Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 13

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Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

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4.3 Entropy condition

In general, the IVP has no unique weak solution, i.e. the IVP for weak solutions is not well-posed.

→ we will define a "selection criterion" (entropy condition) to pick the "correct" physical solution

Example (non-uniqueness):

$$u_t + \partial_x\left(\frac{u^2}{2}\right) = 0 \,, \quad x \in \mathbb{R}, \; t > 0$$

$$u_0(x) = \begin{cases} 0 \,, & x \leq 0 \\ 1 \,, & x > 0 \end{cases}$$

Define:

$$u_1(x,t) \equiv \begin{cases} 0 \,, & x < \frac{t}{2} \\ 1 \,, & x > \frac{t}{2} \end{cases}$$

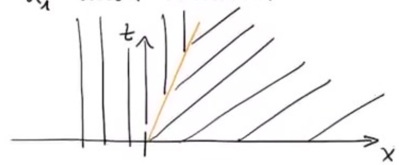
$$u_2(x,t) \equiv \begin{cases} 0 \,, & x < 0 \\ x/t \,, & 0 \leq x \leq t \\ 1 \,, & t < x \end{cases}$$

- u_1 satisfies (RH) conditions along $\Sigma = (\sigma(t), t)$ with $\sigma(t) = \frac{1}{2}t$:

$$\sigma'(t) = \frac{1}{2} \quad \text{and} \quad \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{\frac{1}{2}u_l^2 - \frac{1}{2}u_r^2}{u_l - u_r}$$

$$= \frac{1}{2}(u_l + u_r) = \frac{1}{2} \quad \checkmark$$

$\Rightarrow$ u_1 weak solution



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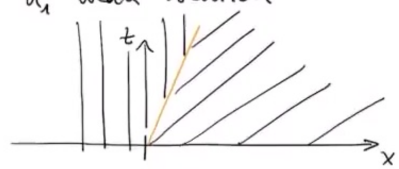
$\sigma'(t) = \frac{1}{2}$ and $\frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{\frac{1}{2}u_l^2 - \frac{1}{2}u_r^2}{u_l - u_r}$

$= \frac{1}{2}(u_l + u_r) = \frac{1}{2}$ ✓

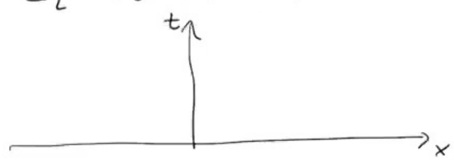
$\Rightarrow$ u_1 weak solution

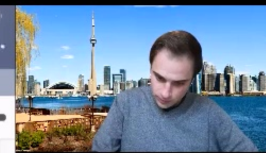


- u_2 can check it satisfies RH conditions along

$\Sigma_1 = (\sigma_1(t), t) = (t, t)$ and

$\Sigma_2 = (\sigma_2(t), t) = (0, t)$





$$\sigma'(t) = \frac{1}{2} \quad \text{and} \quad \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{\frac{1}{2}u_l^2 - \frac{1}{2}u_r^2}{u_l - u_r}$$

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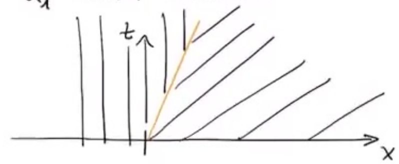

- u_2 can check it satisfies RH conditions along

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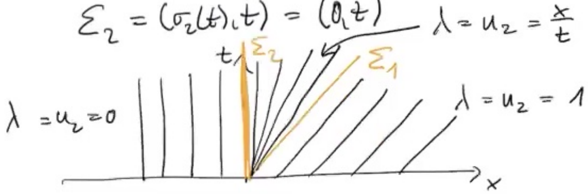

$\Rightarrow$ u_2 also weak solution

$\Rightarrow$ u_2 also weak solution

In reality, discontinuities are never arbitrarily sharp, but rather "smeared out"

$\rightsquigarrow$ physically "correct" solutions should arise as the limit of

$$u_t^\varepsilon + f(u^\varepsilon)_x \underbrace{- \varepsilon u_{xx}^\varepsilon}_{\text{"small viscosity effect"}} = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$

as $\varepsilon \to 0$, i.e. as the problem approaches the inviscid limit

$\rightsquigarrow$ "viscosity method"

Theorem:

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Theorem: let $u^0 \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $f_i, B \in C^2(\mathbb{R}^n \times (0,\infty) \times \mathbb{R})$ with bounded derivatives. Then for any $\varepsilon > 0$ there exists a uniquely defined classical solution u^ε of

$$u_t + \sum_{i=1}^n \partial_i f_i(x,t,u) + B(x,t,u) = \varepsilon \Delta u \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

$$u(x,0) = u^0(x) \quad \text{in } \mathbb{R}^n$$

such that $u^\varepsilon \xrightarrow{\varepsilon \to 0} u$ almost everywhere in $\mathbb{R}^n \times (0,\infty)$ for some u that is a weak solution ("viscosity limit") of

$$\partial_t u + \sum_{i=1}^n \partial_i f_i(x,t,u) + B(x,t,u) = 0$$

$$u(x,0) = u^0(x)$$

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<u>Now:</u> want necessary criterion for a weak soln u to be the viscosity limit

<u>Def:</u> Two smooth functions $\Phi, \Psi \in C^2(\mathbb{R}^m; \mathbb{R})$ are called an <u>entropy/entropy-flux pair</u> for the system of CLs $u_t + f(u)_x = 0$ if

(i) Φ is convex $\left(D^2\Phi(z)y \cdot y > 0 \quad \forall z, y \in \mathbb{R}^m\right)$

(ii) $D\Phi(z) \cdot Df(z) = D\Psi(z), \quad z \in \mathbb{R}^m$

<u>Def:</u> A weak solution u of the IVP is called an <u>entropy solution</u> if for any entropy pair Φ, Ψ it satisfies the inequality

$$\Phi(u)_t + \Psi(u)_x \leq 0 \quad \text{in } \mathbb{R} \times (0, \infty)$$

in the distributional sense, i.e.

$$\int_0^\infty \int_{\mathbb{R}} \Phi(u) v_t + \Psi(u) v_x \, dx\, dt \geq 0 \quad \forall v \in C_0^\infty$$

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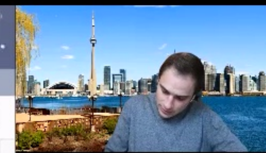

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Theorem: The viscosity limit $u = \lim_{\varepsilon \to 0} u^\varepsilon$

of
$$u_t^\varepsilon + f(u^\varepsilon)_x - \varepsilon u_{xx}^\varepsilon = 0 \quad \text{in } \mathbb{R}\times(0,\infty)$$
$$u^\varepsilon = u^0 \quad \text{on } \mathbb{R}\times\{t=0\}$$

is an entropy solution of
$$u_t + f(u)_x = 0 \quad \text{on } \mathbb{R}\times(0,\infty)$$
$$u = u^0 \quad \text{on } \mathbb{R}\times\{t=0\}$$

provided that u^ε is uniformly bounded in L^∞ and $u^\varepsilon \to 0$ as $|x| \to \infty$ sufficiently rapidly.

Proof: 1) Choose entropy pair Φ, Ψ

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Proof: 1) Choose entropy pair Φ, Ψ

$$D\Phi(u^\varepsilon)\cdot\Big(u^\varepsilon_t + \underbrace{f(u^\varepsilon)_x}_{=Df(u^\varepsilon)\,u^\varepsilon_x} - \varepsilon\, u^\varepsilon_{xx}\Big) = 0$$

$$\Leftrightarrow \quad \Phi(u^\varepsilon)_t + \Psi(u^\varepsilon)_x = \varepsilon\, D\Phi(u^\varepsilon)\, u^\varepsilon_{xx}$$

$$= \varepsilon\, \Phi(u^\varepsilon)_{xx} - \varepsilon \underbrace{\big(D^2\Phi(u^\varepsilon)\, u^\varepsilon_x\big)\cdot u^\varepsilon_x}_{\geq 0}$$

(Φ convex)

$\downarrow$ multiply by $v \in C_0^\infty(\mathbb{R}\times(0,\infty))$, $v \geq 0$

$$\int_0^\infty\!\!\int_{\mathbb{R}} \Phi(u^\varepsilon)_t\, v + \Psi(u^\varepsilon)_x\, v \, dx\, dt$$

$$\overset{\text{integrate by parts}}{=} \int_{\mathbb{R}} \big[\Phi(u^\varepsilon)v\big]_0^\infty dx + \int_0^\infty \big[\Psi(u^\varepsilon)v\big]$$

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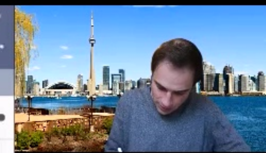

$$\Leftrightarrow \quad \Phi(u^\varepsilon)_t + \Psi(u^\varepsilon)_x = \varepsilon \, D\Phi(u^\varepsilon) u^\varepsilon_{xx}$$

$$= \varepsilon\, \Phi(u^\varepsilon)_{xx} - \varepsilon \underbrace{\left(D^2\Phi(u^\varepsilon) u^\varepsilon_x\right)\cdot u^\varepsilon_x}_{\geq 0 \ (\Phi \text{ convex})}$$

$\Big\downarrow$ multiply by $v \in C_0^\infty(\mathbb{R}\times(0,\infty))$, $v \geq 0$

$$\int_0^\infty \int_{\mathbb{R}} \Phi(u^\varepsilon)_t v + \Psi(u^\varepsilon)_x v \, dx\, dt$$

integrate by parts

$$\cancel{\int_{\mathbb{R}} \underbrace{\left[\Phi(u^\varepsilon) v\right]_0^\infty}_{=0} dx} + \cancel{\int_0^\infty \underbrace{\left[\Psi(u^\varepsilon) v\right]_{-\infty}^{\infty}}_{=0} dt}$$

$v \in C_0^\infty$

$$- \int_0^\infty \int_{\mathbb{R}} \left[\Phi(u^\varepsilon) v_t + \Psi(u^\varepsilon) v_x\right] dx\, dt$$

$$\Rightarrow \int_0^\infty \int_{\mathbb{R}} \left[\Phi(u^\varepsilon) v_t + \Psi(u^\varepsilon) v_x\right] dx\, dt$$

$$\geq - \int_0^\infty \int_{\mathbb{R}} \varepsilon\, \Phi(u$$

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multiply by $v \in C_0^\infty(\mathbb{R}\times(0,\infty))$, $v \geq 0$ $\qquad \geq 0$ (Φ convex)

$$\int_0^\infty\!\!\int_{\mathbb{R}} \Phi(u^\varepsilon)_t\, v + \Psi(u^\varepsilon)_x\, v \; dx\,dt$$

integrate by parts

$$= \int_{\mathbb{R}} \cancel{\underbrace{\left[\Phi(u^\varepsilon)v\right]_0^\infty}_{=0}}\,dx + \int_0^\infty \cancel{\underbrace{\left[\Psi(u^\varepsilon)v\right]_{-\infty}^\infty}_{=0}}\,dt$$

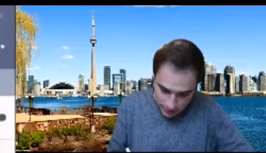
$v \in C_0^\infty$

$$- \int_0^\infty\!\!\int_{\mathbb{R}} \left[\Phi(u^\varepsilon)v_t + \Psi(u^\varepsilon)v_x\right] dx\,dt$$

$$\Longrightarrow \int_0^\infty\!\!\int_{\mathbb{R}} \left[\Phi(u^\varepsilon)v_t + \Psi(u^\varepsilon)v_x\right] dx\,dt$$

$$\underset{\Phi \text{ convex}}{\geq} - \int_0^\infty\!\!\int_{\mathbb{R}} \varepsilon\, \Phi(u^\varepsilon)_{xx}\, v \; dx\,dt$$

$$=$$



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$$\Longrightarrow \int_0^\infty \int_{\mathbb{R}} \left[\Phi(u^\varepsilon) v_t + \Psi(u^\varepsilon) v_x \right] dx\, dt$$

$$\underset{\substack{\uparrow \\ \Phi \text{ convex}}}{\geq} - \int_0^\infty \int_{\mathbb{R}} \varepsilon\, \Phi(u^\varepsilon)_{xx}\, v \; dx\, dt$$

$$= - \int_0^\infty \int_{\mathbb{R}} \varepsilon\, \Phi(u^\varepsilon)\, v_{xx} \; dx\, dt$$

$$\Big\downarrow \begin{array}{l} \varepsilon \to 0 \\ u^\varepsilon \to u \end{array}$$

$$\int_0^\infty \int_{\mathbb{R}} \left[\Phi(u) v_t + \Psi(u) v_x \right] dx\, dt \geq 0$$

$\Longrightarrow$ $u = \lim_{\varepsilon \to 0} u^\varepsilon$ satisfies the entropy condition

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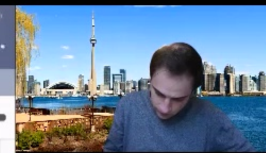

2) Choose $v \in C_0^\infty(\mathbb{R}\times[0,\infty);\mathbb{R}^m)$:

$$\int_0^\infty\int_{\mathbb{R}} \left[u^\varepsilon_t + f(u^\varepsilon)_x - \varepsilon u^\varepsilon_{xx}\right]\cdot v\,dx\,dt = 0$$

$\Big\downarrow$ integrate by parts

$$-\int_0^\infty\int_{\mathbb{R}} u^\varepsilon\cdot v_t\,dx\,dt + \underbrace{\int_{\mathbb{R}}\left[u^\varepsilon\cdot v\right]_0^\infty dx}_{= -\int_{\mathbb{R}} u^\varepsilon(x,0)\cdot v\,dx}$$

$$-\int_0^\infty\int_{\mathbb{R}} f(u^\varepsilon)\, v_x\,dx\,dt - \int_0^\infty\int_{\mathbb{R}} \varepsilon\, u^\varepsilon v_{xx}\,dx\,dt = 0$$

$$\iff \int_0^\infty\int_{\mathbb{R}} \left[u^\varepsilon\cdot v_t + f(u^\varepsilon)\cdot v_x + \varepsilon u^\varepsilon\cdot v_{xx}\right]dx\,dt + \int_{\mathbb{R}} u_0^\varepsilon\cdot v\,dx = 0$$

$\Big\downarrow$ $\varepsilon\to 0$, $u^\varepsilon \to u$

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Theorem: There exists – up to a measure zero – at most one entropy solution for the scalar conservation law

$$u_t + f(u)_x = 0 \quad \mathbb{R}\times(0,\infty)$$
$$u = u^0(x) \quad \mathbb{R}\times\{t=0\}$$

Proof: Evans Sec. 11.4.3

Remark: Interpretation of entropy condition

$$\Phi(u)_t + \Psi(u)_x \le 0$$

In physical applications ($\rightarrow$ Euler eqns)
$\Phi(u)$ will be the thermodynamic entropy and $\Psi(u)$ the entropy flux.

4.4 The Riemann problem & Lax entropy condition

$$u_t + f(u)_x = 0$$

$$u(x,0) = u_0(x) = \begin{cases} u_l, & x<0 \\ u_r, & x>0 \end{cases}$$

with u_l, u_r constants.

4.4.1 First example(s)

Consider simple example:

$$u_t + f(u)_x = 0, \quad f(u) = \frac{u^2}{2}$$

$$u(x,0) = u^0(x) = \begin{cases} u_l, & x<0 \\ u_r, & x>0 \end{cases}$$

$\rightsquigarrow$ const. characteristic speeds:

$$\lambda'(u) = f''(u) = 1 > 0$$

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$$u_t + f(u)_x = 0 \, , \quad f(u) = \frac{u^2}{2}$$

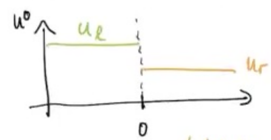
$$u(x,0) = u^0(x) = \begin{cases} u_l \, , & x<0 \\ u_r \, , & x>0 \end{cases}$$

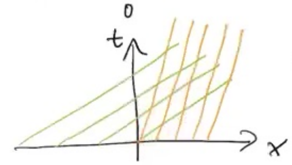
⇝ const. characteristic speeds:

$$\lambda'(u) = f''(u) = 1 > 0$$

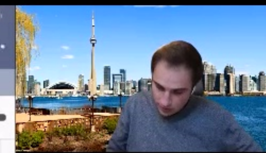
① assume $u_l > u_r$: "compressive data"

$$\lambda_l = \lambda(u_l) = u_l > u_r = \lambda(u_r) = \lambda_r$$





characteristic



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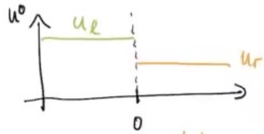
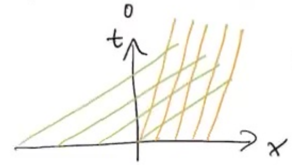
$$\lambda_l = \lambda(u_l) = u_l > u_r = \lambda(u_r) = \lambda_r$$





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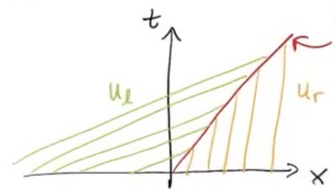

"shock" Σ
with speed
$\sigma'(t) = s$
immediate crossing of characteristics

RH jump conditions:

$$s = \sigma'(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{1}{2}\,\frac{u_l^2 - u_r^2}{u_l - u_r}$$

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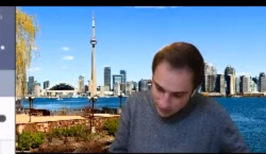
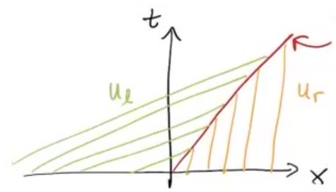




"shock" Σ with speed $\sigma'(t) = s$

immediate crossing of characteristics

RH jump conditions:

$$s = \sigma'(t) = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} = \frac{1}{2}\,\frac{u_\ell^2 - u_r^2}{u_\ell - u_r}$$

$$= \frac{1}{2}(u_\ell + u_r)$$

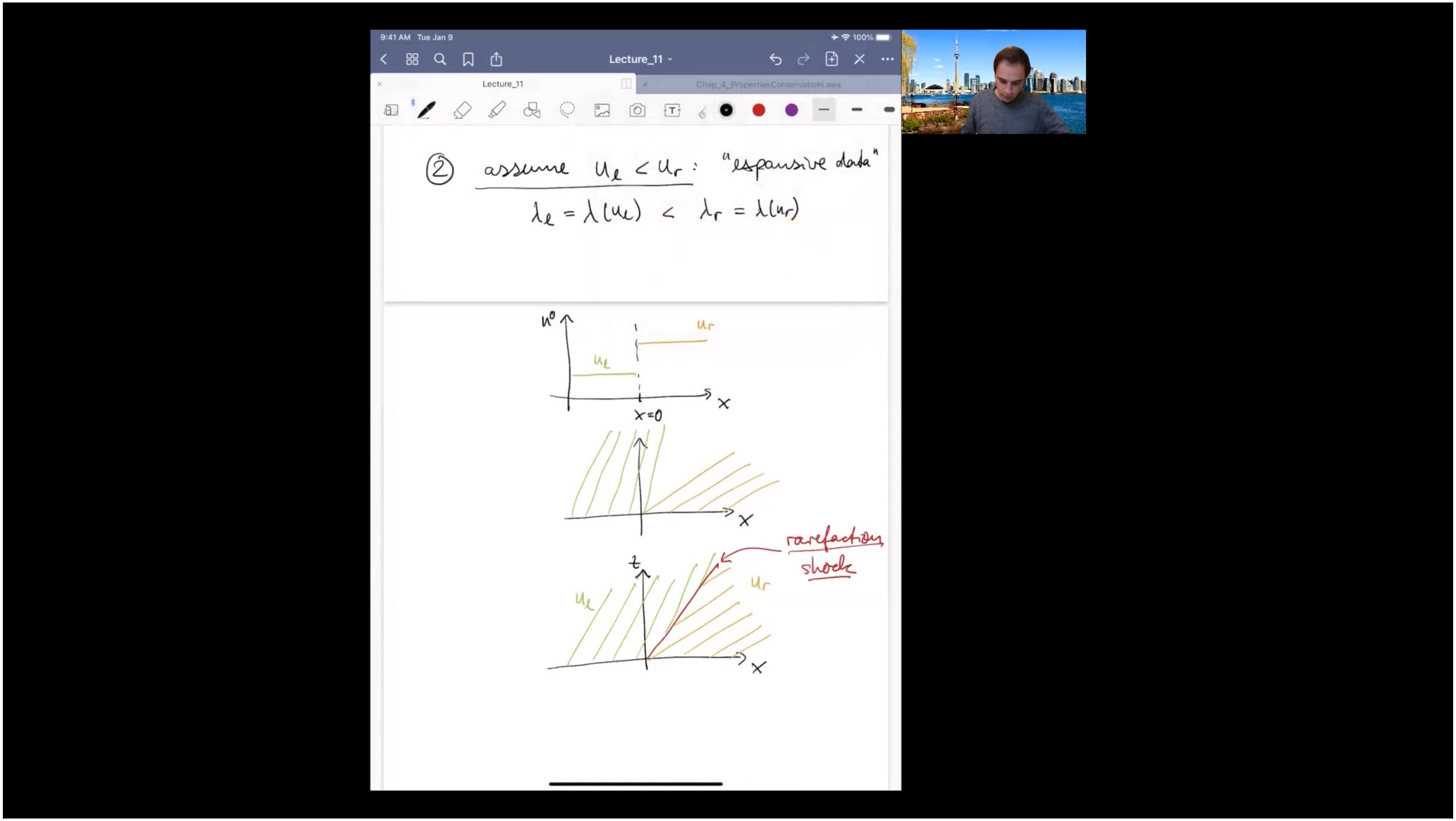
weak solution:

$$u(x,t) = \begin{cases} u_\ell, & x - st < 0 \\ u_r, & x - st > 0 \end{cases}$$

$$\boxed{\lambda(u_\ell) > s > \lambda(u_r)}$$

Lax entropy condition

② assume $u_\ell < u_r$: "expansiv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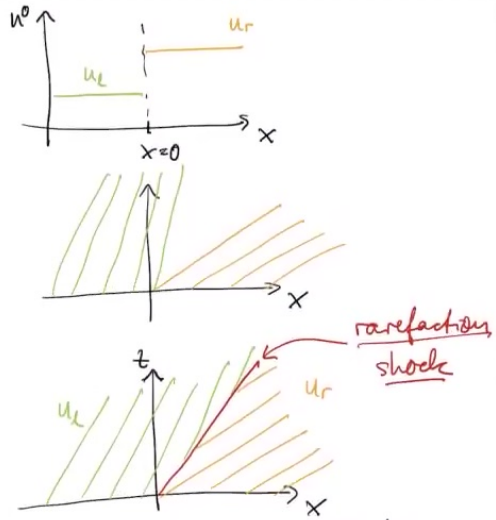
② assume $u_\ell < u_r$: "expansive data"
$\lambda_\ell = \lambda(u_\ell) < \lambda_r = \lambda(u_r)$
u^0
u_r
u_ℓ
x
$x=0$
x
rarefaction
shock
t
u_ℓ
u_r
x



Similar to above, one weak solution is

$$s = \frac{1}{2}(u_l + u_r), \quad u(x,t) = \begin{cases} u_l, & x - st > 0 \\ u_r, & x - st < 0 \end{cases}$$

But: $\boxed{\lambda_l < s < \lambda_r}$ entropy-violating shock

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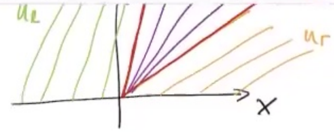

$$\begin{cases} u(x,t) = u_\ell\,, & \frac{x}{t} \le \lambda_\ell = u_\ell \\ u(x,t) = \frac{x}{t}\,, & \lambda_\ell < \frac{x}{t} < \lambda_r \\ u(x,t) = u_r\,, & \frac{x}{t} \ge \lambda_r = u_r \end{cases}$$

centred rarefaction wave

(3) Complete solution:

$$u_\ell > u_r : \quad u(x,t) = \begin{cases} u_\ell\,, & x < st \\ u_r\,, & x > st\,, \end{cases} \quad s = \tfrac{1}{2}(u_\ell + u_r)$$

$$u_\ell \le u_r : \quad u(x,t) = \begin{cases} u_\ell\,, & \frac{x}{t} \le u_\ell \\ \frac{x}{t}\,, & u_\ell < \frac{x}{t} < u_r \\ u_r\,, & \frac{x}{t} \ge u_r \end{cases}$$