Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 9

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Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

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Remarks: 1) note the theorem makes use of the fact that boundedness of the L^2 norm of u^n follows from the boundedness of L^2-norm of $\hat{u}^n$ via Parseval's identity. The latter is much easier to show since while all $\{u_j^n\}_{j \in N}$ are coupled via the difference eqns, each $\hat{u}(\xi)$ satisfies

$$\hat{u}(\xi)^{n+1} = g(\xi)\, \hat{u}(\xi)^n$$

and thus is decoupled from all other wavenumbers ξ.

2) One can make the replacement

$$|g(\xi)| \leq e^{\alpha \Delta t}$$

and the theorem still holds.

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and the theorem still holds.

Also: $|g(\xi)|^n \leq (1+\alpha\Delta t)^n \overset{n=\frac{T}{\Delta t}}{=} \left(1+\frac{\alpha T}{n}\right)^n$

$$\xrightarrow[\Delta t \to 0]{n\to\infty} e^{\alpha \Delta t} \qquad (\text{for } T>0)$$

3) The stability bound allows for exponential growth of modes,

e.g. $\quad u_t + a u_x = u$

but sometimes it is not appropriate

e.g. $\quad u_t + a u_x = \varepsilon u_{xx} \quad , \varepsilon > 0$

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(all modes damped)

Central in space, forward in time discretization, leads to:

$$|g| \leq 1 + \underbrace{\frac{1}{2}\frac{a^2}{\varepsilon}}_{\alpha} \Delta t$$

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$\leadsto$ "strict / strong stability"

$$|\hat{u}(\xi, t+\Delta t)| \leq e^{\alpha \Delta t} |\hat{u}(\xi, t)| \ \forall \xi$$

continuous problem

$\Rightarrow g(\xi) \leq e^{\alpha \Delta t} \ \forall \xi$ (discrete problem)

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4) If one wants $\|u^n\| \leq c\,\|u^0\|$ (cf. 3))
for stability, the requirement on
$g(\xi)$ in the theorem sharpens to

$$|g(\xi)| \leq 1 \quad \forall \xi$$

5) alternative estimation of
amplification factor

1) $\Rightarrow$ it suffices to consider
a arbitrary single wave number
ξ: $\quad u_j^n = e^{i\xi x_j}$

to evalute amplification factor.

6)

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6) 3+ - level schemes: either

(i) use method in 5)

(ii) <u>amplification matrix</u> analysis:

convert multi-level scheme

to effective 2-level scheme

$$u^{n+1} = G\, u^n$$

$$\begin{pmatrix} u^{n+1} \\ v^{n+1} \end{pmatrix} = G \begin{pmatrix} u^n \\ v^n \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} \hat{u}^{n+1}(\xi) \\ \hat{v}^{n+1}(\xi) \end{pmatrix} = G\left(e^{-i\xi}, e^{i\xi}\right) \begin{pmatrix} \hat{u}^n(\xi) \\ \hat{v}^n(\xi) \end{pmatrix}$$

$$= \begin{pmatrix} g_{11}(\xi) & g_{12}(\xi) \\ g_{21}(\xi) & g_{22}(\xi) \end{pmatrix}$$

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$$= \begin{pmatrix} g_{11}(\xi) & g_{12}(\xi) \\ g_{21}(\xi) & g_{22}(\xi) \end{pmatrix} \begin{pmatrix} \hat{u}^{n} \\ \hat{v}^{n} \end{pmatrix}$$

Assume that $g(\xi)$ has a complete set eigenvectors $g_\ell(\xi)$ and corresponding eigenvalues $\lambda_\ell(\xi)$ and suppose that there exists constants $\alpha, \tau > 0$ such that

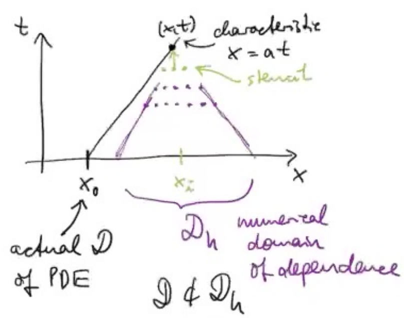
$$|\lambda_\ell(\xi)| \leq 1 + \alpha \Delta t \quad , \quad \ell = 1, 2$$

for all ξ, $0 \leq \Delta t \leq \tau$. Then the scheme will be stable.

CFL condition: A numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as Δt and Δx go to zero.

(Courant, Friedrichs & Lewy, Math. Ann. 100, 32 (1928))

Advection equation:



$\frac{\Delta t}{\Delta x}$ $\boxed{\Delta t \propto (\Delta x)^2}$

Remarks: 1) For linear hyperbolic systems

$$C = \frac{\Delta t}{\Delta x} \max_p |\lambda^p| \leq 1$$

For non-linear conservation laws

$$u_t + f(u)_x = 0$$

$$C = \sup \frac{\Delta t}{\Delta x} |f'(u)| < 1$$

2) A wider stencil $\Rightarrow$ less restrictive CFL condition on Δt

3)

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3) Parabolic equations: diffusion equation

severe constraints on explicit methods

CFL, since $\mathcal{D} = (-\infty, \infty)$

(infinite propagation speed)

so take $\Delta t = \mathcal{O}(\Delta x^2)$ then

explicit method captures $\mathcal{D}$

in limit $\Delta t, \Delta x \to 0$

condition on Δt

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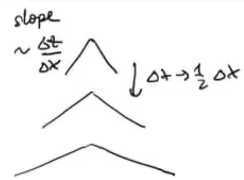

better: <u>use implicit method</u>

Crank – Nicholson method

3.5 Diffusion & dispersion

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Crank – Nicholson method

3.5 Diffusion & dispersion

→ Discretization introduces additional terms not present in continuum PDE

Consider advection equation and let

$v(x,t)$

→ Discretization introduces additional terms not present in continuum PDE

Consider advection equation and let $v(x,t)$ denote an exact solution of the discretized eqn

Example: Upwind method:

$$u_i^{n+1} = u_i^n - \frac{a\,\Delta t}{\Delta x}\big(u_i^n - u_{i-1}^n$$

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Consider advection equation and let $v(x,t)$ denote an exact solution of the discretized eqn

Example: Upwind method:

$$u_i^{n+1} = u_i^n - \frac{a\,\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right), \quad a > 0$$

$\downarrow$ v satisfies

$$v(x, t+\Delta t) = v(x,t) - \frac{a\,\Delta t}{\Delta x}\left[v(x,t) - v(x-\Delta t, t)\right]$$

$\downarrow$ Taylor series

$$\left(v_t + \frac{1}{2}\Delta t\, v_{tt} + \frac{1}{6}(\Delta t)^2 v_{ttt} + \dots\right) + a\left(v_x - \frac{1}{2}\Delta x\, v_{xx} + \frac{1}{6}(\Delta x)^2 v_{xxx} + \dots\right) = 0$$

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$$v(x,t+\Delta t) = v(x,t) - \frac{a\Delta t}{\Delta x}\left[v(x,t) - v(x-\Delta x,t)\right]$$

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$$\boxed{v_t + a v_x = \frac{1}{2}\left(a\,\Delta x\, v_{xx} - \Delta t\, v_{tt}\right) - \frac{1}{6}\left[a(\Delta x)^2 v_{xxx} + (\Delta t)^2 v_{ttt}\right] + \dots}$$

PDE for $v(x,t)$

different from original continuum PDE

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$= 0$

$$\boxed{v_t + a v_x = \frac{1}{2}\left(a\,\Delta x\, v_{xx} - \Delta t\, v_{tt}\right)} \quad \mathcal{O}(\Delta x, \Delta t)$$

$$- \frac{1}{6}\left[\cancel{a(\Delta x)^2 v_{xxx} + (\Delta t)^2 v_{ttt}}\right] + \dots$$

1st order: recover original PDE

2nd order: ∂_t, ∂_x

$\downarrow$

PDE for $v(x,t)$ different from original continuum PDE

$$\boxed{v_t + a v_x = \underbrace{\frac{1}{2} a \Delta x \Big(1 - \underbrace{\frac{a \Delta t}{\Delta x}}_{C}\Big) v_{xx}}_{\varepsilon}}$$

advection diffusion

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$$\boxed{v_t + a v_x = \frac{1}{2}\left(a \Delta x\, v_{xx} - \Delta t\, v_{tt}\right) - \frac{1}{6}\left[\cancel{a(\Delta x)^2 v_{xxx} + (\Delta t)^2 v_{ttt}}\right] + \dots}$$

$\mathcal{O}(\Delta x, \Delta t)$

PDE for $v(x,t)$

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1st order: recover original PDE

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$$\boxed{v_t + a v_x = \underbrace{\frac{1}{2} a \Delta x \Big(1 - \underbrace{\frac{a \Delta t}{\Delta x}}_{C}\Big)}_{\varepsilon} v_{xx}}$$

advection diffusion equation

"numerical diffusion coefficient"

Example 2: Lax-Wendroff method

$$\boxed{v_t + a v_x = \underbrace{\frac{1}{2} a \Delta x \left(1 - \underbrace{\frac{a \Delta t}{\Delta x}}_{C}\right)}_{\varepsilon} v_{xx}}$$

advection diffusion equation

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Example 2: Lax-Wendroff method

$$u_j^{n+1} = u_j^n - \frac{a \Delta t}{2 \Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{a^2 (\Delta t)^2}{2 (\Delta x)^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right)$$

$$\downarrow$$

$$\boxed{v_t + a v_x = -\frac{1}{6} a (\Delta x)^2 \underbrace{\left[1 - \left(\frac{a \Delta t}{\Delta x}\right)^2\right] v_{xxx}}_{\text{dispersive behavior}}}$$

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LeVeque_FiniteDifferenceMethods

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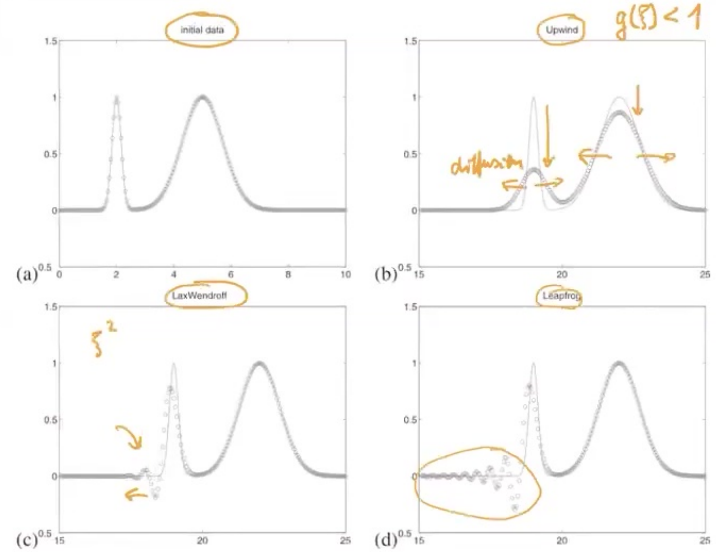

Figure 10.4. *The numerical experiments on the advection equation described in Section* 10.8.

The idea is to ask the following question: is there a PDE $v_t = \cdots$ such that our numerical approximation U_j^n is actually the *exact* solution to this PDE, $U_j^n = v(x_j, t_n)$? Or, less ambitiously, can we at least find a PDE that is better satisfied by U_j^n than the original PDE we were attempting to model? If so, then studying the behavior of solutions to this PDE should tell us much about how the numerical approximation is behaving. This can be advantageous because it is often easier to study the behavior of PDEs than of finite difference formulas.

In fact it is possible to find a PDE that is exactly satisfied by the U_j^n by doing Taylor

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$$-\frac{1}{6}\left[\cancel{a(\Delta x)^2 v_{xxx}} + \cancel{(\Delta t)^2 v_{ttt}}\right] + \dots$$

1st order: recover original PDE

2nd order: ∂_t, ∂_x

PDE for $v(x,t)$ different from original continuum PDE

$$\boxed{v_t + a v_x = \underbrace{\frac{1}{2} a \Delta x \Big(1 - \underbrace{\frac{a \Delta t}{\Delta x}}_{C}\Big)}_{\varepsilon} v_{xx}}$$

$C = 1$

advection diffusion equation

"numerical diffusion coefficient"

Example 2: Lax-Wendroff method

$$u_j^{n+1} = u_j^n - \frac{a \Delta t}{2 \Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{a^2 (\Delta t)^2}{2 (\Delta x)^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right)$$

$\downarrow$