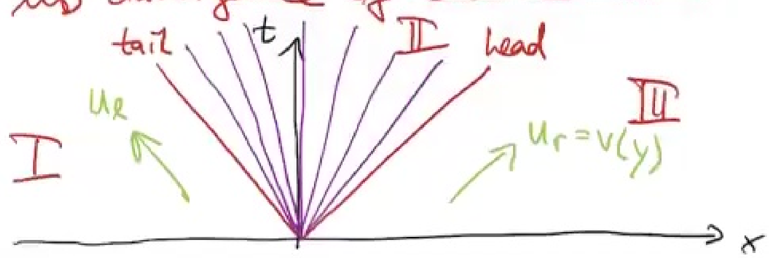
$\rightsquigarrow$ divergence of characteristics

satisfying the entropy condition.

2) Note that from $\frac{d}{dy}\lambda_k(v(y)) > 0$ it follows that

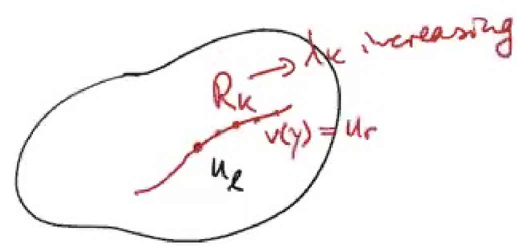
$$\lambda_k(u_l) < \lambda_k(u_r)$$

$\rightsquigarrow$ divergence of characteristics



$\rightsquigarrow$ continuous, self-similar solution that "smooths out" the initial discontinuity.

$\mathbb{R}^m$



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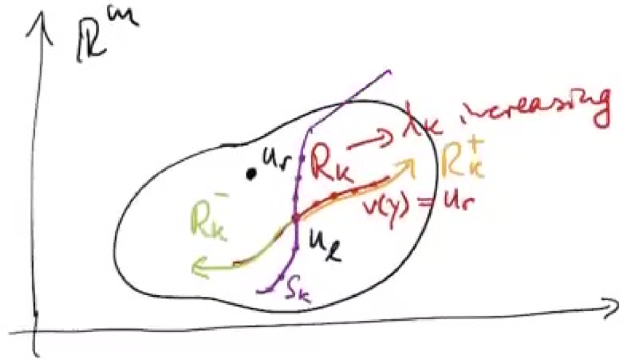
no continuous, self-similar solution that "smooths out" the initial discontinuity.

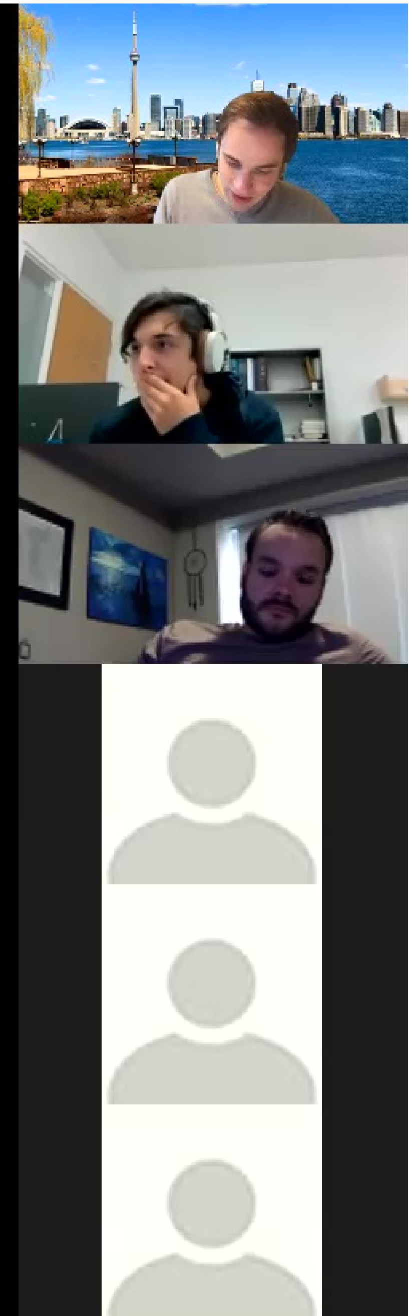


Notation:

$$R_k^+(u_\ell) \equiv \{ z \in R_k(u_\ell) \mid \lambda_k(z) > \lambda_k(u_\ell) \}$$

$$R_k^-(u_\ell) \equiv \{ z \in R_k(u_\ell) \mid \lambda_k(z) < \lambda_k(u_\ell) \}$$

4.4.4 Special solutions: shocks and the lax



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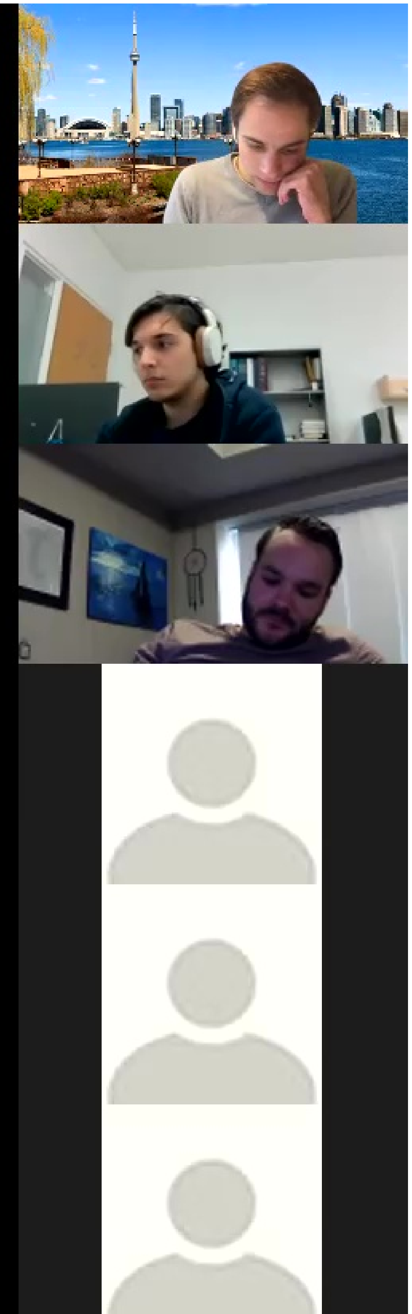
4.4.4 Special solutions: shocks and the Lax entropy condition

Lax (1957):



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4.4.4 Special solutions: shocks and the lax entropy condition

Lax (1957):

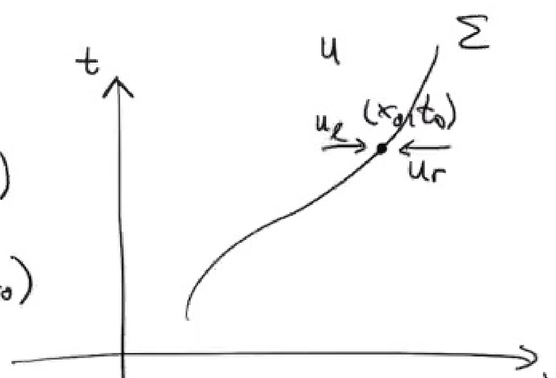
$u_t + f(u)_x = 0$ strictly hyperbolic

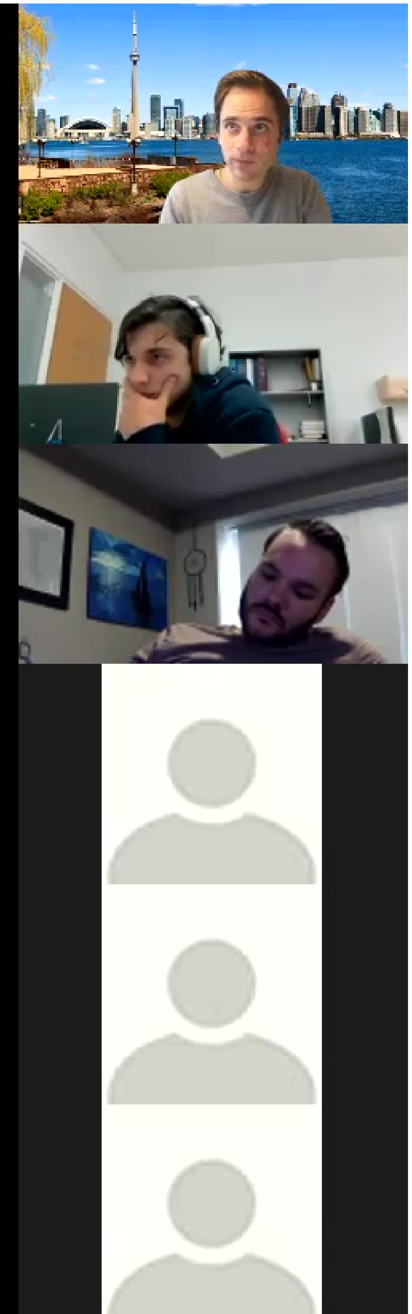
u: weak solution piecewise smooth with a jump along $\Sigma = (\sigma(t), t)$ for a curve σ with $\sigma' = s$ the shock velocity

$(x_0, t_0) \in \Sigma$

$u_\ell = \lim_{x \nearrow x_0} u(x, t_0)$

$u_r = \lim_{x \searrow x_0} u(x, t_0)$





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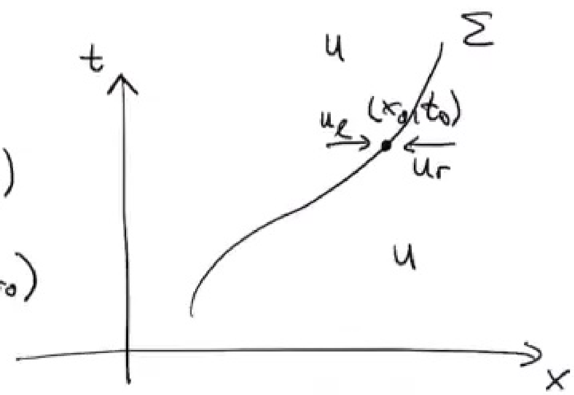
velocity

$(x_0, t_0) \in \Sigma$

$$u_\ell = \lim_{x \nearrow x_0} u(x, t_0)$$

$$u_r = \lim_{x \searrow x_0} u(x, t_0)$$

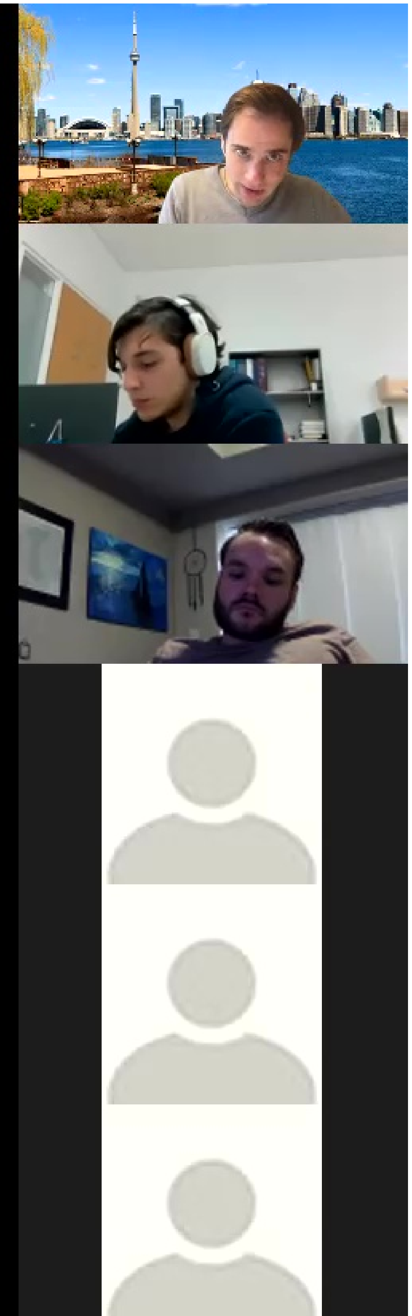


Assume:

$$\lambda_1(u_r) < \dots < \lambda_k(u_r) < s < \lambda_{k+1}(u_r) < \dots < \lambda_m(u_r)$$

$$\lambda_1(u_\ell) < \dots < \lambda_j(u_\ell) < s < \lambda_{j+1}(u_\ell) \dots < \lambda_m(u_\ell)$$

$\rightsquigarrow$ k-conditions on u_r at (x_0, t_0) from initial data at $t=0$ on right side due to impinging ℓ-

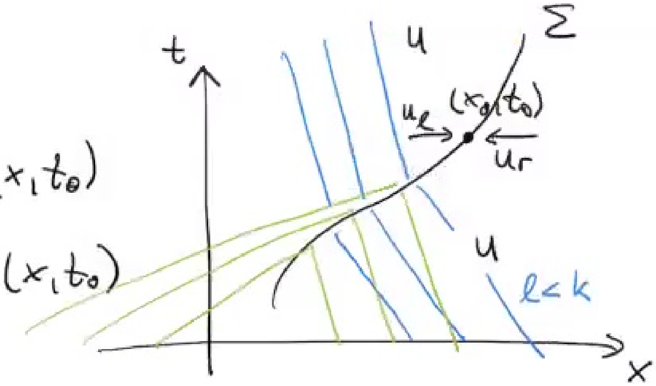


a jump along $\Sigma = (\sigma(t), t)$ for a curve σ with $\sigma' = s$ the shock velocity

$(x_0, t_0) \in \Sigma$

$$u_\ell = \lim_{x \nearrow x_0} u(x, t_0)$$

$$u_r = \lim_{x \searrow x_0} u(x, t_0)$$

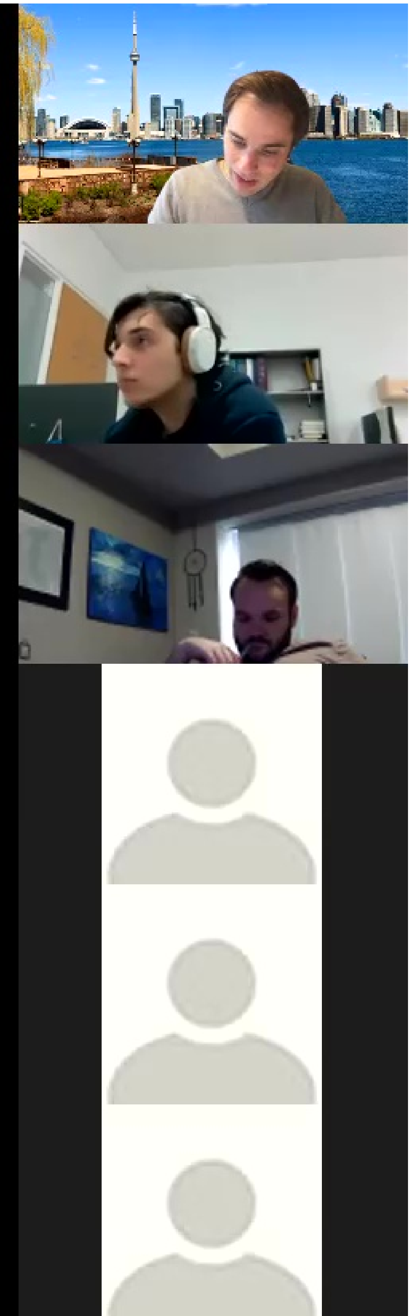


Assume:

$$\lambda_1(u_r) < \dots < \lambda_k(u_r) < s < \lambda_{k+1}(u_r) < \dots < \lambda_m(u_r)$$

$$\lambda_1(u_\ell) < \dots < \lambda_j(u_\ell) < s < \lambda_{j+1}(u_\ell) \dots < \lambda_m(u_\ell)$$

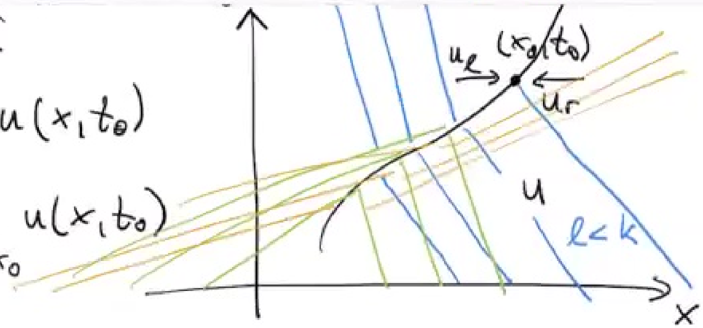
$\rightsquigarrow$ k-conditions on u_r at (x_0, t_0) from initial data at $t=0$ on right side due to impinging ℓ-characteristics $\lambda_1, \dots, \lambda_k < s$



$(x_0, t_0) \in \Sigma$

$u_\ell = \lim_{x \nearrow x_0} u(x, t_0)$

$u_r = \lim_{x \searrow x_0} u(x, t_0)$



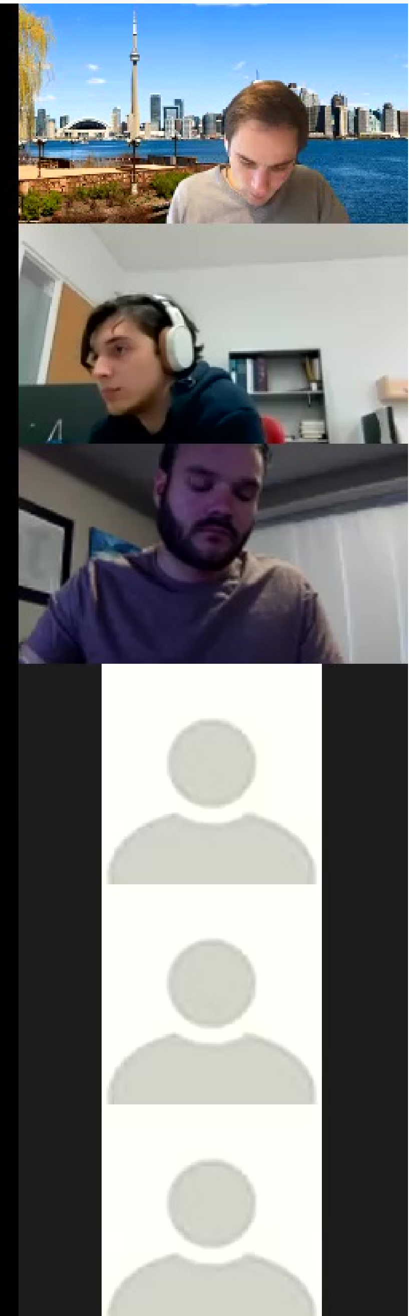
Assume:

$$\lambda_1(u_r) < \dots < \lambda_k(u_r) < s < \lambda_{k+1}(u_r) < \dots < \lambda_m(u_r)$$

$$\lambda_1(u_\ell) < \dots < \lambda_j(u_\ell) < s < \lambda_{j+1}(u_\ell) \dots < \lambda_m(u_\ell)$$

⇝ k-conditions on u_r at (x_0, t_0) from initial data at $t=0$ on right side due to impinging ℓ-characteristics $\lambda_1, \dots, \lambda_k < s$

$m-j$ conditions on u_ℓ at (x_0, t_0) from ℓ-characteristics $\lambda_{j+1}, \dots, \lambda_m > s$



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$m-j$ conditions on u_ℓ at (x_0, t_0)

from ℓ-characteristics

$\lambda_{j+1}, \dots, \lambda_m > s$

m conditions relating u_r & u_ℓ

(jump conditions)

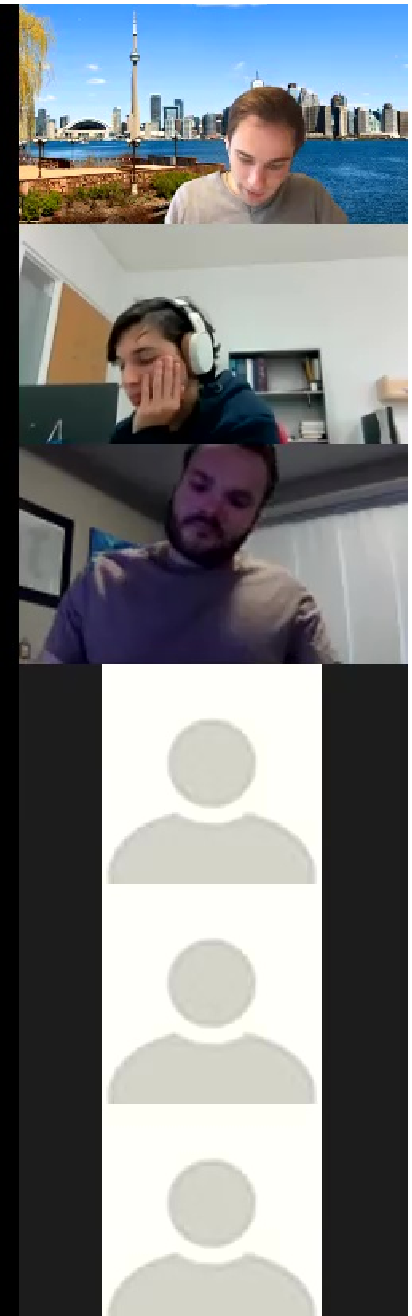
$\rightsquigarrow$ $2m+1$ unknowns

$$u_\ell = \begin{pmatrix} u_{\ell 1} \\ \vdots \\ u_{\ell m} \end{pmatrix}, \quad u_r = \begin{pmatrix} u_{r1} \\ \vdots \\ u_{rm} \end{pmatrix}, \quad s$$

Necessary condition for computing the unknowns for arbitrary initial data is

$$k + m - j + m = 2m + k - j = 2m + 1$$

$$\Leftrightarrow \quad j = k - 1$$



the unknowns for arbitrary initial data is

$$k + m - j + m = 2m + k - j = 2m + 1$$

$$\Leftrightarrow \quad j = k - 1$$

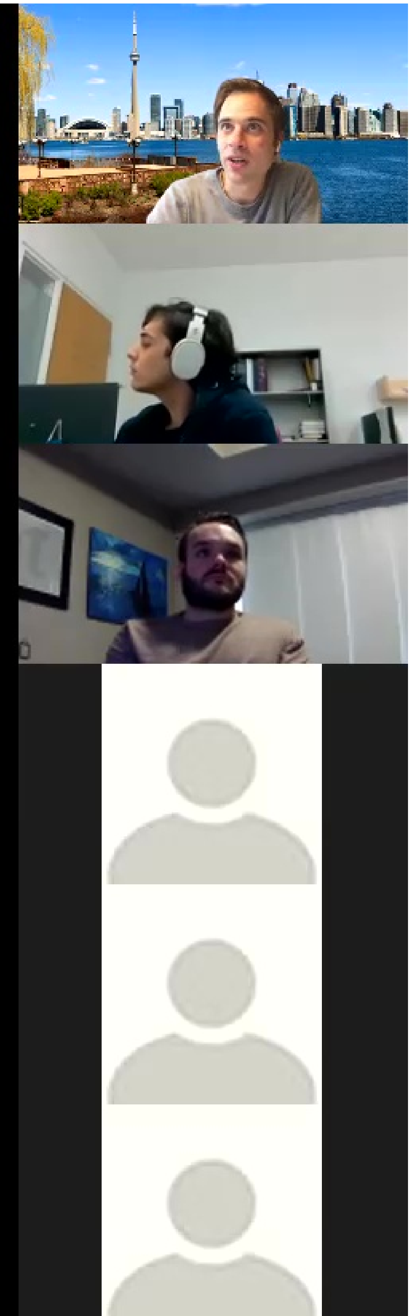
$$\Leftrightarrow \quad \lambda_k(u_r) < s < \lambda_{k+1}(u_r)$$

$$\lambda_{k-1}(u_l) < s < \lambda_k(u_l)$$

or (*)

$$\boxed{\lambda_k(u_r) < s < \lambda_k(u_l)}$$

Lax entropy condition



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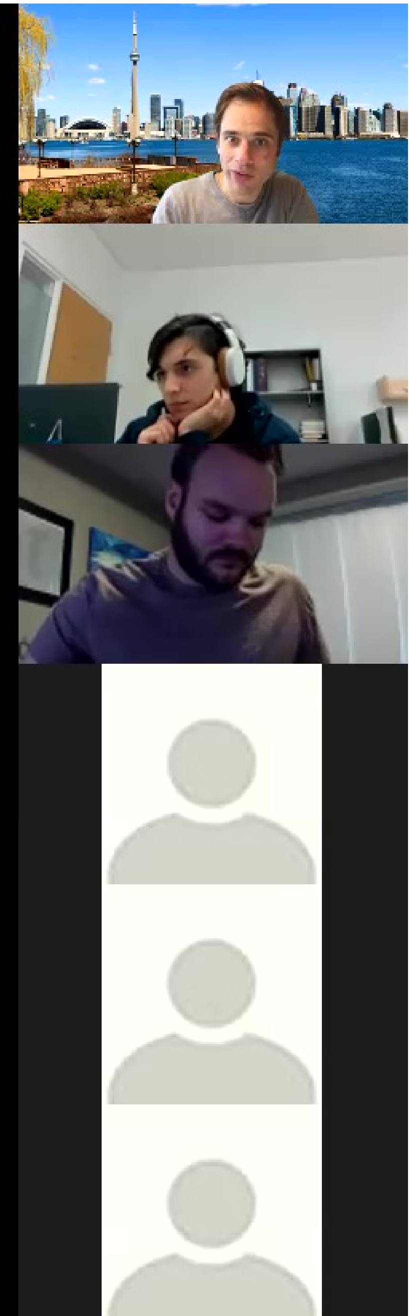
or (*) $\lambda_k(u_r) < s < \lambda_k(u_l)$ Lax entropy condition

Def (Lax entropy condition, k-shock)

A discontinuous weak solution $u(x,t)$ statisfies the RH jump conditions and the Lax entropy condition (*) is called a k-shock.

Remarks: 1) The Lax entropy condition states that k-characteristics impinge on the discontinuity while the other characteristics cross it:

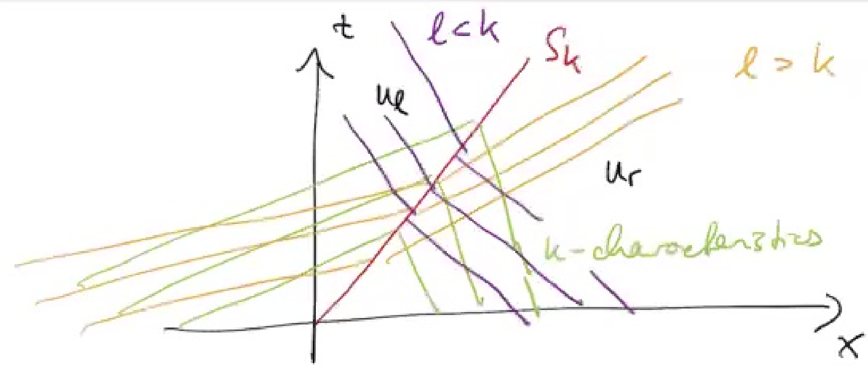
$\lambda_l(u_l) < s$ & $\lambda_l(u_r) < s$ $l < k$

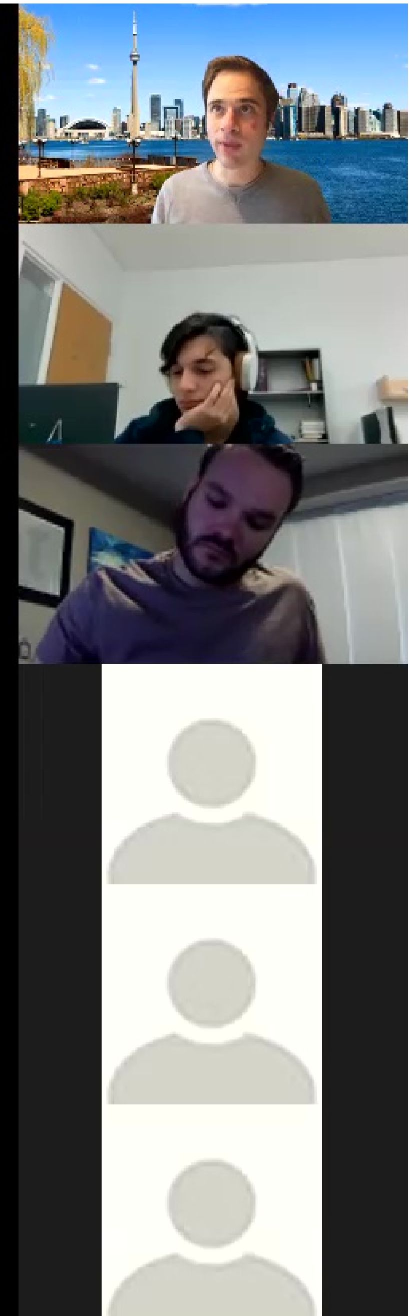


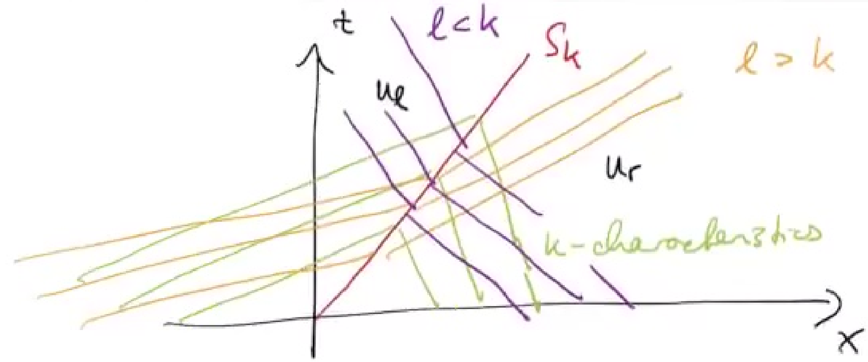
Remarks: 1) The Lax entropy condition states that k-characteristics impinge on the discontinuity while the other characteristics cross it:

$$\lambda_\ell(u_\ell) < s \quad \& \quad \lambda_\ell(u_r) < s \qquad \ell < k$$

$$\lambda_\ell(u_\ell) > s \quad \& \quad \lambda_\ell(u_r) > s \qquad \ell > k$$







Existence of k-shocks:

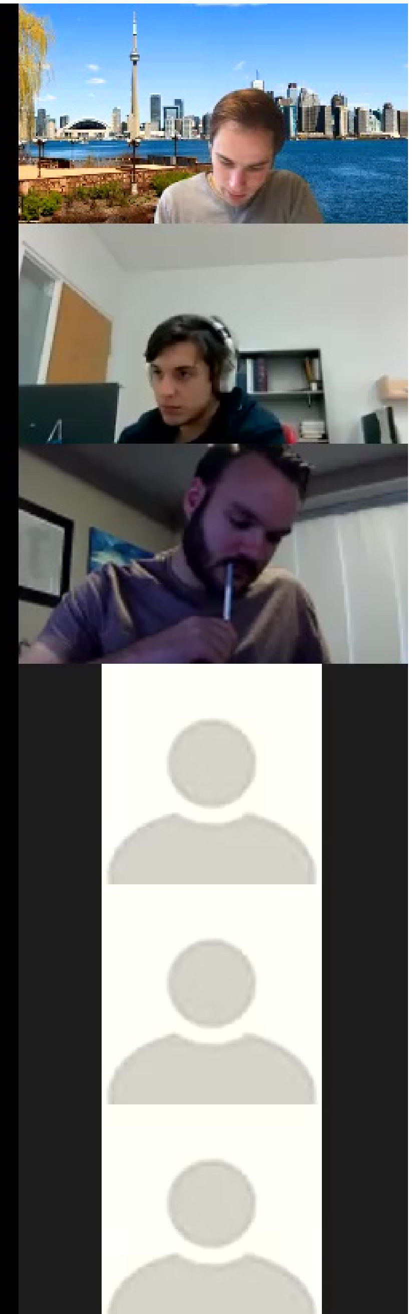
Def: Given state $u_\ell \in \mathbb{R}^m$, define shock set

$$S(u_\ell) \equiv \left\{ z \in \mathbb{R}^m \;\middle|\; f(z) - f(u_\ell) = s(z - u_\ell) \text{ for a constant } s = s(z, u_\ell) \right\}$$

where f is the flux of a system of conservation laws.

Theorem (properties of shock set):

Consider RP for a strictly hyperbolic system of CL, $u_t + f(u)_x = 0$



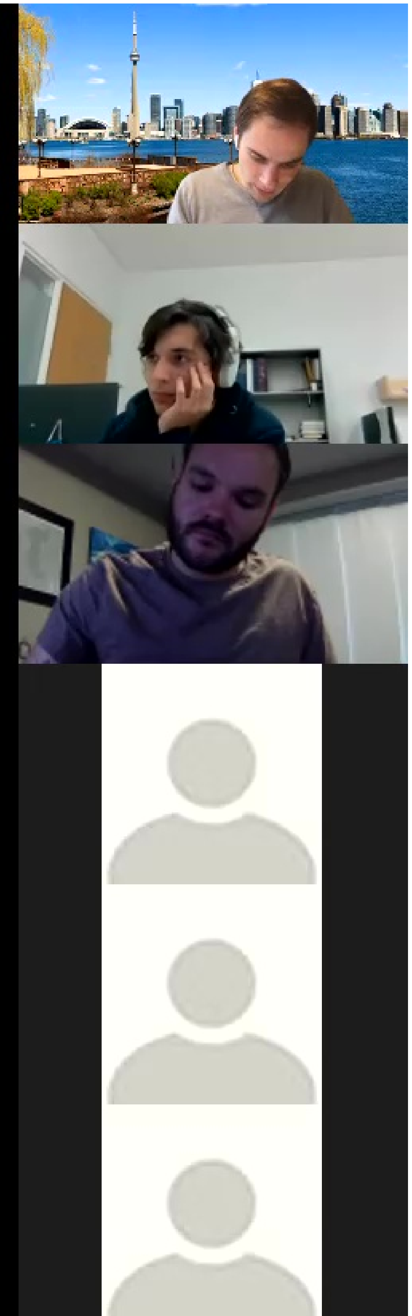
of conservation laws.

Theorem (properties of shock set):

Consider RP for a strictly hyperbolic system of CL, $u_t + f(u)_x = 0$, let $u_\ell \in \mathbb{R}^m$. Then there exists a neighborhood of u_ℓ

in which $S(u_\ell)$ is the union of m smooth curves $S_k(u_\ell)$ $(k = 1, \dots, m)$ with

(i) $S_k(u_\ell)$ pass through u_ℓ with tangent $r_k(u_\ell)$, $\quad s(0) = u_\ell$
$\quad s'(0) = r_k(u_\ell)$

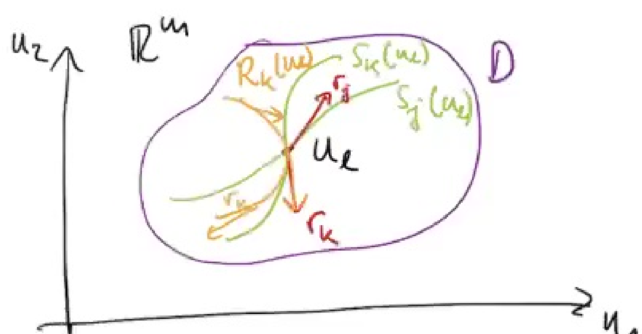


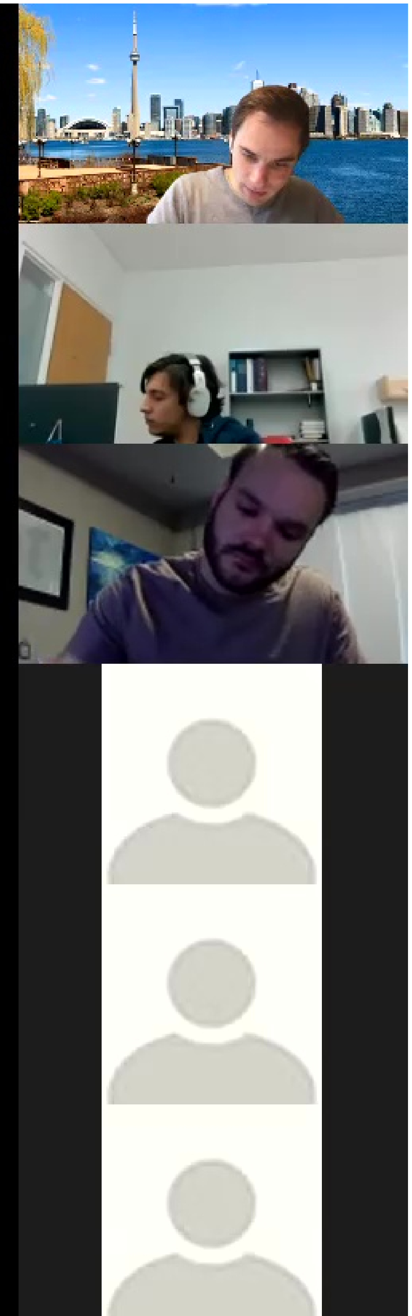
in which $S(u_e)$ is the union of m smooth curves $S_k(u_e)$ $(k=1,\dots,m)$ with

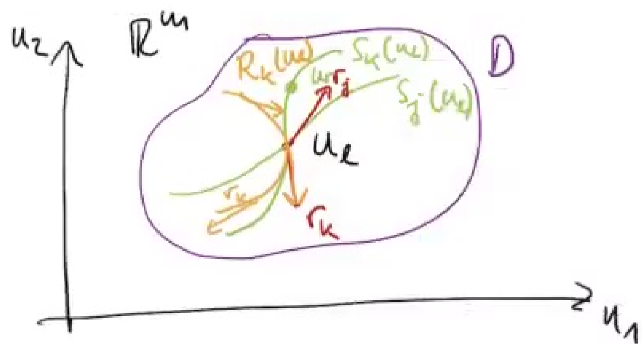
(i) $S_k(u_e)$ pass through u_e with tangent $r_k(u_e)$, $s(0)=u_e$, $s'(0)=r_k(u_e)$

(ii) $$\lim_{\substack{z\to u_e \\ z\in S_k}} s(z,u_e) = \lambda_k(u_e)$$

(iii) $$s(z,u_e) = \frac{\lambda_k(z)+\lambda_k(u_e)}{2} + O(|z-u_e|^2)$$
as $z\in S_k(u_e)\to u_e$







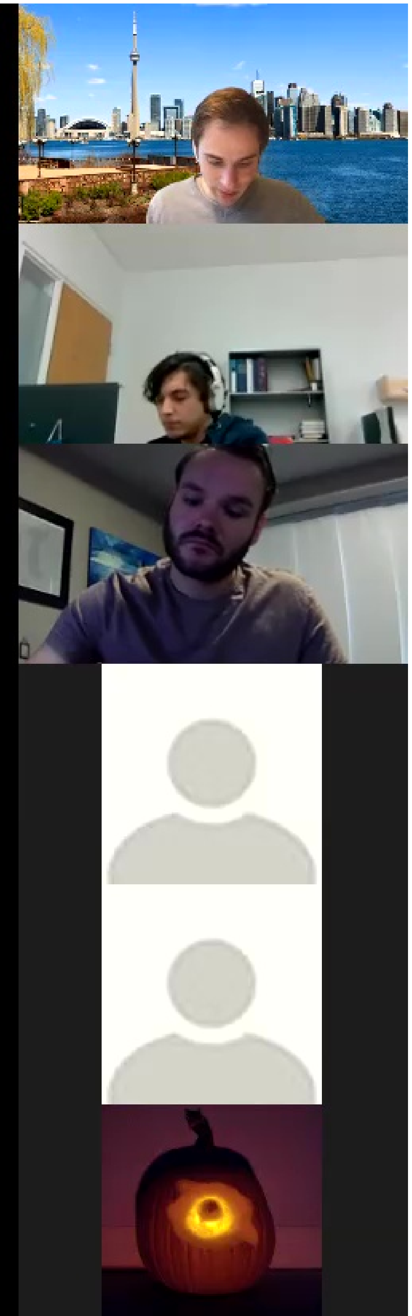
Structure of k-shocks

Let (λ_k, r_k) be genuinely non-linear in $D \subseteq \mathbb{R}^m$ with $u_r \in S_k(u_\ell)$, then

$$u(x,t) = \begin{cases} u_\ell & x < st \\ u_r & x > st \end{cases}$$

is a weak solution to the RP with $s = s(u_r, u_\ell)$.

$$\frac{d}{dt}\Big|_{t=0} \lambda_k(S_k(t)) = \nabla\lambda_k(S_k(t))\, S_k'(t)\Big|_{t=0}$$



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in $D \subseteq \mathbb{R}^m$ with $u_r \in S_k(u_\ell)$, then

$$u(x,t) = \begin{cases} u_\ell & x < st \\ u_r & x > st \end{cases}$$

is a weak solution to the RP with

$s = s(u_r, u_\ell)$.

$$\frac{d}{dt}\Big|_{t=0} \lambda_k(S_k(t)) = \nabla\lambda_k(S_k(t))\, S_k'(t)\Big|_{t=0}$$

$$= \nabla\lambda_k(u_\ell)\, r_k(u_\ell) \neq 0$$

(genuinely non-linear)

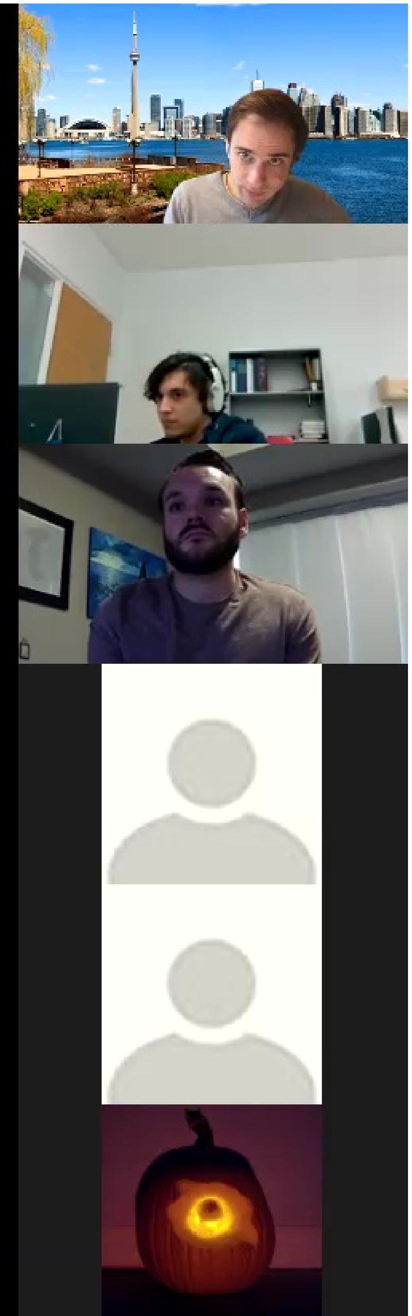
λ_k cannot be constant

$\Rightarrow \lambda_k(u_r) < s(u_r, u_\ell) < \lambda_k(u_\ell)$

or $\lambda_k(u_\ell) < s(u_r, u_\ell) < \lambda_k(u_r)$ Lax entropy violating

we can only admit

u_r satisfy



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$$\frac{d}{dt}\Big|_{t=0} \lambda_k(S_k(t)) = \nabla\lambda_k(S_k(t))\, S_k'(t)\Big|_{t=0}$$

$$= \nabla\lambda_k(u_\ell)\, r_k(u_\ell) \neq 0$$

(genuinely non-linear)

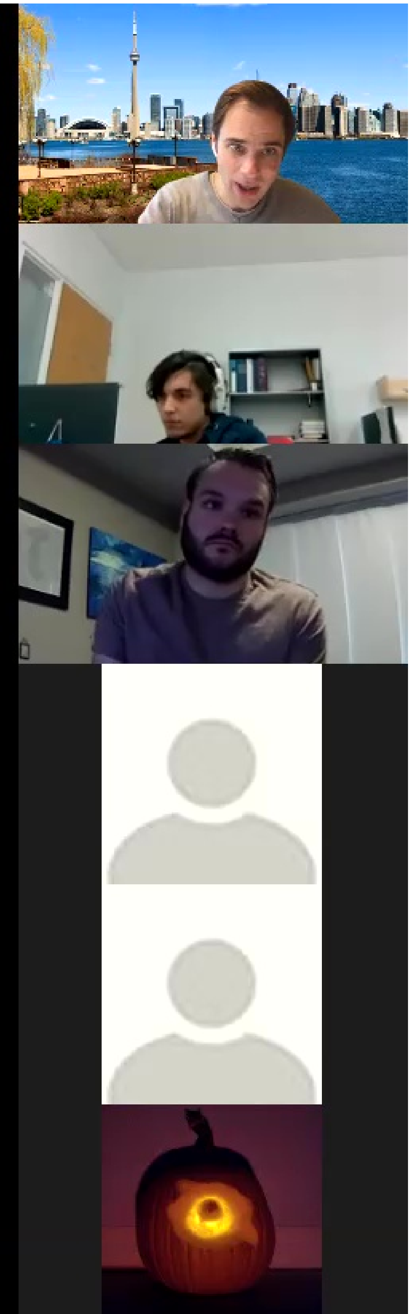
λ_k cannot be constant

$$\Rightarrow \quad \lambda_k(u_r) < s(u_r, u_\ell) < \lambda_k(u_\ell)$$

$$\text{or} \quad \lambda_k(u_\ell) < s(u_r, u_\ell) < \lambda_k(u_r)$$ Lax entropy violating

so can only admit

u_r satisfying first relation!



Notation:

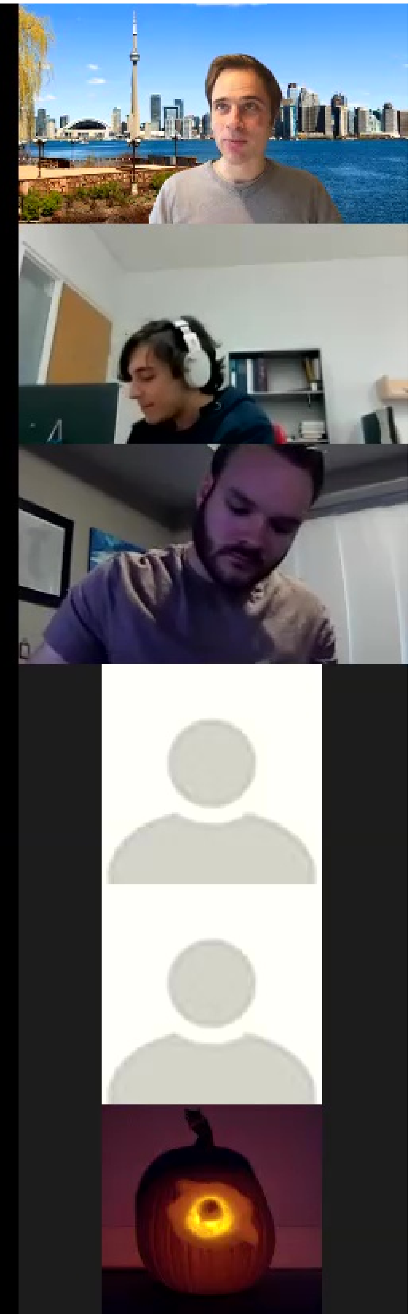
$$S_k^+(u_\ell) \equiv \{ z \in S_k(u_\ell) \mid \lambda_k(u_\ell) < \sigma(z, u_\ell) < \lambda_k(z) \}$$

$$S_k^-(u_\ell) \equiv \{ z \in S_k(u_\ell) \mid \lambda_k(z) < \sigma(z, u_\ell) < \lambda_k(u_\ell) \}$$

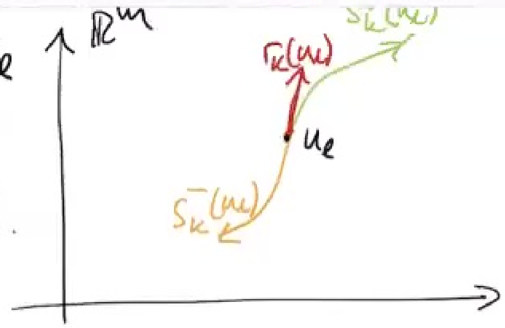
$$S_k(u_\ell) \equiv S_k^+(u_\ell) \cup \{u_\ell\} \cup S_k^-(u_\ell)$$

Note: (u_r, u_ℓ) admissible if and only if $u_r \in S_k^-(u_\ell)$.

Theorem:



Note: (u_r, u_ℓ) admissible if and only if $u_r \in S_k^-(u_\ell)$.



Theorem (equivalence of entropy condition): Consider hyperbolic and genuinely non-linear system $u_t + f(u)_x = 0$ and assume u weak shock solution with sufficiently small jump $u_\ell - u_r$.

Then the entropy condition along the discontinuity is equivalent to the Lax entropy condition.