Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 7

Speakers: Daniel Siegel

Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

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Lecture 7

Lecture 7 | Chap_3_FiniteDifferenceMethods

Chap. 3: Finite Difference Methods

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Lecture 7

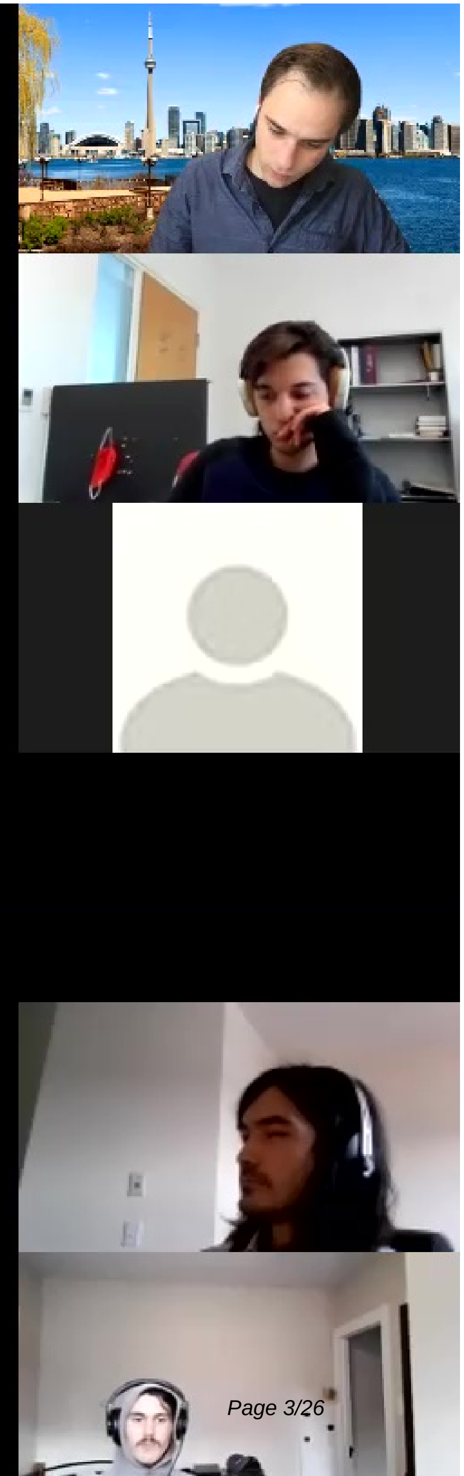
3.1 Basic notions of discretization

$$
\begin{tikzcd}
L: & X \arrow[rr] & & Y \\
 & u \arrow[rr, mapsto] \arrow[d, "D_h^X"] & & Lu = S \arrow[d] \\
L_h: & X_h \arrow[rr] & & Y_h
\end{tikzcd}
$$

L differential operator

continuum problem: $Lu = S$ continuum PDE

For example: $X = \{ u \in C^2(\Omega) \mid \|Lu\| < \infty \}$



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$$
\begin{tikzcd}
L: & X \arrow[rr] \arrow[d, "D_h^X"] & & Y \arrow[d] \\
L_h: & X_h \arrow[rr] & & Y_h
\end{tikzcd}
$$

$$u \longmapsto Lu = S$$

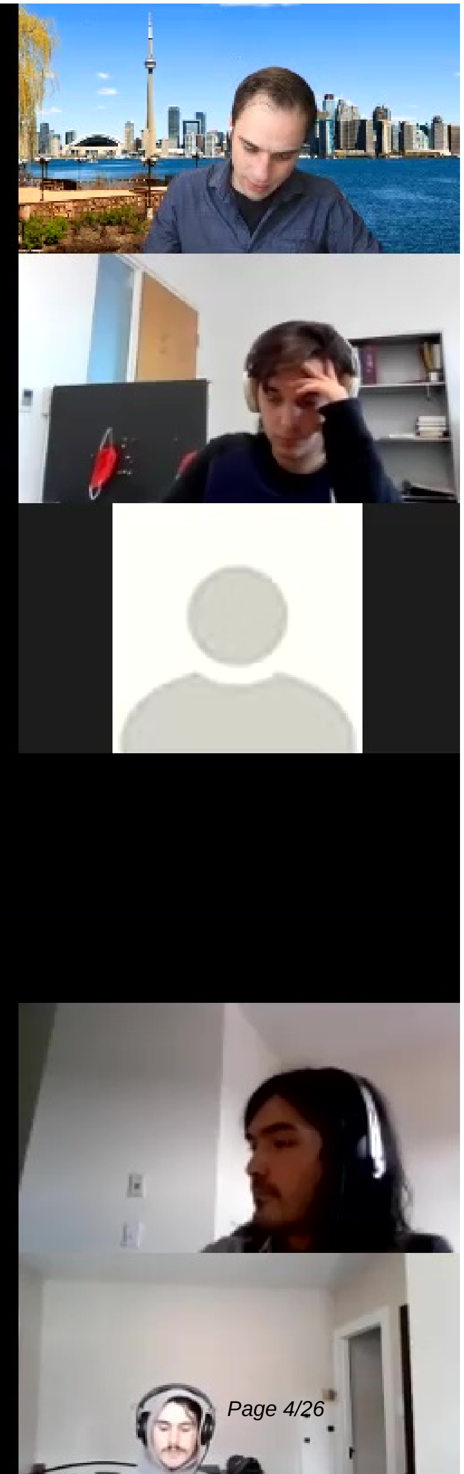
differential operator

continuum problem: $Lu = S$ continuum PDE

For example: $X = \{ u \in C^2(\Omega) \mid |Lu| < \infty \}$

$$L = \Delta, \Box, \partial_t - \partial_x f(\cdot), \ldots$$

discretized problem: $L_h u_h = S_h$ discretized problem



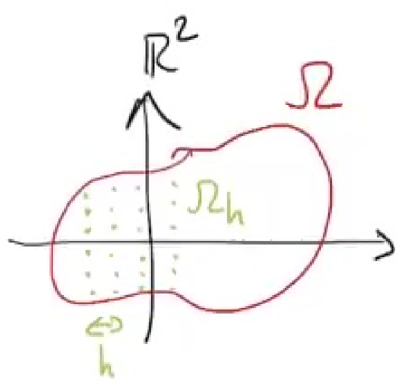
discretized problem: $L_h u_h = S_h$ discretized problem

typically: X_h = "{grid functions}"

$= \{ v : \Omega_h \to \mathbb{R}^n \}$

Ω_h = "grid" : $\Omega \cap h\mathbb{Z}^n$

h: discretization parameter

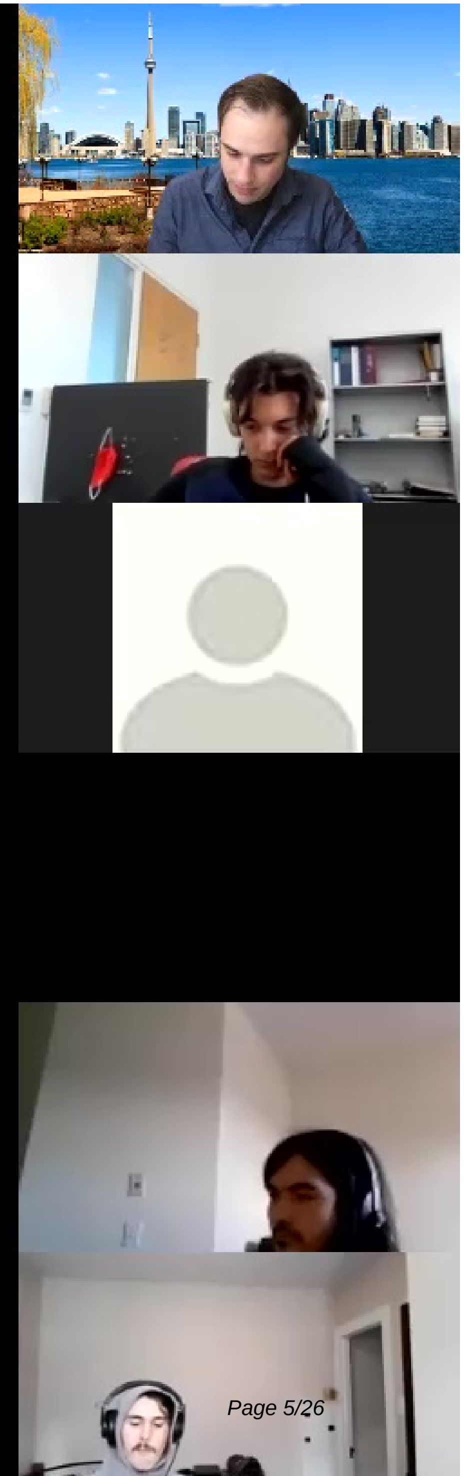


Notation: $x \to x_i \in \{x_1, \dots, x_{N_x}\}$

$t \mapsto t_n \in \{t_1, \dots, t_N\}$

D_h^{x}: discretization operator

$D_h^{x} : X \to X_h$



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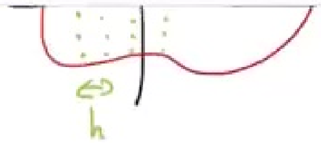


$$t \mapsto t_n \in \{t_1, \dots, t_N\}$$

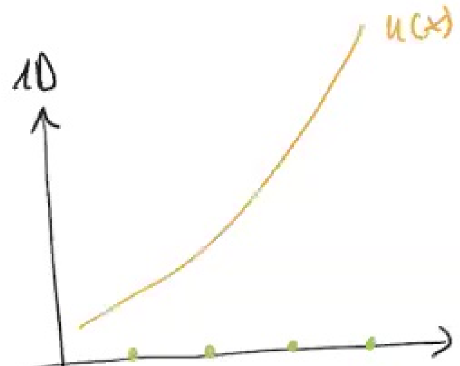
D_h^{X} : discretization operator

$$D_h^{X} : X \longrightarrow X_h$$

$$u \longmapsto D_h^{X} u \equiv u_h$$

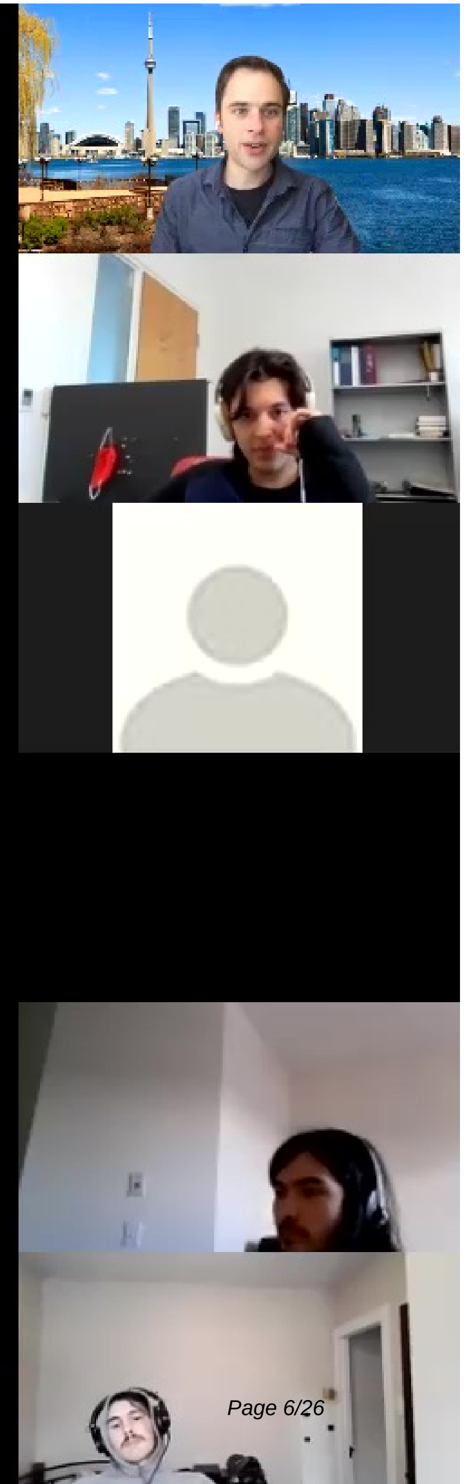
$$D_h^{X} u = u_h = \begin{cases} u|_{\Omega_h} \quad (u_h)_i^n = u(x_i, t^n) \\ \end{cases}$$

finite differencing methods



$$(u_h)_i^n = \underbrace{\frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x_i, t_n)\, dx}_{\text{spatial average across } \Delta x}$$

finite volume



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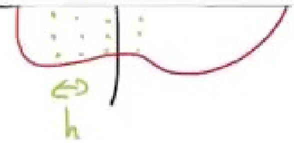
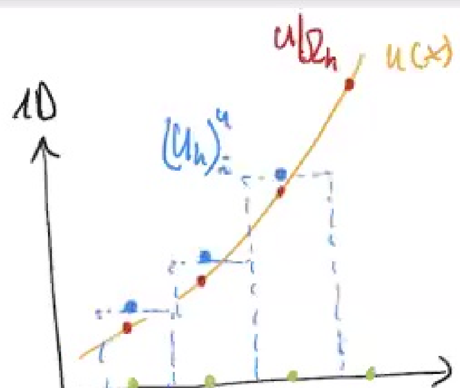


$$t \mapsto t_n \in \{t_1, \dots, t_N\}$$

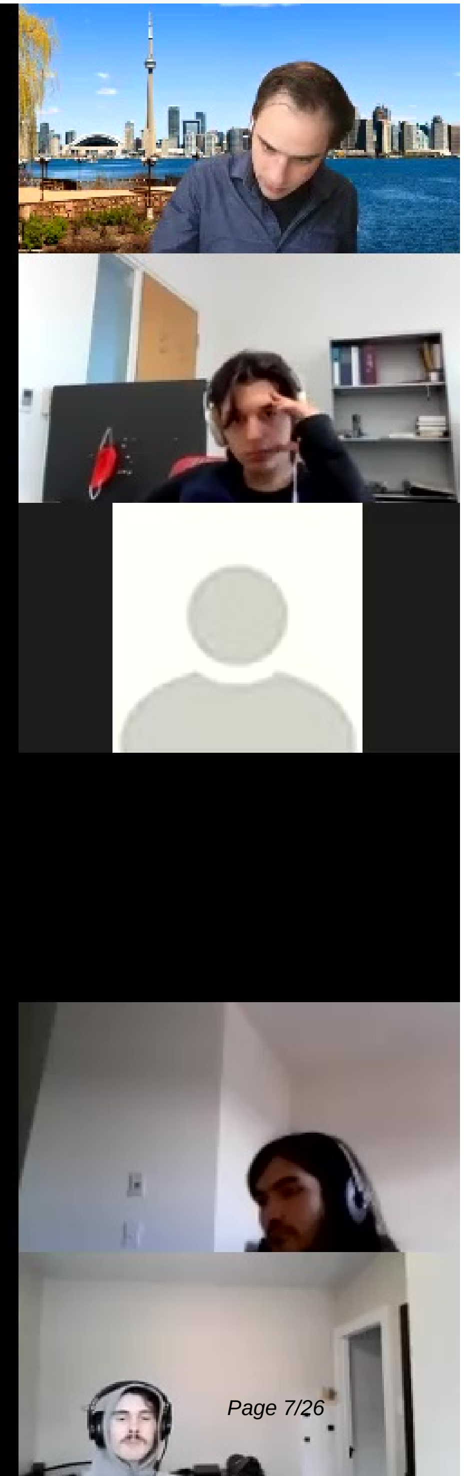
D_h^X: discretization operator

$$D_h^X : X \longrightarrow X_h$$
$$u \longmapsto D_h^X u \equiv u_h$$

$$D_h^X u = u_h = \begin{cases} u|_{\Omega_h} \quad (u_h)_i^n = u(x_i, t^n) \\ \qquad \text{finite differencing methods} \\ \end{cases}$$



$$(u_h)_i^n = \underbrace{\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, t_n)\,dx}_{\text{spatial average across } \Delta x}$$



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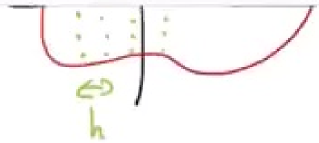


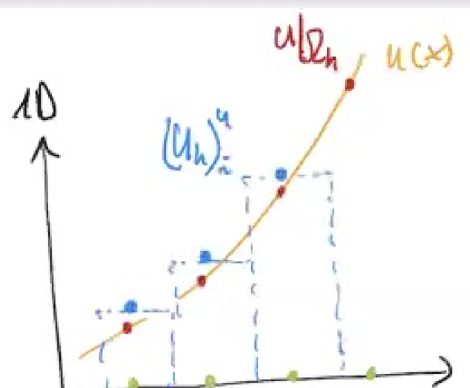
$$t \mapsto t_n \in \{t_1, \dots, t_N\}$$

$D_h^{\mathcal{X}}$: discretization operator

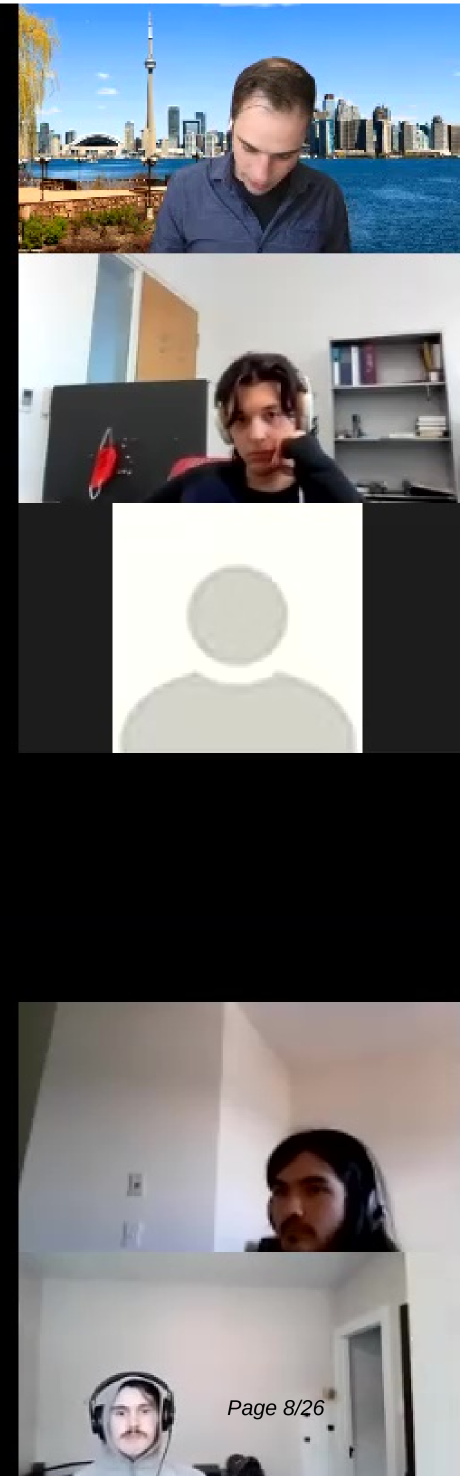
$$D_h^{\mathcal{X}}: \mathcal{X} \longrightarrow \mathcal{X}_h$$
$$u \longmapsto D_h^{\mathcal{X}} u \equiv u_h$$

$$D_h^{\mathcal{X}} u = u_h = \begin{cases} u|_{\Omega_h} \quad (u_h)_i^n = u(x_i, t^n) \\ \\ \end{cases}$$

finite differencing methods



$$(u_h)_i^n = \underbrace{\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x_i, t_n)\, dx}_{\text{spatial average across } \Delta x}$$

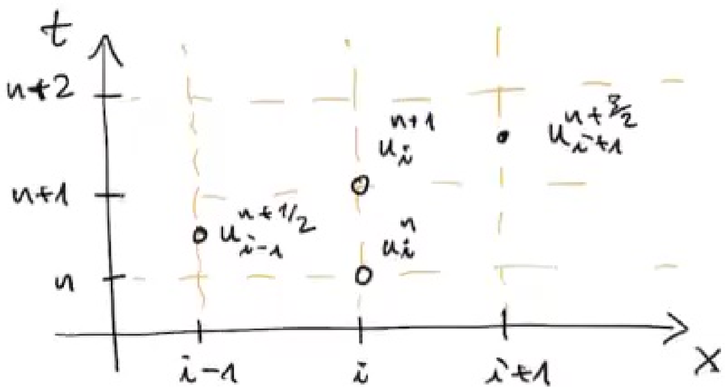


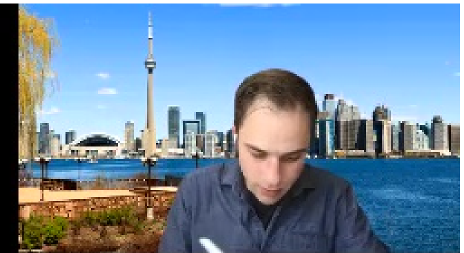
3.2 Finite difference approximations

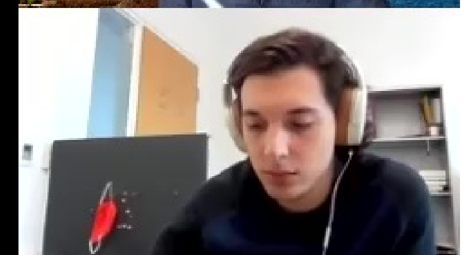
Notation:

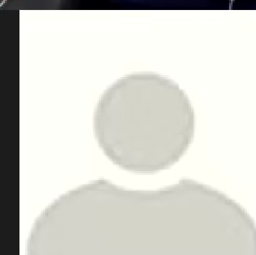
$u(x_i, t^n) \equiv u_i^n$

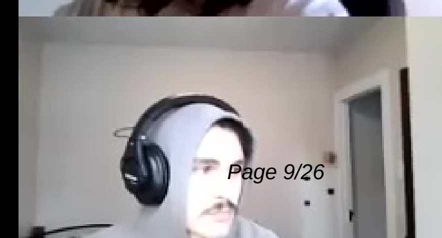
$u(x_i + \frac{\Delta x}{2}, t^n) = u^n_{i+\frac{1}{2}}$











3.2.1 Partial derivatives

Consider Taylor expansion:

u^n_{i+1}

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3.2.1 Partial derivatives

Consider Taylor expansion:

$$u_{i+1}^{n} = u_i^n + \frac{\partial u}{\partial x}\bigg|_i \Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\bigg|_i (\Delta x)^2 + \mathcal{O}((\Delta x)^3)$$

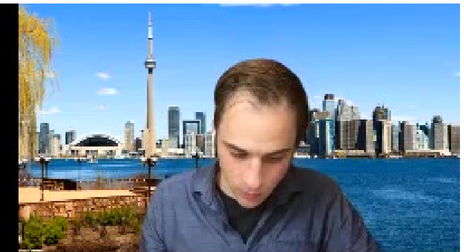
Solve for $\frac{\partial u}{\partial x}$:

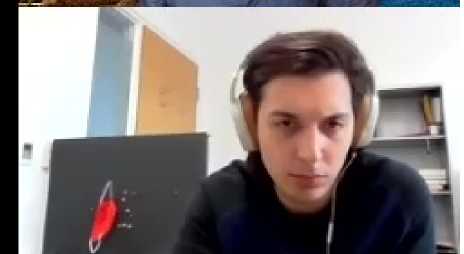
$$\frac{\partial u}{\partial x}\bigg|_{i,n} = \frac{u_{i+1}^n - u_i^n}{\Delta x} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\bigg|_i \Delta x + \mathcal{O}((\Delta x)^2)$$

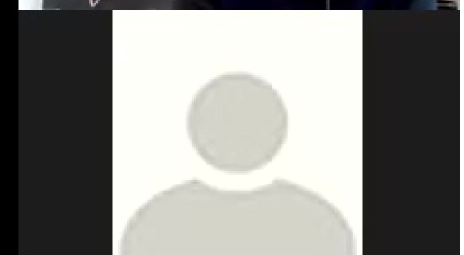
$$\boxed{D_1^+ u = \frac{u_{i+1}^n - u_i^n}{\Delta x}}$$

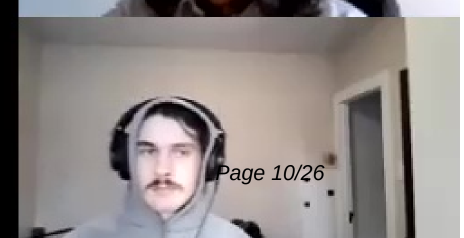
"forward difference approximation"

"stencil size"









Solve for $\frac{\partial u}{\partial x}$:

$$(*)\quad \left.\frac{\partial u}{\partial x}\right|_{i,n} = \frac{u^n_{i+1} - u^n_i}{\Delta x} - \frac{1}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_i \Delta x + \mathcal{O}((\Delta x)^2)$$

$$\boxed{D_1^+ u = \frac{u^n_{i+1} - u^n_i}{\Delta x}}$$

"forward difference approximation"

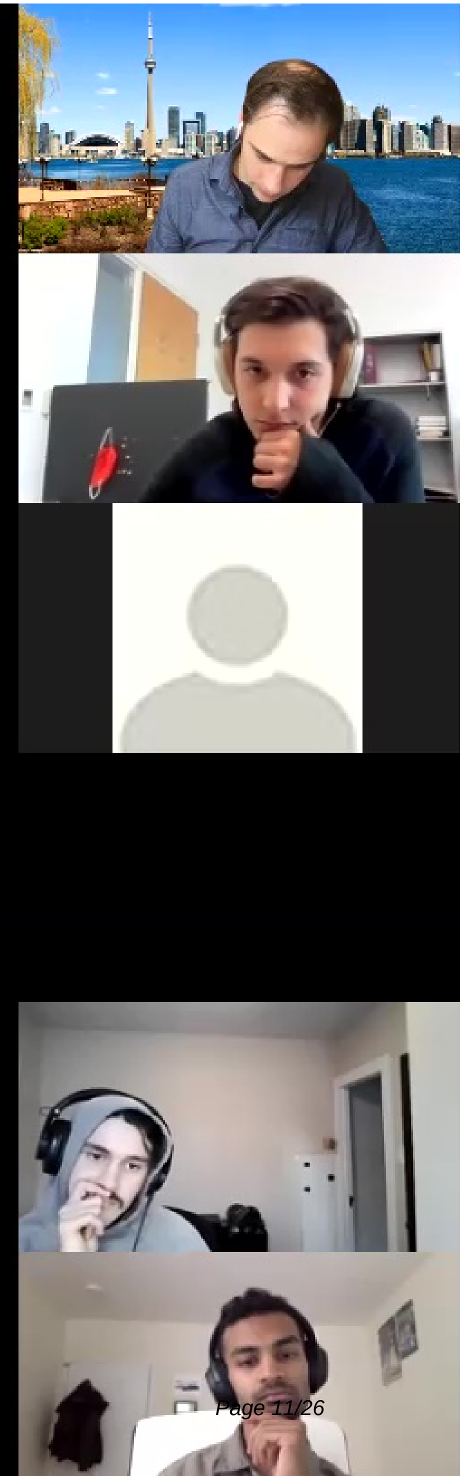
"stencil size"

accuracy / truncation error:

$$D_1^+ u - \frac{\partial u}{\partial x} \overset{(*)}{=} \frac{1}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_i \Delta x + \mathcal{O}((\Delta x)^2)$$

first-order accurate

Similarly:



Similarly:

$$\boxed{D_1^- u = \frac{u_i^n - u_{i-1}^n}{\Delta x}}$$

"backward difference approximation"

first-order accurate

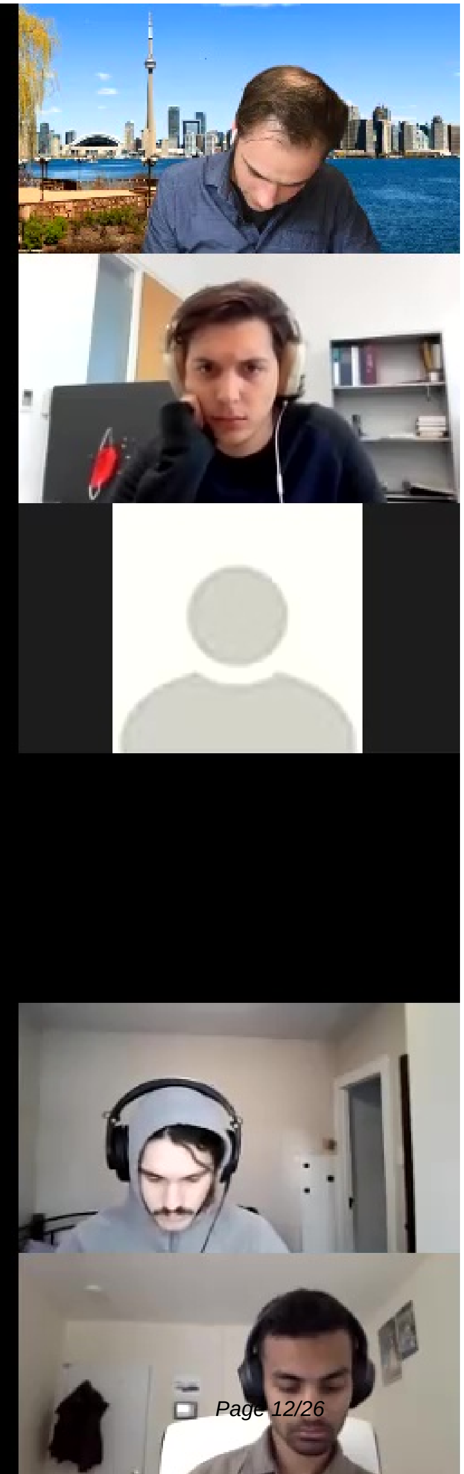
Improve accuracy:

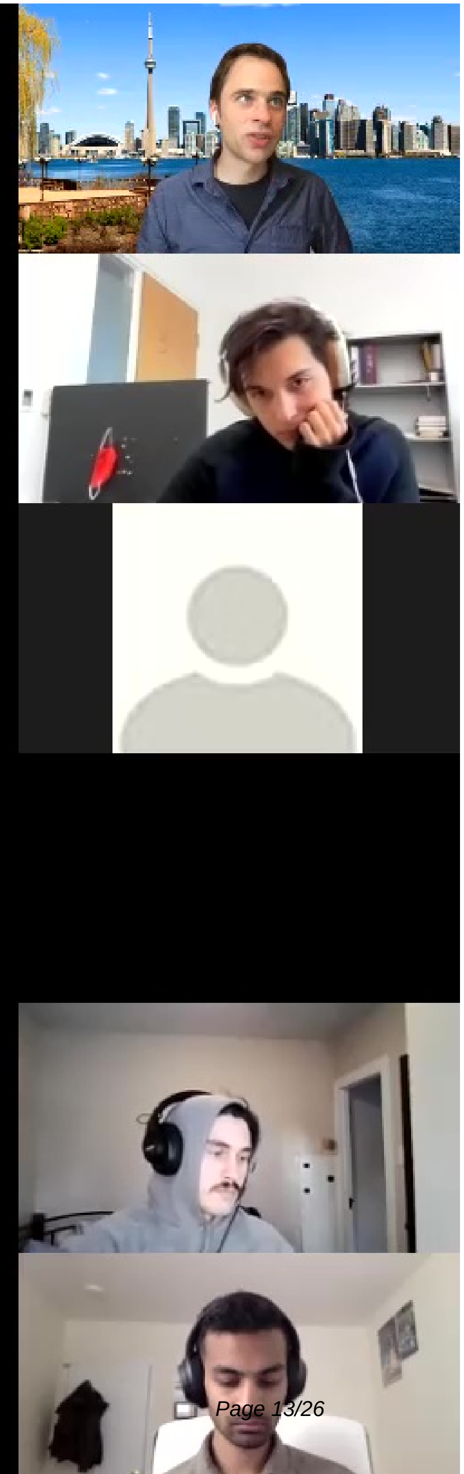
forward: $$u_{i+1}^n = u_i^n + \frac{\partial u}{\partial x}\Big|_i \frac{\Delta x}{2} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Big|_i \left(\frac{\Delta x}{2}\right)^2 + \dots$$

backward: $$u_{i-1}^n = u_i^n - \frac{\partial u}{\partial x}\Big|_i \frac{\Delta x}{2} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Big|_i \left(\frac{\Delta x}{2}\right)^2 + \dots$$

$\downarrow$ substract

$$\rightsquigarrow \frac{\partial u}{\partial x}\Big|_i = \frac{u_{i+1}^n + u_{i-1}^n}{2\Delta x} -$$





Improve accuracy:

forward: $$u^n_{i+\frac{1}{2}} = u^n_i + \frac{\partial u}{\partial x}\Big|_i \frac{\Delta x}{2} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Big|_i \left(\frac{\Delta x}{2}\right)^2 + \dots$$

backward: $$u^n_{i-\frac{1}{2}} = u^n_i - \frac{\partial u}{\partial x}\Big|_i \frac{\Delta x}{2} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Big|_i \left(\frac{\Delta x}{2}\right)^2 + \dots$$

$\downarrow$ substract

$$\rightsquigarrow \frac{\partial u}{\partial x}\Big|_i = \frac{u^n_{i+\frac{1}{2}} - u^n_{i-\frac{1}{2}}}{\Delta x} + \mathcal{O}((\Delta x)^2)$$

$$\boxed{D^0_1 u = \frac{u^n_{i+\frac{1}{2}} - u^n_{i-\frac{1}{2}}}{\Delta x}}$$

"centred difference approximation"

Remark:

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forward: $u^n_{i+\frac{1}{2}} = u^n_i + \frac{\partial u}{\partial x}\Big|_i \frac{\Delta x}{2} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Big|_i \left(\frac{\Delta x}{2}\right)^2 + \dots$

backward: $u^n_{i-\frac{1}{2}} = u^n_i - \frac{\partial u}{\partial x}\Big|_i \frac{\Delta x}{2} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Big|_i \left(\frac{\Delta x}{2}\right)^2 + \dots$

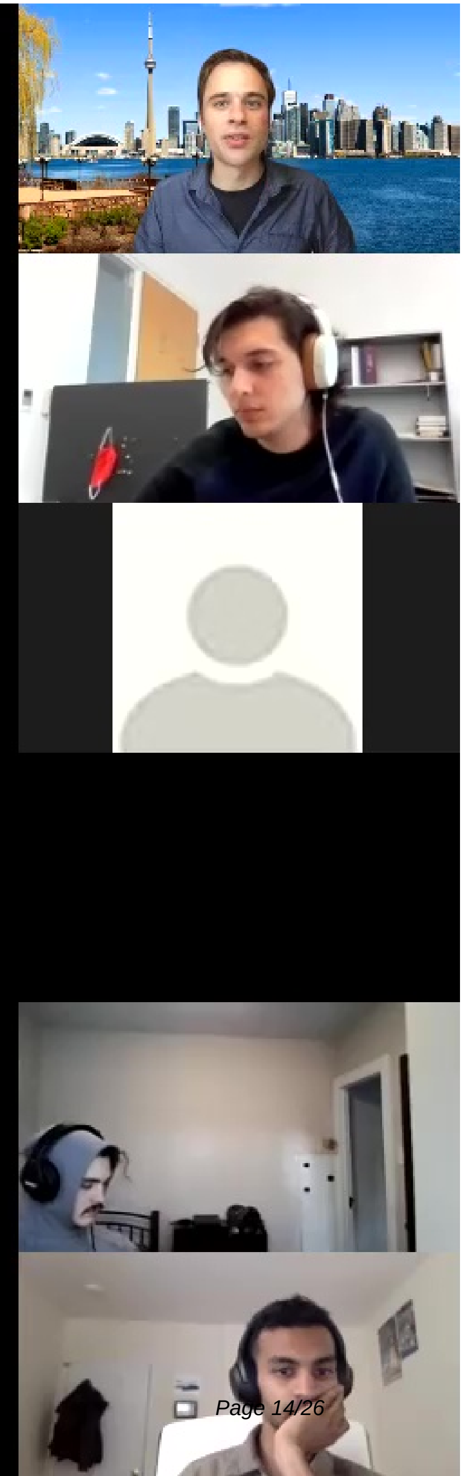
$\downarrow$ substract

$$\leadsto \frac{\partial u}{\partial x}\Big|_i = \frac{u^n_{i+\frac{1}{2}} - u^n_{i-\frac{1}{2}}}{\Delta x} + \mathcal{O}\left((\Delta x)^2\right)$$

$$\boxed{D_1^0 u = \frac{u^n_{i+\frac{1}{2}} - u^n_{i-\frac{1}{2}}}{\Delta x}}$$

"centred difference approximation"

Remark: By virtue of Taylor expansion we assume that



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the function u is continuously differentiable.

Higher-order derivatives:

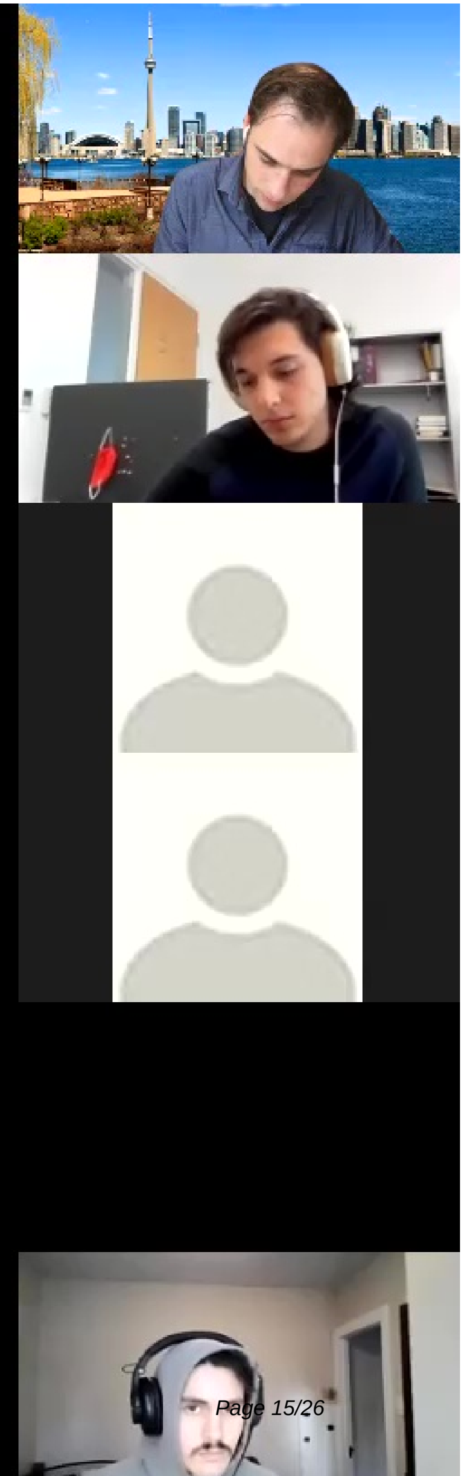
Example: $\frac{\partial^2 u}{\partial x^2} \approx D_1^+ D_1^- u = \frac{1}{\Delta x}\left(D_1^+ u_i^n - D_1^+ u_{i-1}^n\right)$

$$= \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

$$\equiv D^2 u$$

General approach:

Goal: compute appro



<u>Example:</u> $$\frac{\partial^2 u}{\partial x^2} \approx D_1^+ D_1^- u = \frac{1}{\Delta x}\left(D_1^+ u_i^n - D_1^+ u_{i-1}^n\right)$$

$$= \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

$$=: D^2 u$$

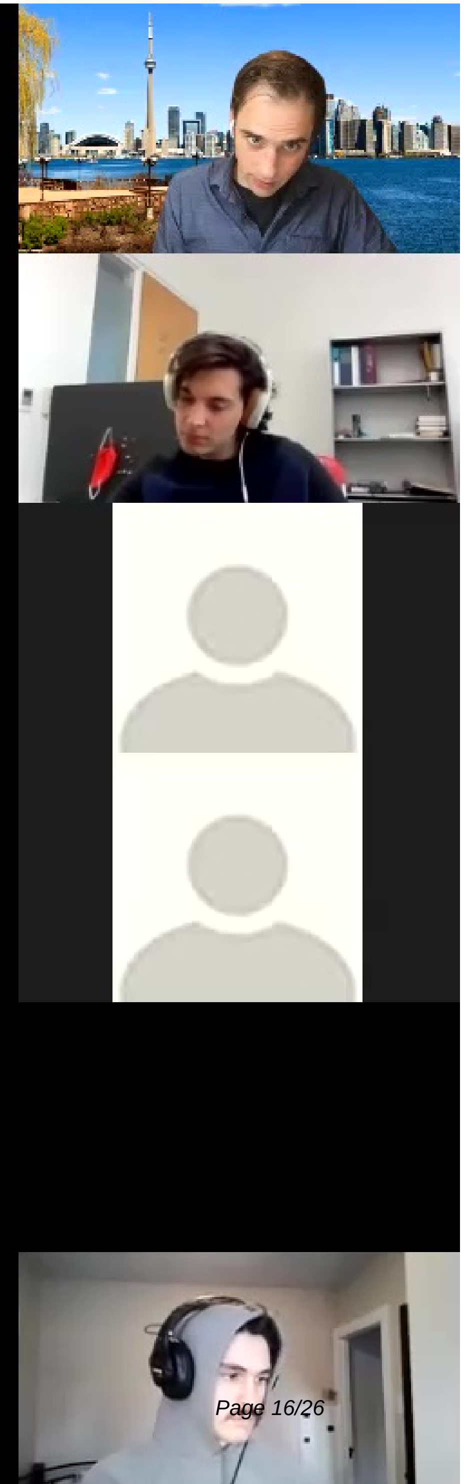
<u>General approach:</u>

Goal: compute approximation to

$$u^{(k)} = \frac{\partial^k u}{\partial x^k}$$

based on a stencil of $n \geq k+1$ points $x_1, \dots, x_n$

<u>Example:</u> want one-sided approximation to $\left.\frac{\partial u}{\partial x}\right|_i$ based on $u_i^n, u_{i-1}^n, u_{i-2}^n$



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Example: want one-sided approximation to $\left.\frac{\partial u}{\partial x}\right|_i$ based on $u_i^n, u_{i-1}^n, u_{i-2}^n$

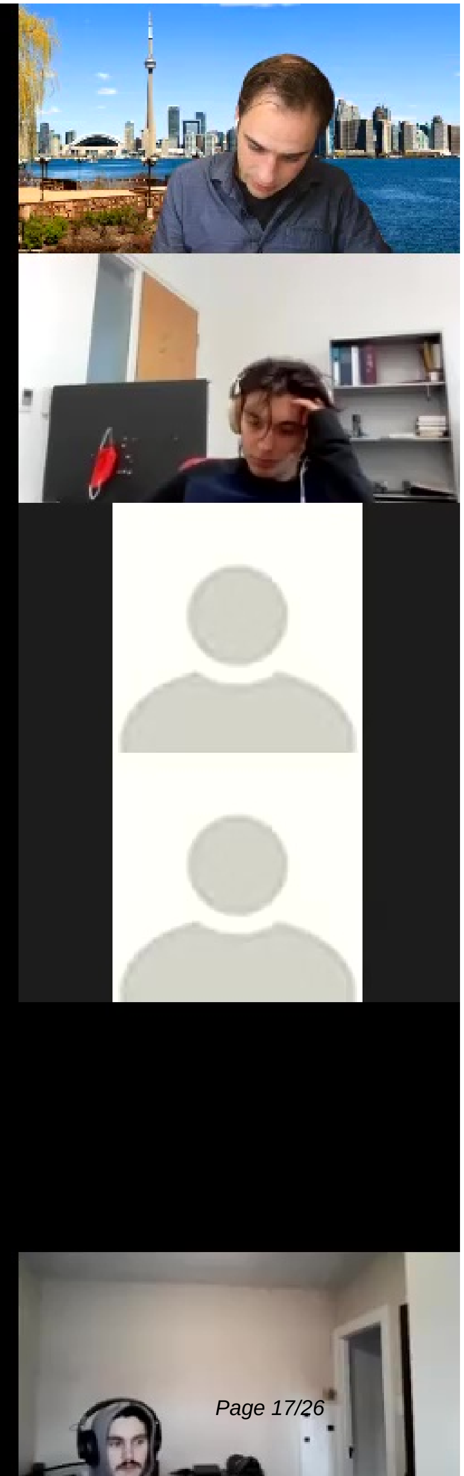
$$\leadsto u_{i-1} = u_i - u^{(1)}\big|_i \Delta x + \frac{1}{2} u^{(2)}\big|_i (\Delta x)^2 - \frac{1}{6} u^{(3)}\big|_i (\Delta x)^3 + \mathcal{O}(\Delta x^4)$$

$$u_{i-2} = u_i - u^{(1)}\big|_i (2\Delta x) + \frac{1}{2} u^{(2)}\big|_i (2\Delta x)^2 - \frac{1}{6} u^{(3)}\big|_i (2\Delta x)^3 + \mathcal{O}(\Delta x^4)$$

Ansatz:

$$D_2 u\big|_i = c_1 u_i + c_2 u_{i-1} + c_3 u_{i-2}$$

$$= (c_1 + c_2 + c_3) u_i - (c_2 + 2c_3) \Delta x\, u^{(1)} + \frac{1}{2}(c_2 + 4c_3)(\Delta x)^2$$



$$+\mathcal{O}(\Delta x^4)$$

Ansatz:

$$D_2 u\big|_i = c_1 u_i + c_2 u_{i-1} + c_3 u_{i-2}$$

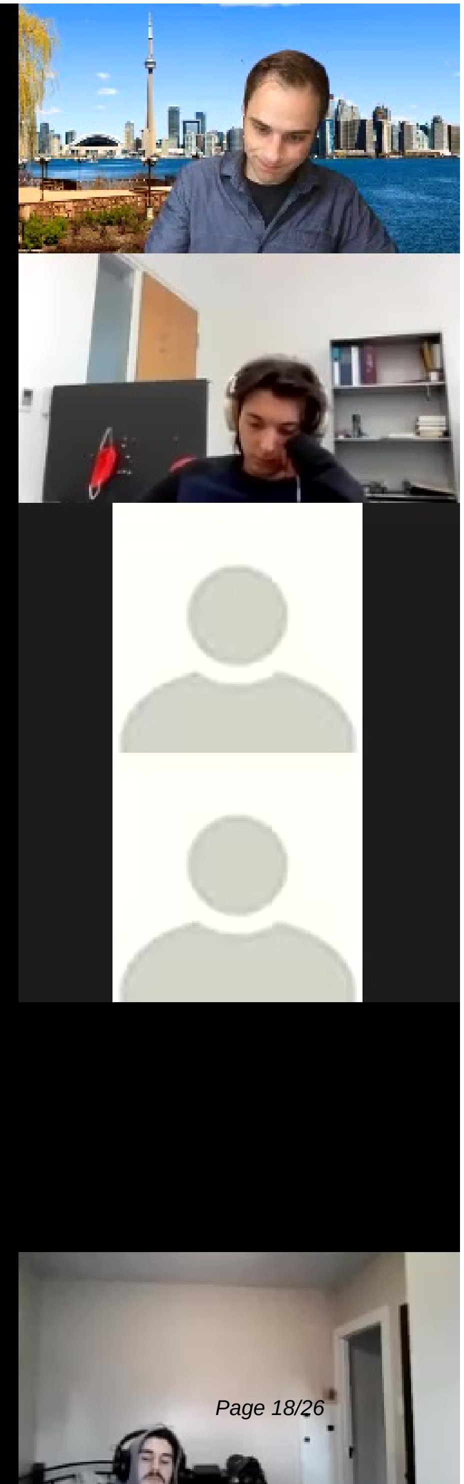
$$= \underbrace{(c_1 + c_2 + c_3)}_{=0} u_i - \underbrace{(c_2 + 2c_3)}_{=1} \Delta x \, u^{(1)}$$

$$+ \frac{1}{2}\underbrace{(c_2 + 4c_3)}_{=0} (\Delta x)^2 u^{(2)}$$

$$- \frac{1}{6}(c_2 + 8c_3)(\Delta x)^3 u^{(3)} + \dots$$

note: coefficients are of the form $\frac{1}{(l-1)!} \sum_{m=1}^{n} c_m (x_m - x_i)^{l-1}$

$$\left.\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -(c_2 + c_3)\Delta x &= 1 \\ \frac{1}{2}(c_2 + 4c_3)(\Delta x)^2 &= 0 \end{aligned}\right\} \quad \begin{aligned} c_1 &= \frac{3}{2\Delta x} \\ c_2 &= -\frac{2}{\Delta x} \\ c_3 &= \frac{1}{2\Delta x} \end{aligned}$$



Truncation Error:

$$D_2 u\big|_i - u^{(1)}\big|_i = -\frac{1}{6}(c_2+8c_3)(\Delta x)^3 u^{(3)}\big|_i + \dots$$

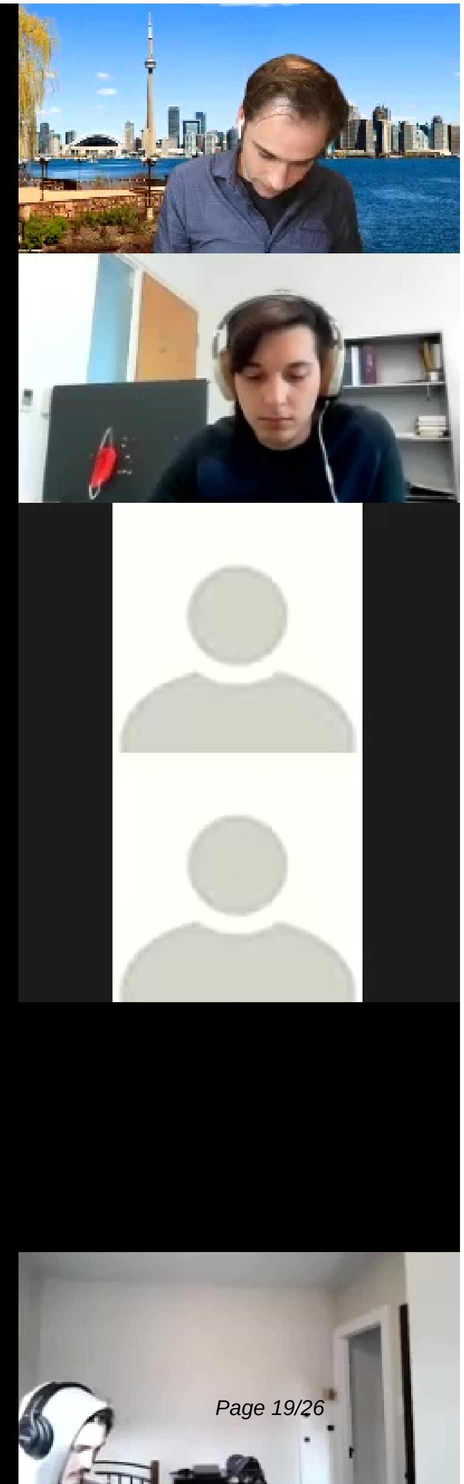
$$= -\frac{1}{3}(\Delta x)^2 u^{(3)}\big|_i + \dots$$

$$\longrightarrow \underline{O(\Delta x^2)}$$

Generalize procedure: Consider stencil $\{x_\ell\}_{\ell=1,\dots,n}$ around x_i, $n \geq k+1$.

Consider n Taylor series $\ell=1,\dots,n$ and obtain $u^{(k)}$ that is

$O((\Delta x)^p)$ $(p = n-k)$ accurate.



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$$D_2 u\big|_i - u^{(1)}\big|_i = -\frac{1}{6}(c_2 + 8c_3)(\Delta x)^3 u^{(3)}\big|_i + \dots$$

$$= -\frac{1}{3}(\Delta x)^2\, u^{(3)}\big|_i + \dots$$

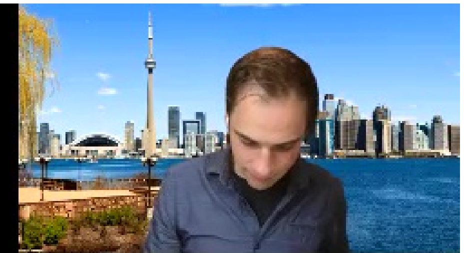
$$\longrightarrow \underline{\mathcal{O}(\Delta x^2)}$$

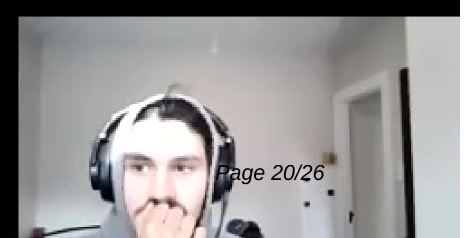
<u>Generalize procedure</u>: Consider stencil $\{x_\ell\}_{\ell=1,\dots,n}$ around x_i, $n \geq k+1$.

Consider n Taylor series $\ell = 1,\dots,n$ and obtain $u^{(k)}$ that is

$\mathcal{O}((\Delta x)^p)$ $(p = n-k)$ accurate.

$p > n-k$ achievable if additional higher order terms cancel ($\rightarrow$ centered differences)





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General: increasing stencil size increases accuracy.

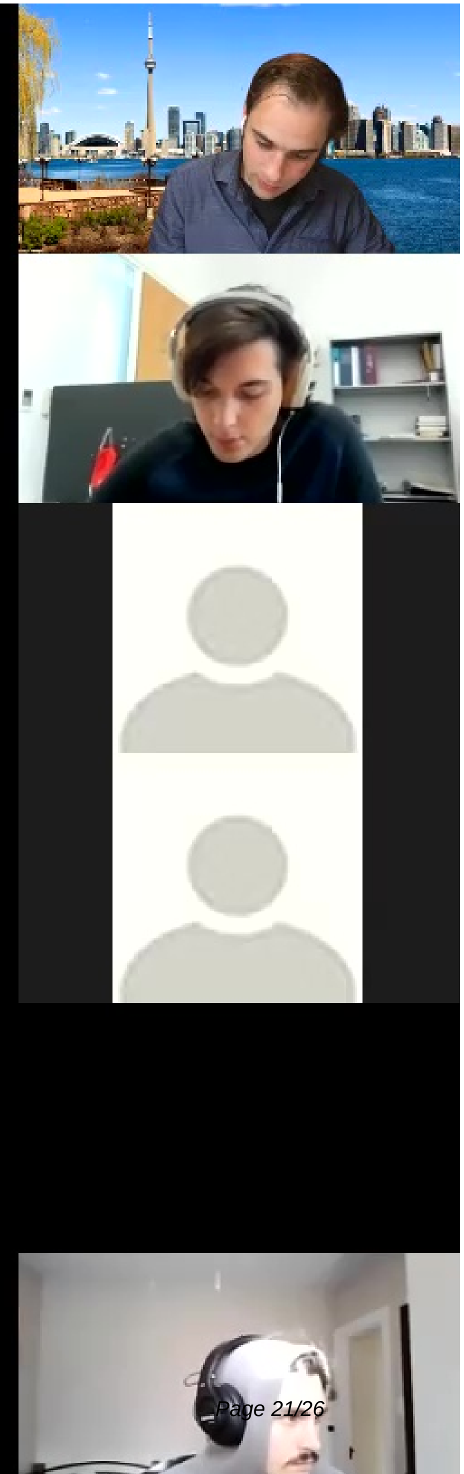
3.2.2 Sample Discretizations

1D advection equation

$$u_t + a u_x = 0$$

$$u(x,0) = u_0(x)$$

$$\left(u(0,t) = u(1,t) = 0 \right)$$



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3.2.2 Sample Discretizations

1D advection equation

$$u_t + a u_x = 0 \qquad 0 \leq x \leq 1,\ t \geq 0$$

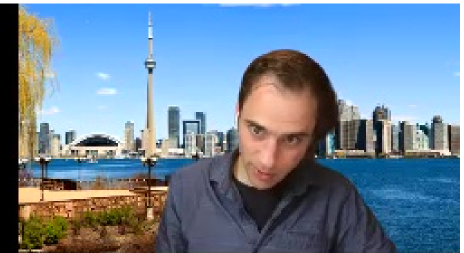
$$u(x,0) = u_0(x)$$

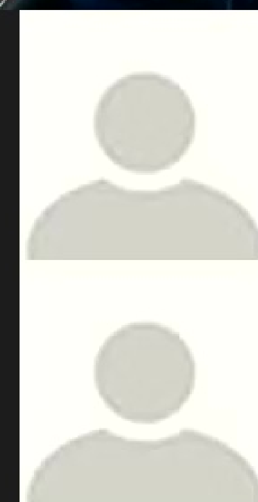
$$\left(u(0,t) = u(1,t) = 0 \right)$$

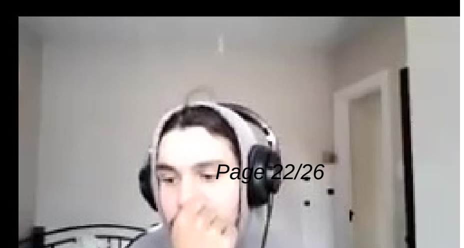
(i) Consider simple <u>forward in time</u>, <u>centred in space</u> FDA:

$$u_x \Big|_i^n \approx \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$$

$$u_t \Big|_i^n = \frac{u_i \ldots}{}$$







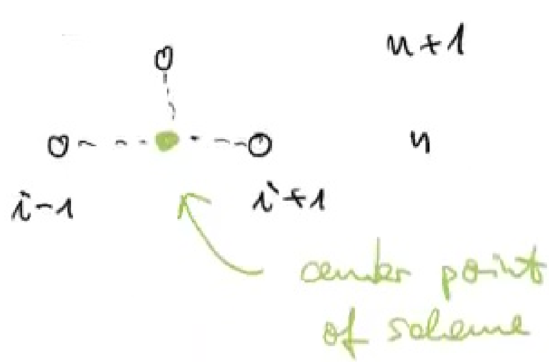
$$\left(u(0,t) = u(1,t) = 0 \right)$$

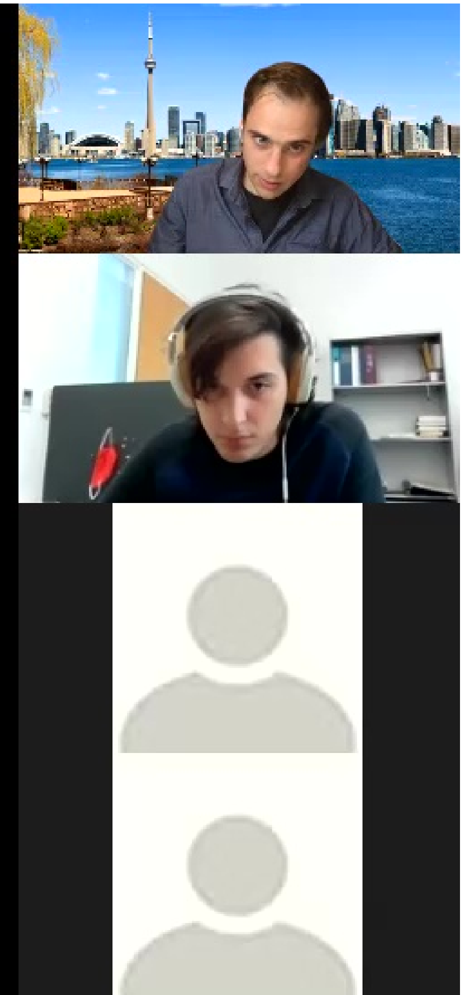
(i) Consider simple forward in time, centred in space FDA:

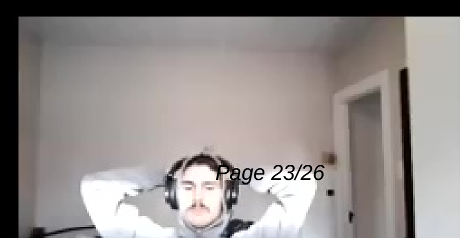
$$u_x\Big|_i^n \approx \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$$

$$u_t\Big|_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

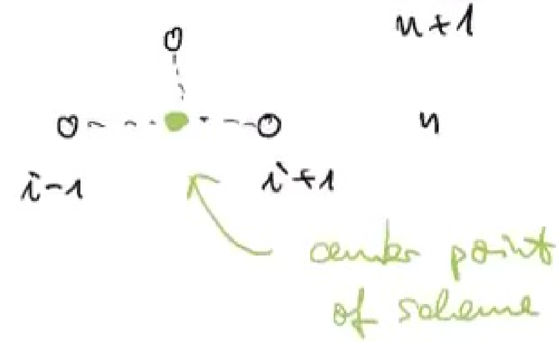
stencil:







stencil:



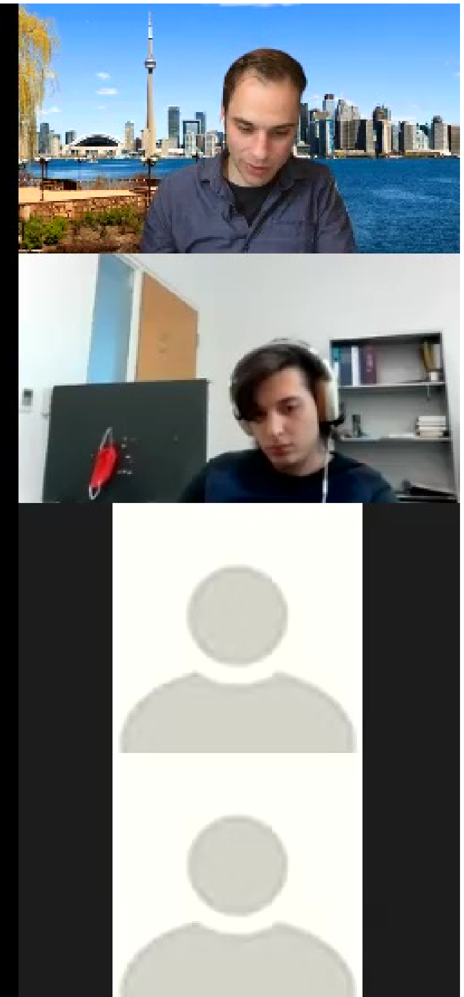
$$\rightsquigarrow \quad \frac{u_i^{n+1}-u_i^{n}}{\Delta t} + \frac{a}{2\Delta x}\left(u_{i+1}^{n} - u_{i-1}^{n}\right) = 0$$

$$\Leftrightarrow \quad u_i^{n+1} = u_i^{n} - \frac{a\Delta t}{2\Delta x}\left(u_{i+1}^{n} - u_{i-1}^{n}\right)$$

linear system for u^{n+1} : $A u^{n+1} = b$

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} u_1^{n+1} \\ \vdots \\ u_I^{n+1} \end{pmatrix} = \begin{pmatrix} u_i^{n} - \lambda\left(u_{i+1}^{n} - u_{i-1}^{n}\right) \end{pmatrix}$$

$x_i \in \{x_1, \dots, x_I\}$



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Lecture 7

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$$\Leftrightarrow \quad u_i^{n+1} = u_i^n - \frac{a\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)$$

linear system for u^{n+1}: $\quad A u^{n+1} = b$

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} u_1^{n+1} \\ \vdots \\ u_I^{n+1} \end{pmatrix} = \begin{pmatrix} u_i^n - \lambda\left(u_{i+1}^n - u_{i-1}^n\right) \end{pmatrix}$$

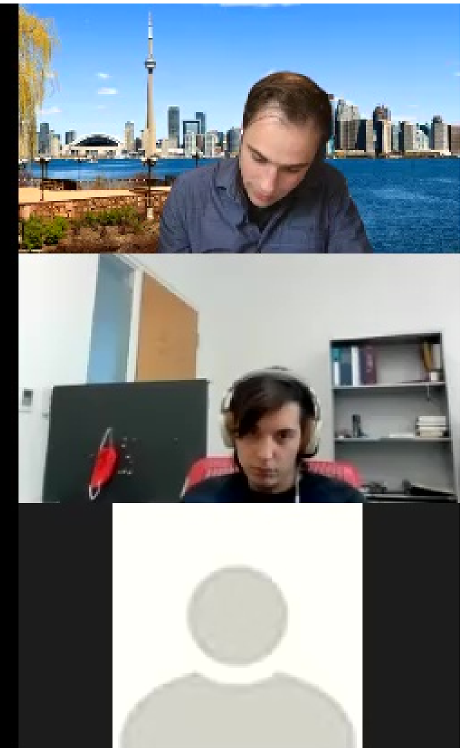
$\uparrow \; x_i \in \{x_1, \dots, x_I\}$ $\qquad O(\Delta t, (\Delta x)^2)$

$\rightsquigarrow$ explicit scheme

will see: unstable for any fixed $\frac{\Delta t}{\Delta x}$

(ii) Lax-Friedrichs method

replace: $\quad u_i^n \longrightarrow \frac{1}{2}\left(u_{i-1}^n + u_{i+1}^n\right)$



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Lecture 7

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$$\dots \frac{\dots}{2\Delta x}\left(u_{i+1} - u_{i-1}\right)$$

linear system for u^{n+1}: $\quad A u^{n+1} = b$

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} u_1^{n+1} \\ \vdots \\ u_I^{n+1} \end{pmatrix} = \begin{pmatrix} u_i^n - \lambda\left(u_{i+1}^n - u_{i-1}^n\right) \end{pmatrix}$$

$\uparrow \; x_i \in \{x_1, \dots, x_I\}$ $\qquad \mathcal{O}\left(\Delta t, (\Delta x)^2\right)$

$\rightsquigarrow$ explicit scheme

will see: unstable for any fixed $\frac{\Delta t}{\Delta x}$

(ii) Lax-Friedrichs method

replace: $\quad u_i^n \longrightarrow \frac{1}{2}\left(u_{i-1}^n + u_{i+1}^n\right)$

$$\rightsquigarrow \quad u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{a\,\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)$$