Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 12

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Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

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Lecture_12

Lecture_12 | Chap_4_PropertiesConservationLaws | Lecture_11

Further remarks on entropy solutions:

Definition: A weak solution u of the (IVP) is called an entropy solution if for any entropy pair Φ, Ψ it satisfies the inequality

$$\boxed{\Phi(u)_t + \Psi(u)_x \leq 0 \quad \text{in } \mathbb{R} \times (0,\infty)}$$

in the distributional sense, i.e.

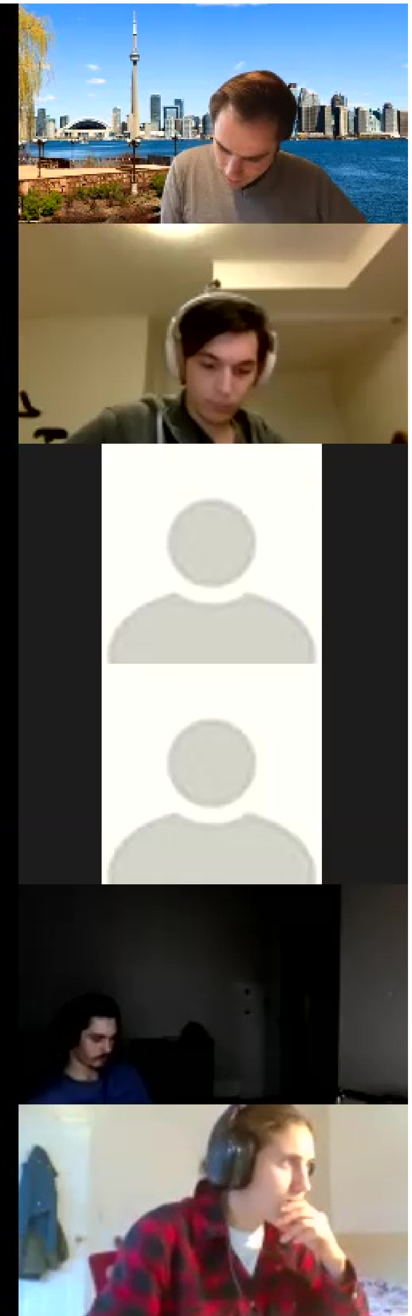
partial integration reverses sign

$$\int_0^\infty \int_{\mathbb{R}} \Phi(u) v_t + \Psi(u) v_x \, dx\, dt \geq 0 \quad \forall v \in C_0^\infty(\mathbb{R} \times (0,\infty)), \; v \geq 0.$$

Remarks:

1) Uniqueness:

- Note entropy solution is weak soln that satisfies the entropy inequalities for ANY entropy pair Φ, Ψ of the CL
- 1D scalar equations:
  There exists at most one weak solution



1) Uniqueness:

- Note entropy solution is weak soln that satisfies the entropy inequalities for ANY entropy pair Φ, Ψ of the CL

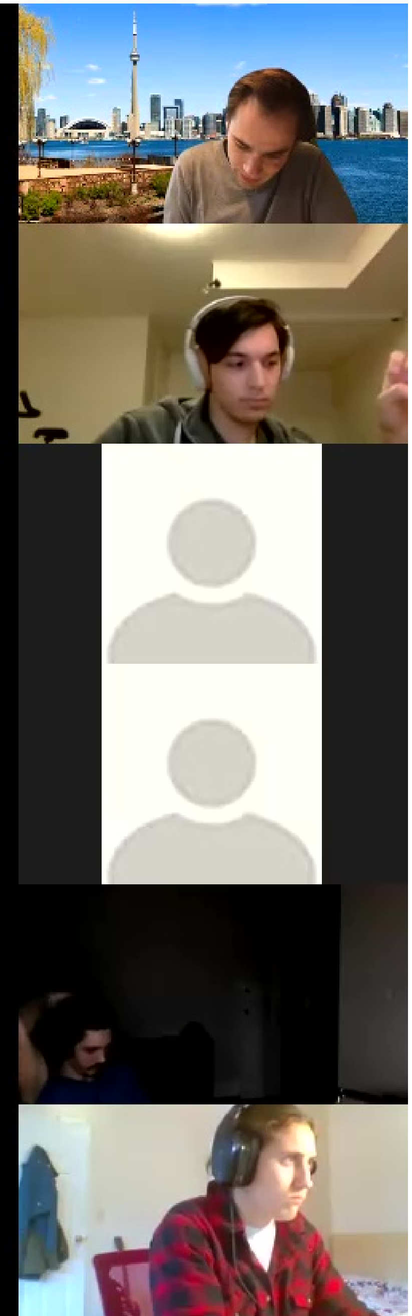
- 1D salar equations:
  There exists at most one weak solution that satisfies the entropy inequalities for ALL entropy pairs of the CL.

- 1D systems of CLs: similar uniqueness theorem

- multi-D system: uniqueness still an open problem

2) Existence:

- 1D sca



9:41 AM Tue Jan 9

Lecture_12

Lecture_12 | Chap_4_PropertiesConservationLaws | Lecture_11

ALL entropy pairs of the CL.

- 1D systems of CLs: similar uniqueness theorem
- multi-D system: uniqueness still an open problem

2) Existence:

- 1D scalar CL $(m=1)$: any convex Φ

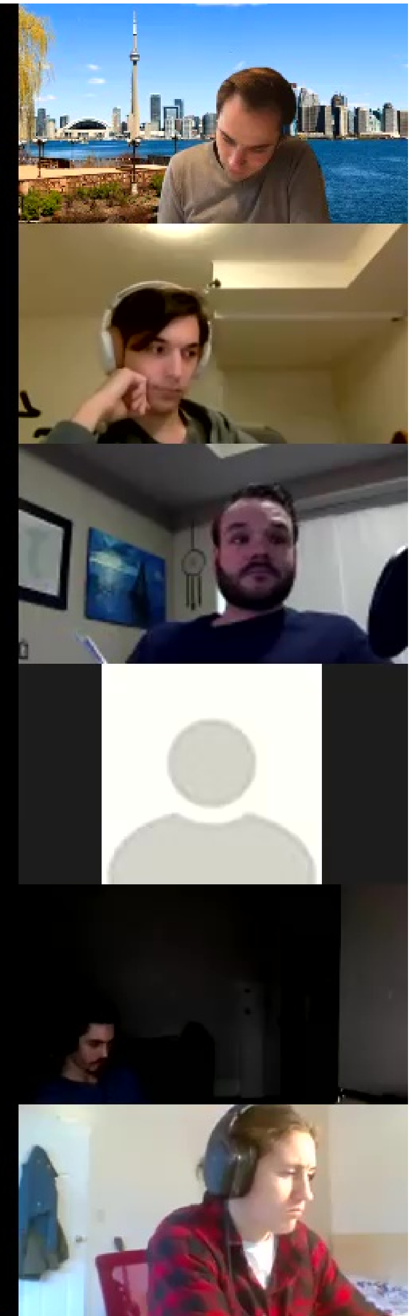
$\rightsquigarrow$ obtain corresponding flux

$$\Psi(z) \equiv \int_{z_0}^{z} \Phi'(y)\, f'(u)\, dy \quad , z \in \mathbb{R}$$

- $m=2$: find Φ, Ψ, with Φ convex, and

$$\underbrace{(\Phi_{z_1}, \Phi_{z_2})}_{D\Phi}\, Df(z) = \underbrace{\begin{pmatrix} \Psi_{z_1} \\ \Psi_{z_2} \end{pmatrix}}_{D\Psi} \quad , z \in \mathbb{R}^2$$

$m > 2$: over-determined



z_0

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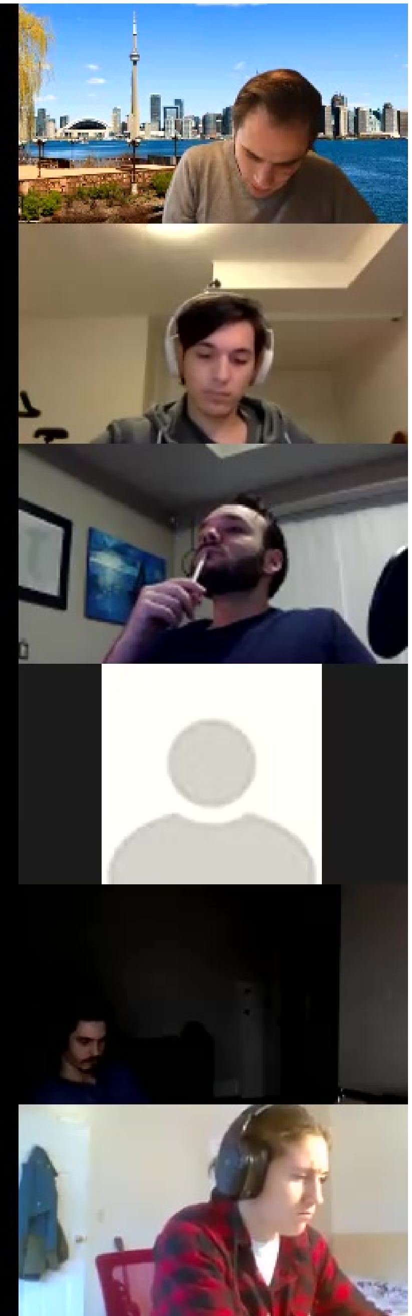
$m > 2$: over-determined

- hydrodynamics: one has the thermo-dynamic entropy

$\Phi(u) \equiv -S$

$\Psi(u)$: entropy flux

Note: mathematical entropy can only decrease



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$$(\Phi_{z_1}, \Phi_{z_2})\, Df(z) = \begin{pmatrix} \Psi_{z_1} \\ \Psi_{z_2} \end{pmatrix} \quad , z \in \mathbb{R}^2$$

$$\underbrace{(\Phi_{z_1}, \Phi_{z_2}, \Phi_{z_3})}_{D\Phi}\, Df(z) = \overbrace{\begin{pmatrix} \ \end{pmatrix}}^{D\Psi}$$

$m > 2$: over-determined

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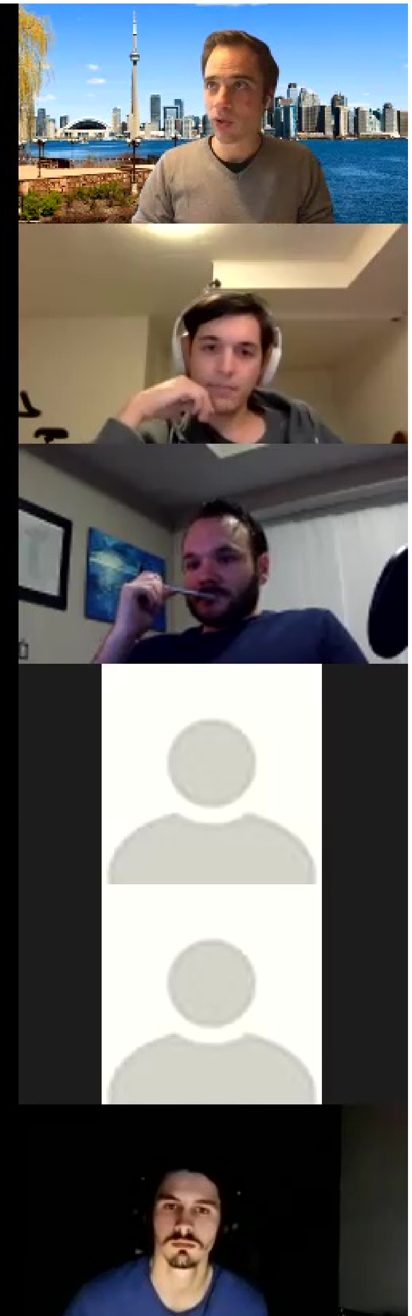


(unique)

solution
(unique)

Riemann problem

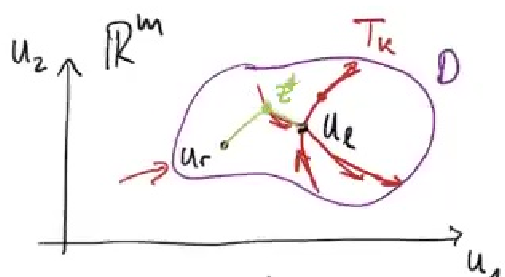
Riemann problem for systems of CLs

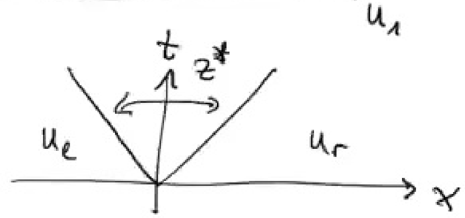
(Riemann ~1860)

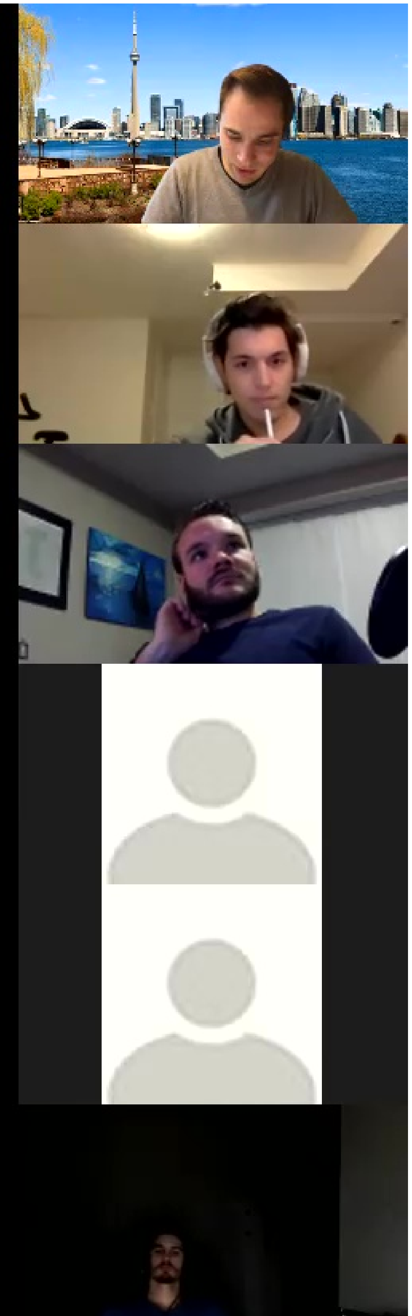
$$u_t + f(u)_x = 0$$

$$u(x,0) = u_0(x) = \begin{cases} u_l, & x<0 \\ u_r, & x>0 \end{cases}$$

with u_l, u_r constants.







4.4.2 Riemann Invariants & characteristic fields

<u>Def (k-characteristic)</u>: let $u = u(x,t) \in \mathbb{R}^m$ be a smooth solution of the hyperbolic system

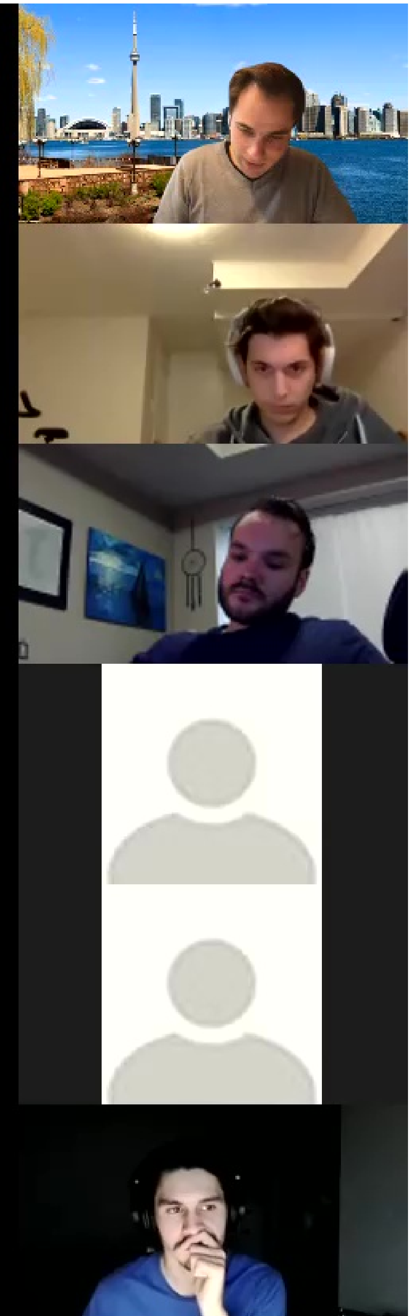
$$u_t + f(u)_x = 0, \quad x \in \mathbb{R}, \quad t \in [0,T]$$

and let $\lambda_1(u) \le \lambda_2(u) \le \dots \le \lambda_m(u)$ be the eigenvalues of $Df(u)$. A curve $\Gamma_k = (\gamma_k(t), t) \in C^1([0,\tau])$ with

$$\gamma_k'(t) = \lambda_k(u(\gamma(t),t)), \quad 0 \le t \le \tau$$

is called a <u>k-characteristic</u>.

In general: $u(x,t)$ is not constant along



9:41 AM Tue Jan 9

Lecture_12

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a k-characteristic, but:

Def Riemann invariants:

Let

- $u = u(x,t)$ be a smooth solution
- $\Gamma_k^t = (\gamma_k(t), t)$ k-characteristics $k = 1, \dots, m$
- let $w \in C^1(\mathbb{R}^m, \mathbb{R})$ with

$$Df(u)^T \cdot \nabla w = \lambda_k \nabla w$$

then $w(u)$ is constant along Γ_k^t

$$\frac{d}{dt} w(u(\gamma_k(t), t)) = 0$$

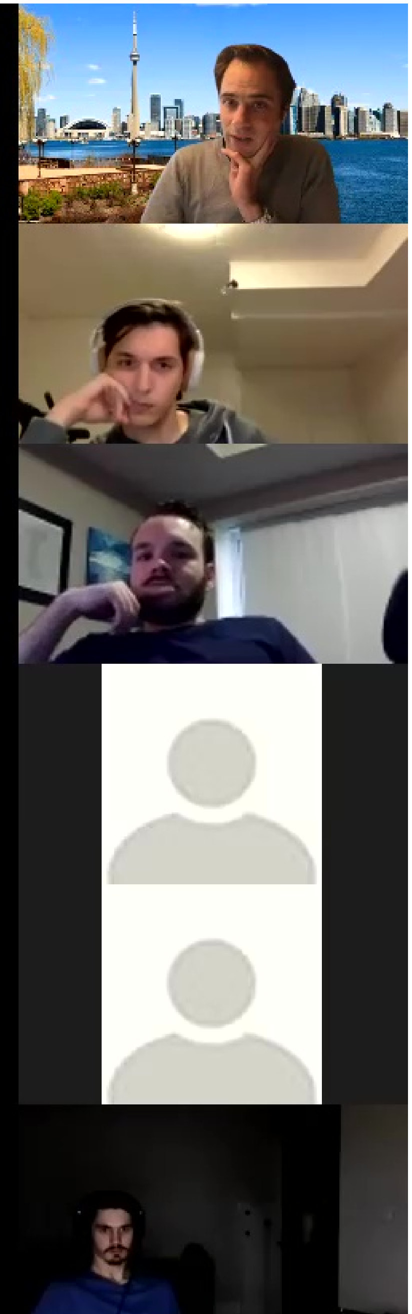
w is called a Riemann invariant.

Proof: $$\frac{d}{dt} w(u(\gamma_k(t), t)) = \sum_{l=1}^{m} [\partial_x u_l \, \gamma'(t) + \partial_t u_l] \, \partial_l w$$

$= \dots$

$\square$

Def



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$\square$

Def (k-Riemann invariants)

let • $u_t + f(u)_x = 0$

• $r_k(u)$ denotes eigenvectors of $Df(u)$ corresponding to the eigenvalues $\lambda_k(u)$

• consider an integral curve ($v \in C^1(\mathbb{R};\mathbb{R}^m)$, $\xi \mapsto v(\xi)$) of r_k in state space:

$$v'(\xi) = \mu(\xi)\, r_k(v(\xi)).$$

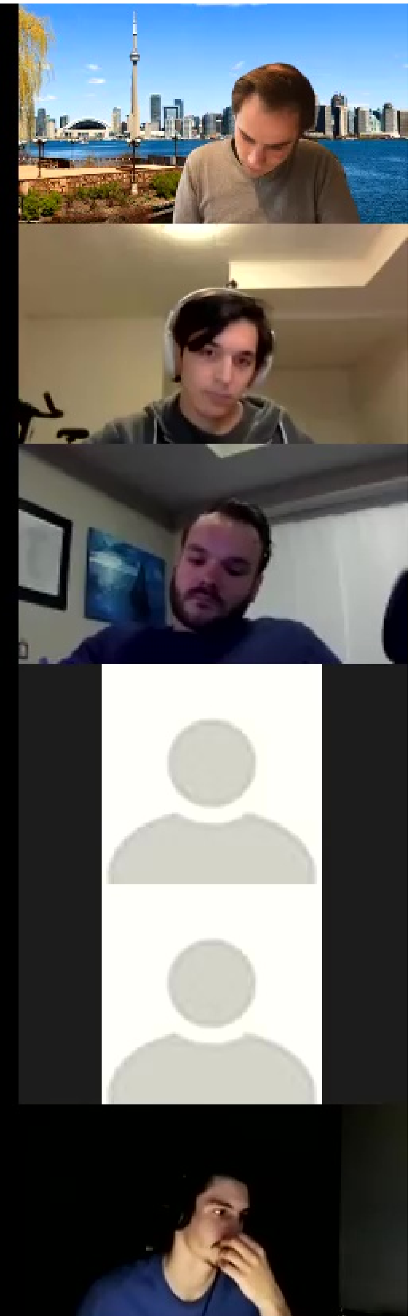
A function $w \in C^1(\mathbb{R}^m;\mathbb{R})$ with

$$\nabla w(v(\xi)) \cdot r_k(v(\xi)) = 0$$

for an integral curve v of r_k is constant along the integral curve,

$$\frac{d}{dt} w(v(\xi)) = 0$$

Such a w is called a k-Riema



let
- $u_t + f(u)_x = 0$
- $r_k(u)$ denotes eigenvectors of $Df(u)$ corresponding to the eigenvalues $\lambda_k(u)$
- consider an integral curve $\left(v \in C^1(\mathbb{R}; \mathbb{R}^m) \quad \xi \mapsto v(\xi) \right)$ of r_k in state space:

$$v'(\xi) = \mu(\xi)\, r_k(v(\xi)).$$

A function $w \in C^1(\mathbb{R}^m; \mathbb{R})$ with

$$\nabla w(v(\xi)) \cdot r_k(v(\xi)) = 0 \quad \leftarrow$$

for an integral curve v of r_k is constant along the integral curve,

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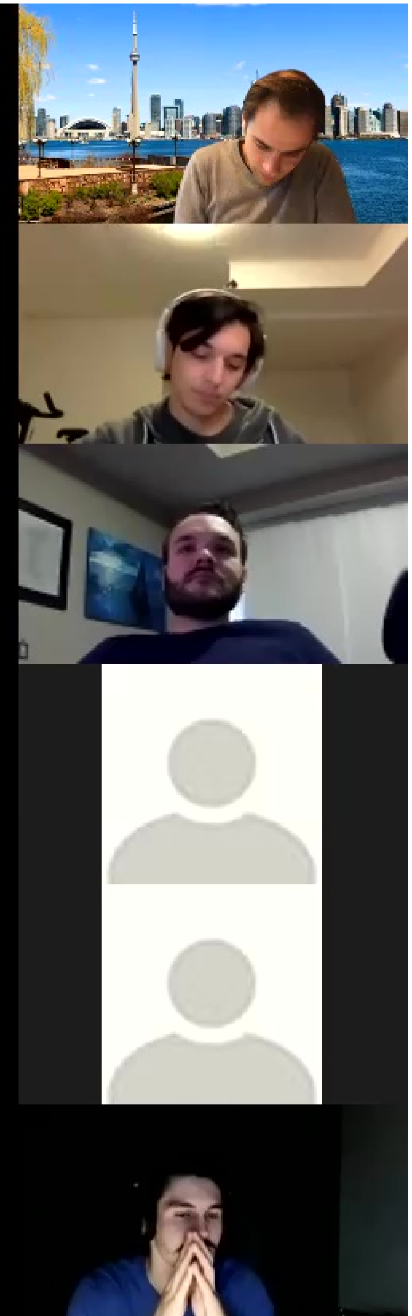
Such a w is called a k-Riemann invariant.

Note: $\frac{d}{\ }$



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Lecture_12

Lecture_12 | Chap_4_PropertiesConservationLaws | Lecture_11

corresponding to the eigenvalues $\lambda_k(u)$

- consider an <u>integral curve</u> ($v \in C^1(\mathbb{R};\mathbb{R}^m)$, $\xi \mapsto v(\xi)$) of r_k in <u>state space</u>:

$$v'(\xi) = \mu(\xi)\, r_k(v(\xi)).$$

A function $w \in C^1(\mathbb{R}^m;\mathbb{R})$ with

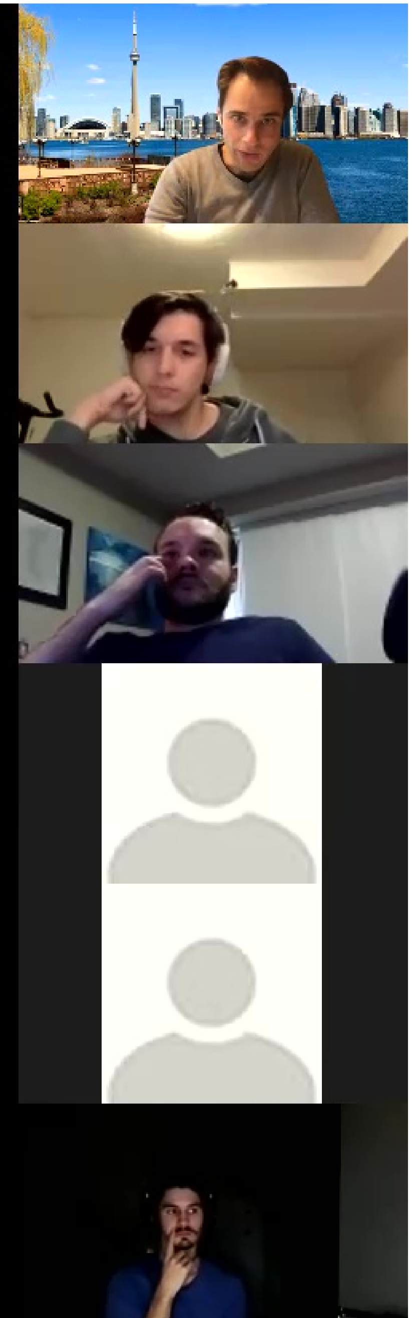
$$\nabla w(v(\xi)) \cdot r_k(v(\xi)) = 0$$

for an integral curve v of r_k is constant along the integral curve,

$$\frac{d}{dt} w(v(\xi)) = 0$$

Such a w is called a k-Riemann invariant.

Note: $$\frac{d}{dt} w(v(\xi)) = \nabla w(v(\xi)) \cdot v'(\xi)$$
$$= \nabla w(v(\xi)) \cdot \mu(\xi)\, r_k(v(\xi))$$
$$= 0 \qquad \square$$



$$= \nabla w(v(s)) \cdot \mu(s)\, r_k(v(s))$$

$$= 0 \qquad \square$$

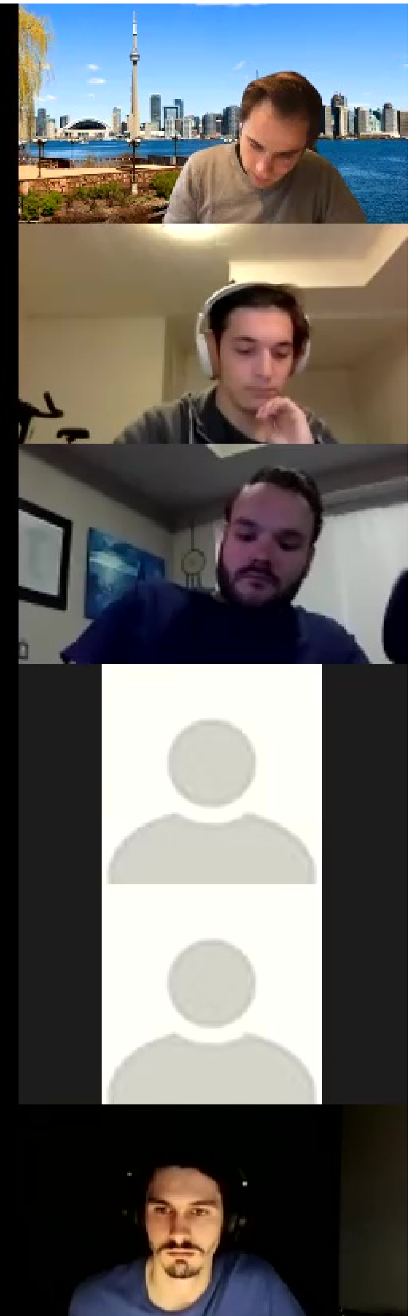
Remarks:

1) Existence: one can show (Kroener Theorem 4.1.12) There are $m-1$ k-Riemann invariants $w_1, \dots, w_{m-1} \in C^1(\mathbb{R}^m, \mathbb{R})$ such that their gradient $\nabla w_1, \dots, \nabla w_{m-1}$ are linearly independent

2) Riemann invariant $\Rightarrow$ k-Riemann invariant. Let w be RI wrt. $\lambda_j(u)$. Then w is a k-Riemann invariant for all $k \neq j$:

$$\lambda_j(v) \cdot r_k(v)^T \nabla w(v)$$

$$= r_k(v)^T \cdot Df(v)^T \cdot \nabla w(v)$$



9:41 AM Tue Jan 9

Lecture_12

$w_1, \dots, w_{m-1} \in C^1(\mathbb{R}^m, \mathbb{R})$ such that their gradient $\nabla w_1, \dots, \nabla w_{m-1}$ are linearly independent

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$$\underline{\lambda_j(v) \cdot r_k(v)^T \nabla w(v)}$$

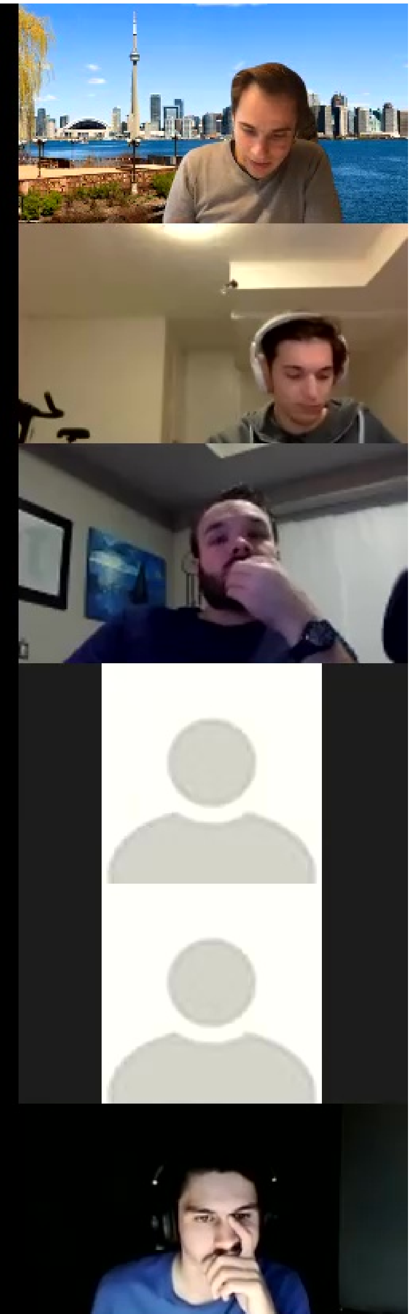
$$= r_k(v)^T \cdot Df(v)^T \cdot \nabla w(v)$$

$$= [Df \cdot r_k]^T \cdot \nabla w$$

$$= \underline{\lambda_k r_k^T \cdot \nabla w}$$

$$\Rightarrow (\lambda_j - \lambda_k) \nabla w \cdot r_k = 0$$

$$j \neq k: \quad \nabla w \cdot r_k = 0$$



Now: more on characteristics

Note that the change of λ_k along an integral curve $v(\xi)$ of r_k can be computed as:

$$\frac{d}{d\xi}\lambda_k(v(\xi)) = \nabla\lambda_k(v(\xi))\cdot v'(\xi)$$

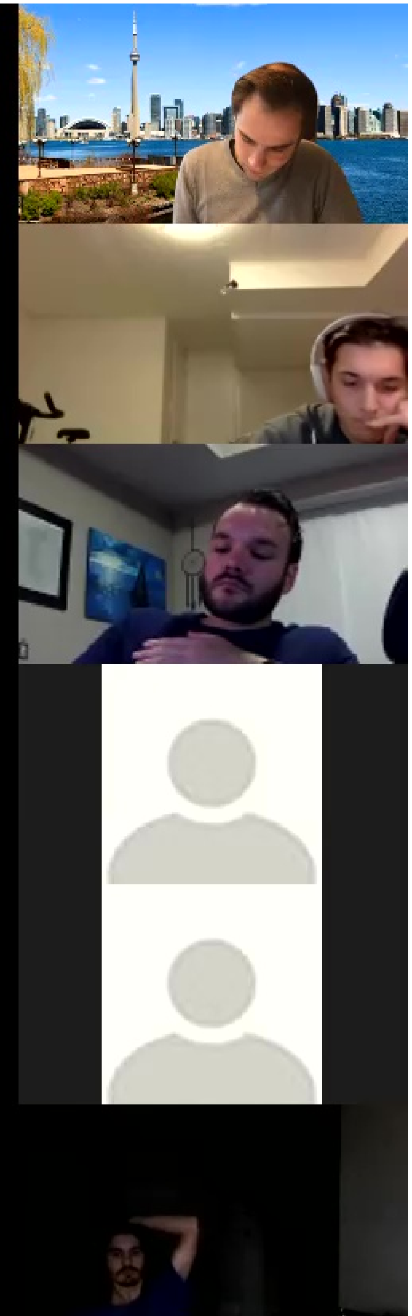
$$= \mu(\xi)\,\underbrace{\nabla\lambda_k(v(\xi))\cdot r_k(v(\xi))}$$

Def: let $D \subseteq \mathbb{R}^m$. A k-characteristic is called

- linearly degenerate in D if and only if $\nabla\lambda_k \cdot r_k = 0$ in D
- genuinely non-linear in D if and only if

$$\nabla\lambda_k \cdot r_k \neq 0 \quad \text{in } D$$

A system $u_t + f(u)_x = 0$ is called linearly degenerate or genuinely



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Def: Let $D \subseteq \mathbb{R}^m$. A k-characteristic is called

- <u>linearly degenerate</u> in D if and only if $\nabla \lambda_k \cdot r_k = 0$ in D

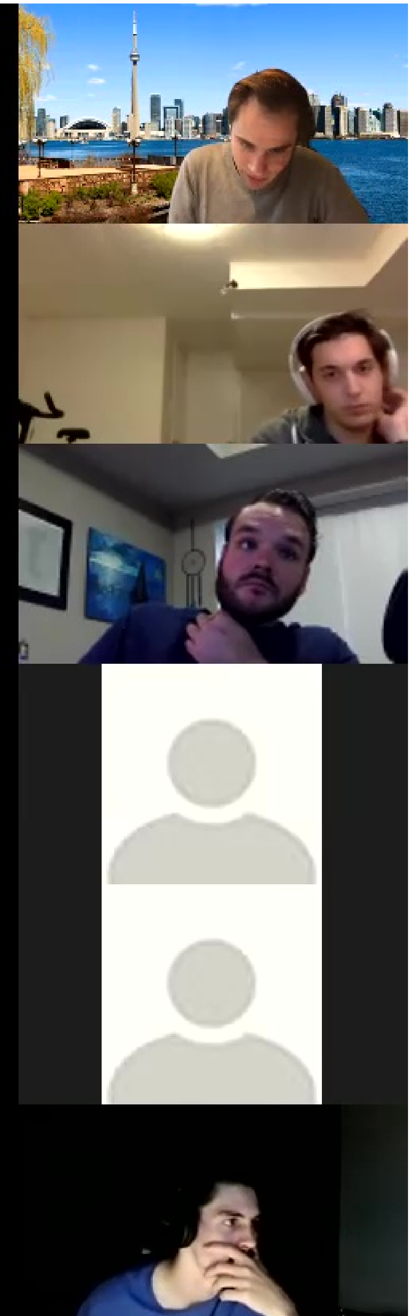
- <u>genuinely non-linear</u> in D if and only if

$$\nabla \lambda_k \cdot r_k \neq 0 \quad \text{in } D$$

A system $u_t + f(u)_x = 0$ is called linearly degenerate or genuinely non-linear in D if the property holds for all k-characteristics

Remark: For const coeff. linear hyperbolic system

u_t



Remark: For const coeff. linear hyperbolic system

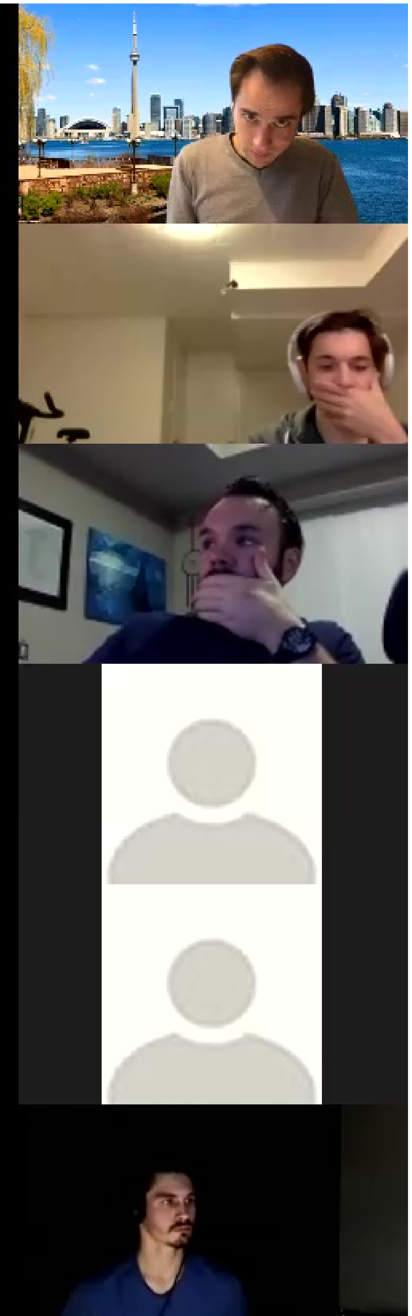
$$u_t + A u_x = 0$$

$$\nabla \lambda_k(v) \equiv 0 \quad \text{everywhere}$$

$$\Rightarrow \text{linearly degenerate}$$

For non-linear linearly degenerate systems λ_k constant along a given integral curve, but takes different values along different integral curves.

4.4.3 Special sol

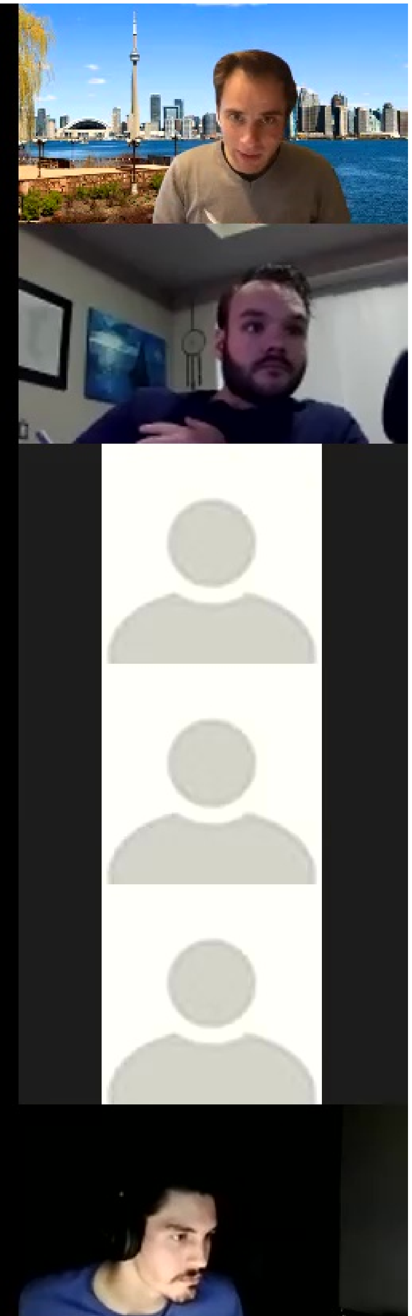


4.4.3 Special solutions: rarefaction waves

Def (k-rarefaction waves): let $D \subseteq \mathbb{R} \times (0,\infty)$ and $u \in C^1(D;\mathbb{R}^m)$ be a solution of the strict hyperbolic system $u_t + f(u)_x = 0$ with $\lambda_k(u)$ eigenvalues of $Df(u)$. If all k-Riemann invariants w_j are constant in D

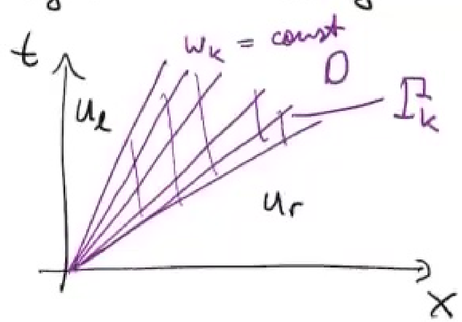
$$\frac{d}{dt} w_j(u(\gamma(t),t)) = 0 \qquad \forall\, j = 1,\dots,m-1$$

with $\gamma'(t) = \lambda_k$ for $0 \le t \le \tau$, $(\gamma(t),t) \in D$ the solution u is called a <u>k-rarefaction wave</u>.



with $\gamma'(t) = \lambda_k$ for $0 \le t \le \tau$, $(\gamma(t), t) \in D$ the solution u is called a <u>k-rarefaction wave</u>.

<u>Theorem:</u> Let the system $u_t + f(u)_x = 0$ be strictly hyperbolic and let u be a k-rarefaction wave. Then the k-characteristics are straight lines along which u is constant.



$u = u(\frac{x}{t})$