Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 6

Speakers: Daniel Siegel

Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

Date: September 29, 2020 - 3:30 PM

URL: http://pirsa.org/20090029

2.6 Relativistic Hydrodynamics

In special relativity the Euler equations can be written as:

$$\partial_\mu (\rho u^\mu) = 0 \qquad \mu = 0,1,2,3$$

$$\partial_\mu T^{\mu\nu} = 0 \qquad i,j,k = 1,2,3$$

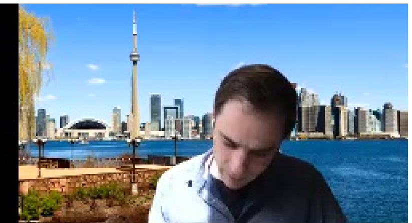
For an ideal/perfect fluid (in local rest frame no energy fluxes & anisotropic stresses)

Minkowski metric

$$T^{\mu\nu} = \rho h\, u^\mu u^\nu + p\, \eta^{\mu\nu}$$ energy-momentum tensor

$$h = 1 + \varepsilon + \frac{p}{\rho}$$ specific enthalpy

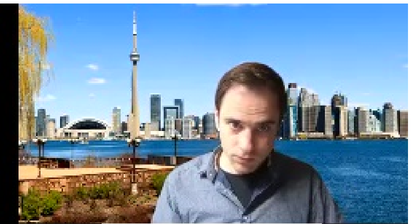
General relativity: invoke equivo



9:41 AM Tue Jan 9

Lecture 6

Lecture 6 | Chap_2_Equations



General relativity: invoke equivalence principle

$$\partial_\mu(\rho u^\mu)=0$$

$$\partial_\mu T^{\mu\nu}=0$$

$\Big\downarrow$ equivalence principle, substitution rules: $\eta_{\mu\nu}\to g_{\mu\nu}$, $\partial_\mu\to\nabla_\mu$

$$\boxed{\begin{aligned}\nabla_\mu(\rho u^\mu)&=0\\ \nabla_\mu T^{\mu\nu}&=0\end{aligned}}$$

$S_{SR},\ X_{SR},\ T_{SR}^{i_1\ldots i_r}$

$\rightarrow S_{GR},\ X^\mu_{GR},\ T^{i_1\ldots i_r}_{j_1\ldots j_s\,GR}$

$$T^{\mu\nu}=h\rho u^\mu u^\nu+pg^{\mu\nu}$$

Goal: want to rewrite the equations as an initial value problem in conservation form

$T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}$

<u>Goal</u>: want to rewrite the equations as an initial value problem in conservation form

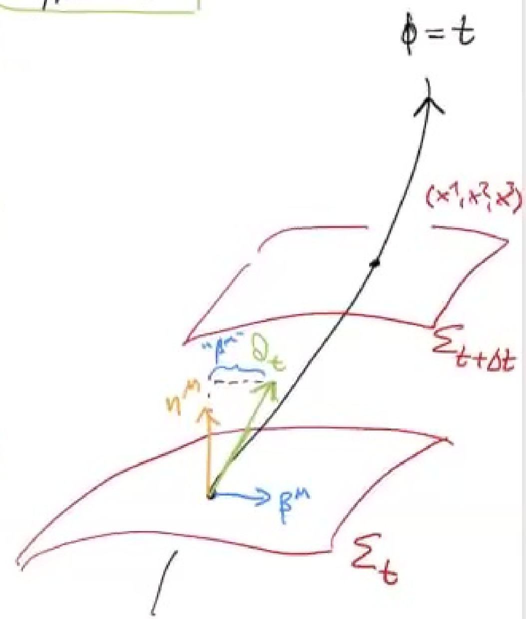
$\longrightarrow$ <u>3+1 decomposition of spacetime</u>

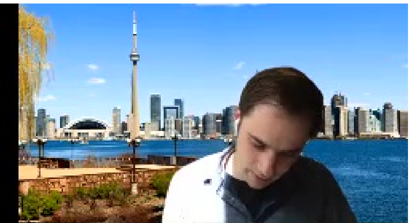
$$(\partial_t)^\mu = \alpha n^\mu + \beta^\mu, \quad \boxed{n^\mu \beta_\mu = 0}$$

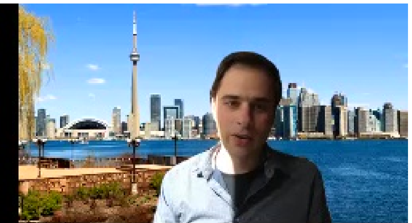
"lapse" "shift"

represent 4 DOF to specify (diffeomorphism invariance)

$u^\mu = n^\mu$: Eulerian observer ("lab frame") no motion wrt spatial coordinate







Adapted coordinates:

$$(\partial_t)^\mu = (1, 0, 0, 0)$$

$$\beta^\mu = (0, \beta^i)$$

$$\rightsquigarrow n^\mu = \frac{1}{\alpha}(\partial_t)^\mu - \frac{1}{\alpha}\beta^\mu = \left(\frac{1}{\alpha}, -\frac{1}{\alpha}\beta^i\right)$$

$$n_\mu = (-\alpha, 0, 0, 0)$$

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu) = \begin{pmatrix} -\alpha^2-\beta^2 & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

Note: a specific choice of α & β^i represent
a specific choice of coordinates (gauge)

a specific choice of coordinates (gauge)

<u>Example:</u> $\alpha \equiv 1$ (Geodesic
$\beta^i \equiv 0$ gauge)

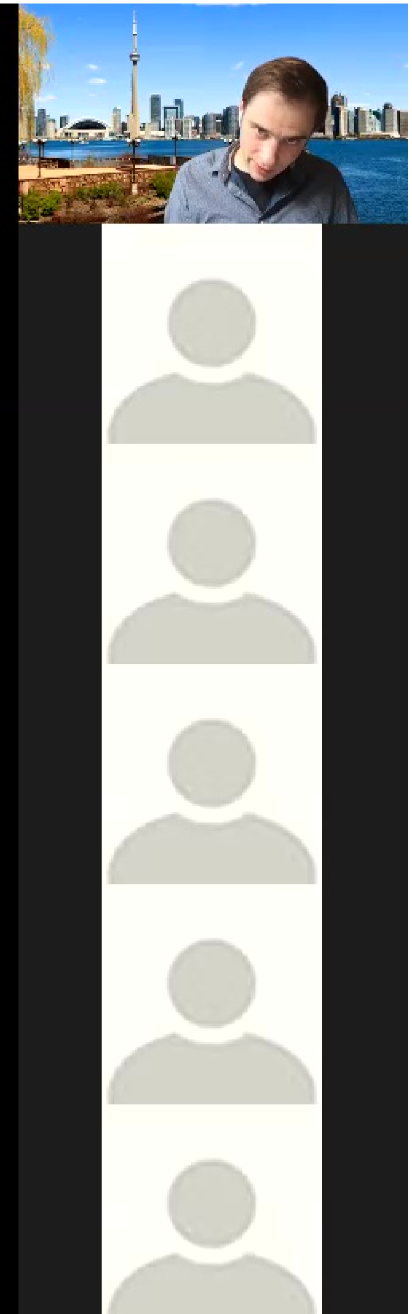
Adapted equations of hydrodynamics

<u>Split fluid velocity:</u>

$$u^\mu = W\,(n^\mu + v^\mu)$$

$W \equiv -u^\mu n_\mu$ Lorentz factor btw. Eulerian observer & comoving (fluid) frame

$$v^i = \frac{u^i}{W} + \frac{1}{\alpha}\beta^i$$

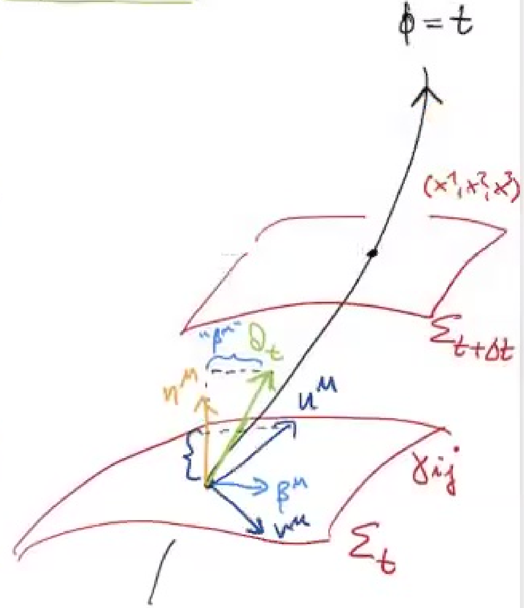


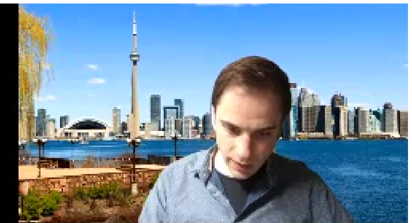
$$(\partial_t)^\mu = \alpha n^\mu + \beta^\mu , \quad \boxed{n^\mu \beta_\mu = 0}$$

"lapse" "shift"

represent 4 DOF to specify (diffeomorphism invariance)

$u^\mu = n^\mu$: Eulerian observer ("lab frame") no motion wrt spatial coordinates





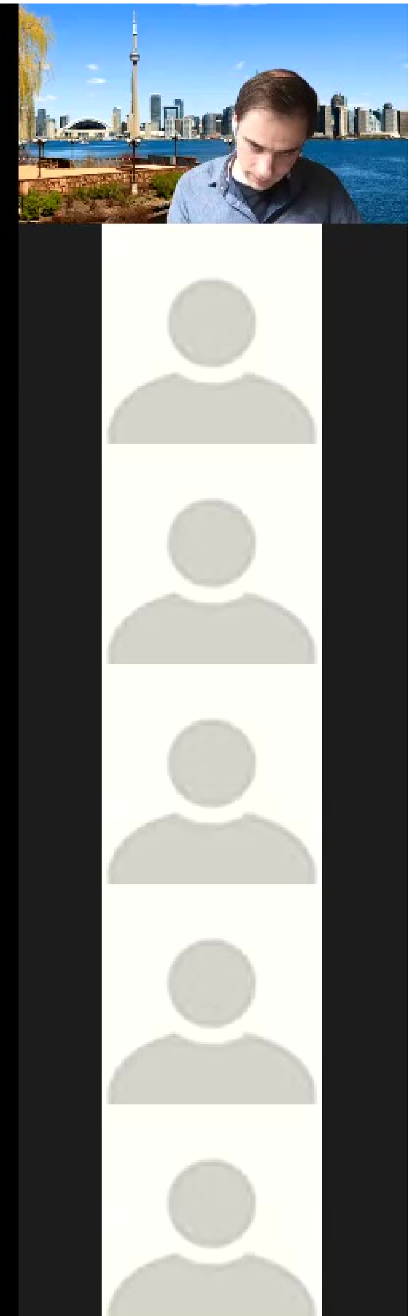
Adapted coordinates:

$$(\partial_t)^\mu = (1, 0, 0, 0)$$

$$\beta^\mu = (0, \beta^i)$$

$$\rightsquigarrow n^\mu = \frac{1}{\alpha} (\partial_t)^\mu - \frac{1}{\alpha} \beta^\mu = \left(\frac{1}{\alpha}, -\frac{1}{\alpha} \beta^i \right)$$

$$n_\mu = (-\alpha, 0, 0, 0)$$



Split fluid velocity:

$$u^\mu = W\left(n^\mu + v^\mu\right)$$

$W \equiv -u^\mu n_\mu$ Lorentz factor btw. Eulerian observer & comoving (fluid) frame

$$v^i = \frac{u^i}{W} + \underbrace{\frac{1}{\alpha}\beta^i}_{-n^i}$$

and define:

$$E_E \equiv n^\mu n^\nu T_{\mu\nu} = (E+p)W^2 - p = h\rho W^2 - p$$

$$S_i \equiv -\gamma_i{}^\mu n^\nu T_{\mu\nu} = (E+p)W^2 v_i = h\rho W^2 v_i$$

$$S_{ij} \equiv \gamma_i{}^\mu \gamma_j{}^\nu T_{\mu\nu} = h\rho W^2 v_i v_j + p\,\gamma_{ij}$$

(total energy dens

$$v^i = \frac{u^i}{W} + \frac{1}{\alpha}\beta^i$$

and define:

$$E_E \equiv n^\mu n^\nu T_{\mu\nu} = (E+p)W^2 - p = h\rho W^2 - p$$

$$S_i \equiv -\gamma_i{}^\mu n^\nu T_{\mu\nu} = (E+p)W^2 v_i = h\rho W^2 v_i$$

$$S_{ij} \equiv \gamma_i{}^\mu \gamma_j{}^\nu T_{\mu\nu} = h\rho W^2 v_i v_j + p\,\gamma_{ij}$$

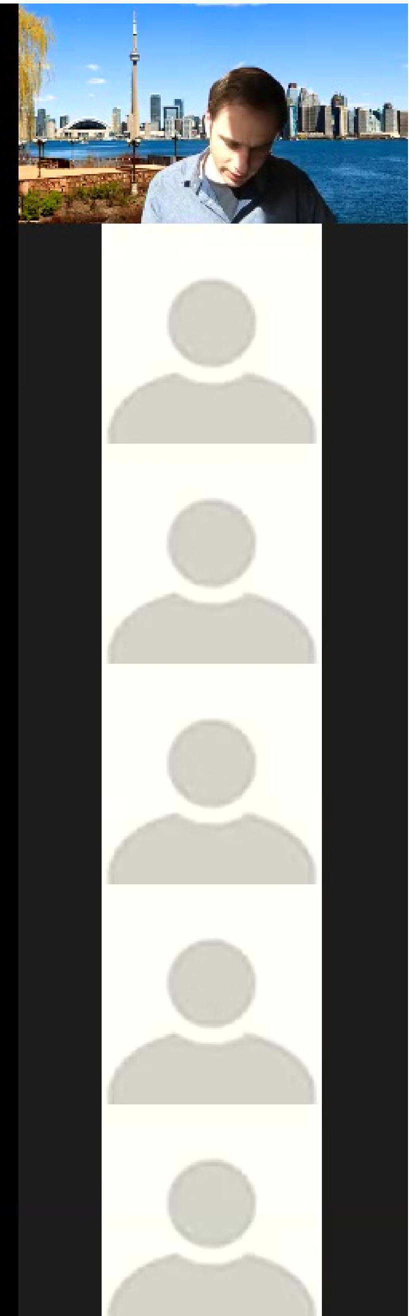
(total energy density, momentum density, stress-energy as measured by Eulerian observer)

Then one obtains:

- $0 = \nabla_\mu(\rho u^\mu) \longrightarrow \boxed{\partial_t(\sqrt{\gamma}\,D) + \partial_i(\sqrt{\gamma}\,D\,\tilde{v}^i) = 0}$

$D \equiv \rho W$ rest mass measured by Eulerian observer

$\tilde{v}^i \equiv \alpha\cdot$



stress-energy as measured by Eulerian observer)

Then one obtains:

- $0 = \nabla_\mu(\rho u^\mu) \longrightarrow \boxed{\partial_t(\sqrt{\gamma}\, D) + \partial_i(\sqrt{\gamma}\, D\, \tilde{v}^i) = 0}$

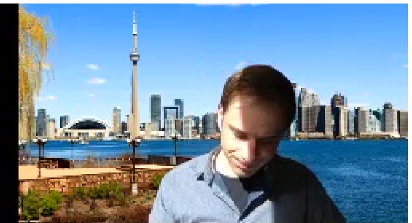
  $D \equiv \rho W$ rest mass measured by Eulerian observer

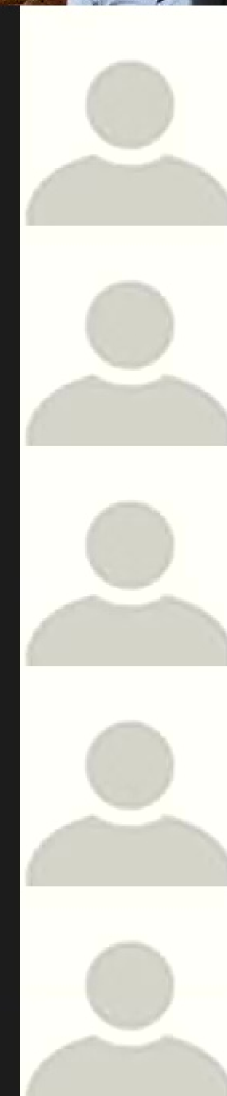
  $\tilde{v}^i \equiv \alpha v^i - \beta^i$

- $0 = n_\mu \nabla_\nu T^{\mu\nu}$ (projection of conservation law along Eulerian observer)

$$\rightarrow \boxed{\partial_t(\sqrt{\gamma}\, E_E) + \partial_j\left[\sqrt{\gamma}\,(\alpha S^j - E_E \beta^j)\right] = \alpha\sqrt{\gamma}\,(K_{ij} S^{ij} - S^k \partial_k \ln\alpha)}$$

extrinsic curvature





9:41 AM Tue Jan 9

100%

Lecture 6

Lecture 6 | Chap_2_Equations

$$\rightarrow \boxed{\partial_t(\sqrt{\gamma}\, E_E) + \partial_j\left[\sqrt{\gamma}\,(\alpha S^j - E_E \beta^j)\right] = \alpha\sqrt{\gamma}\,\left(K_{ij} S^{ij} - S^k \partial_k \ln\alpha\right)}$$

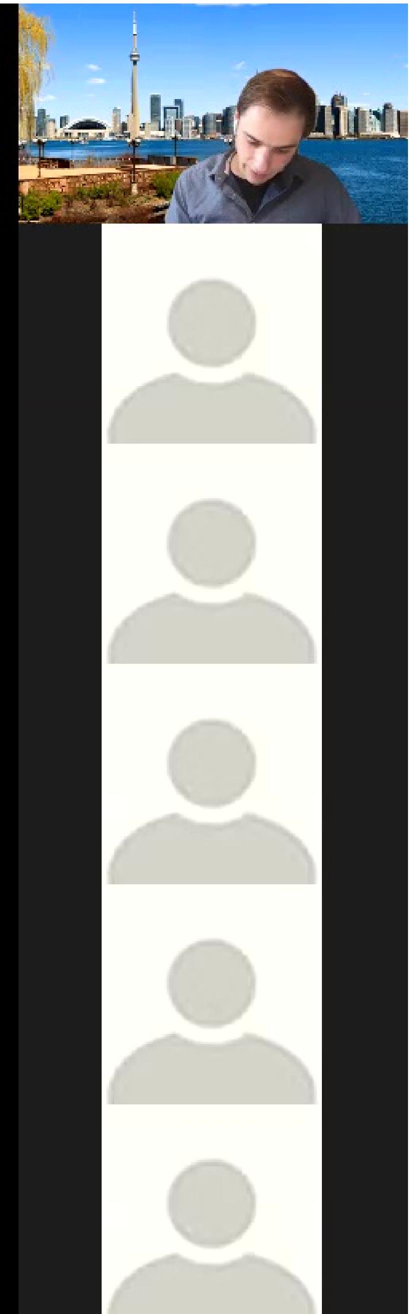
extrinsic curvature

- $0 = \gamma_i{}^{\mu} \nabla_\nu T^\nu{}_\mu$ (spatial projection)

$$\rightarrow \boxed{\partial_t(\sqrt{\gamma}\, S_i) + \partial_j\left[\sqrt{\gamma}\,(S_j \tilde{v}^i + p\,\delta^i{}_j)\right] = \sqrt{\gamma}\left[\frac{\alpha}{2} S^{kl}\partial_i \gamma_{kl} + S_l \partial_i \beta^l - S\,\partial_i \alpha\right]}$$

$\rightsquigarrow$ Conservative formulation:

$$\partial_t u + \partial_i f^i(u) = S(u)$$



9:41 AM Tue Jan 9

Lecture 6

Lecture 6 | Chap_2_Equations

$$= \sqrt{\gamma}\left[\frac{\alpha}{2} S^{kl}\,\partial_i \gamma_{kl} + S_l\,\partial_i \beta^l - S\,\partial_i \alpha\right]$$

⇝ Conservative formulation:

$$\boxed{\partial_t u + \partial_i f^i(u) = S(u)}$$

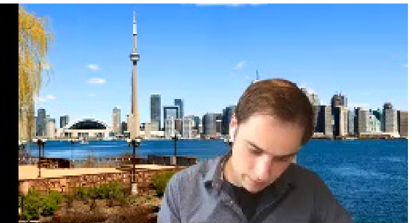
$$u \equiv \sqrt{\gamma}\,(D, S_i, \tau) \qquad \tau \equiv E_E - D$$

det(γ_{ij})

$$f^i \equiv \sqrt{\gamma}\,\left(D\tilde{v}^i,\ S_j \tilde{v}^i + p\,\delta^i{}_j,\ \tau \tilde{v}^i + p v^i\right)$$

$$S(u) \equiv \Big(0,\ \alpha\sqrt{\gamma}\left(K_{ij} S^{ij} - S^k \partial_k \ln\alpha\right),$$
$$\sqrt{\gamma}\left(\frac{\alpha}{2} S^{kl}\,\partial_i \gamma_{kl} + S_l\,\partial_i \beta^l - S\,\partial_i \alpha\right)\Big)$$

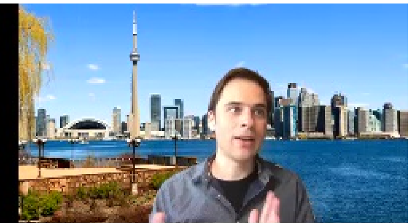
Re



9:41 AM Tue Jan 9

Lecture 6

Chap_2_Equations



$$s(u) = (0, \alpha\sqrt{\gamma}(K_{ij}S^{ij} - S^k\partial_k \ln\alpha),$$

$$\sqrt{\gamma}\left(\frac{\alpha}{2}S^{kl}\partial_i\gamma_{kl} + S_l\partial_i\beta^l - S\partial_i\alpha\right)$$

Remarks: 1) The energy equation is written in terms of $\tau = E_E - D$ to obtain the "correct" Newtonian limit:

$$\tau = E_E - D = \rho h W^2 - p - \rho W$$

$$\overset{\frac{p}{\rho c^2} \ll 1}{=} \rho \underbrace{W(W-1)}_{\approx \left[1+\frac{1}{2}\left(\frac{v}{c}\right)^2 \ldots\right]\left[\frac{1}{2}\left(\frac{v}{c}\right)^2 + \ldots\right]} + \rho\varepsilon W^2 \qquad W^2 \approx 1 + \left(\frac{v}{c}\right)^2 \ldots \qquad h = 1 + \varepsilon + \cancel{\frac{p}{\rho}}^{\,0}$$

$$\approx \underbrace{\frac{1}{2}\rho v^2} + \underbrace{\rho\varepsilon}_{e} = E$$

2) $s(u)$: "geometric source terms"

Note: do not depend on time derivatives

$$\approx \underbrace{\tfrac{1}{2}\rho v^2} + \underbrace{\rho\varepsilon}_{e} = E$$

2) $s(u)$: "geometric source terms"

Note: do not depend on time derivatives of space-time quantities

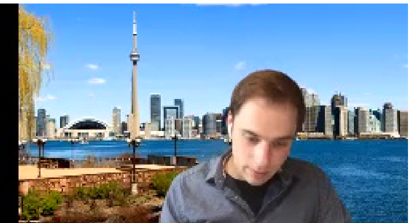
3) EOS: needed to close the system

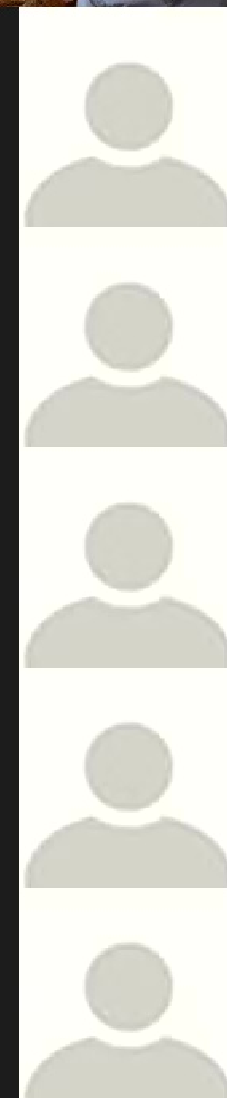
$$p = p(\rho, \varepsilon, \dots)$$

4) Conservatives & primitives:

$u = \sqrt{\gamma}\,(D, S_i, \tau)$ "conservatives"

$w = (\rho, v^i, \varepsilon)$ "primitives"





4) Conservatives & primitives:

$$u = \sqrt{\gamma}\,(D, S_i, \tau) \quad \text{"conservatives"}$$

$$w = (\rho, v^i, \varepsilon) \quad \text{"primitives"}$$

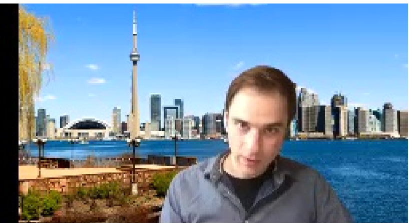
In order to compute flux terms during evolution, one must compute w from u numerically

typically: obtain w via non-linear root finding in some variable z

$$f(z) = z(\rho, \varepsilon, v^i) - z$$

Example: $z = p$

$$\rightsquigarrow \quad v^i = \frac{S^i}{\tau + p}, \quad W = \sqrt{1 - v^i v_i}$$



In order to compute flux terms during evolution, one must compute w from u numerically

typically: obtain w via non-linear root finding in some variable z

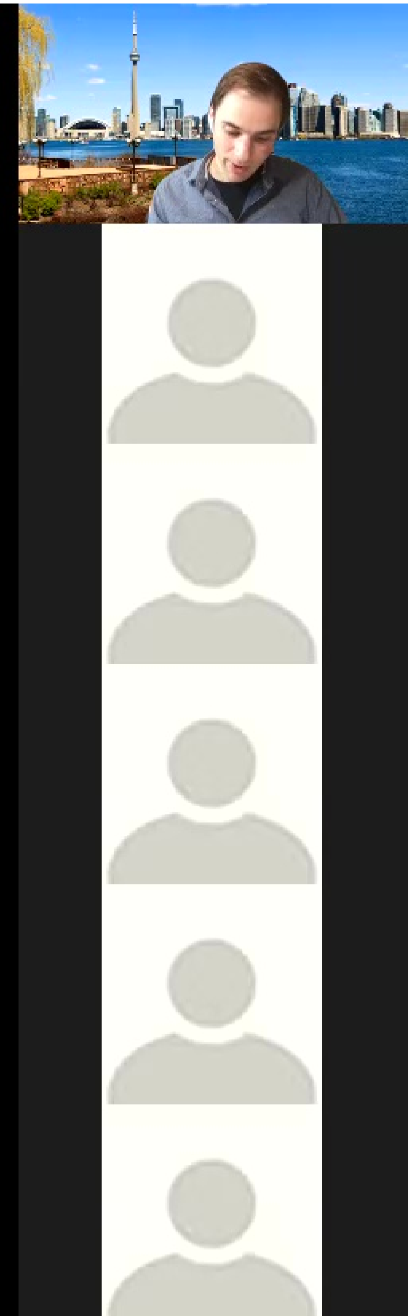
$$f(z) = z(\rho, \varepsilon, v^i) - z$$

Example: $z = p$

$$\rightsquigarrow v^i = \frac{S^i}{\tau + p}, \quad W = \sqrt{1 - v^i v_i}$$

$$\rho = \frac{D}{W}$$

$$\varepsilon = \frac{\tau - DW + p(1-W^2)}{DW}$$



In order to compute flux terms during evolution, one must compute w from u numerically

typically: obtain w via non-linear root finding in some variable z

$$f(z) = z(\rho, \varepsilon, v^i) - z$$

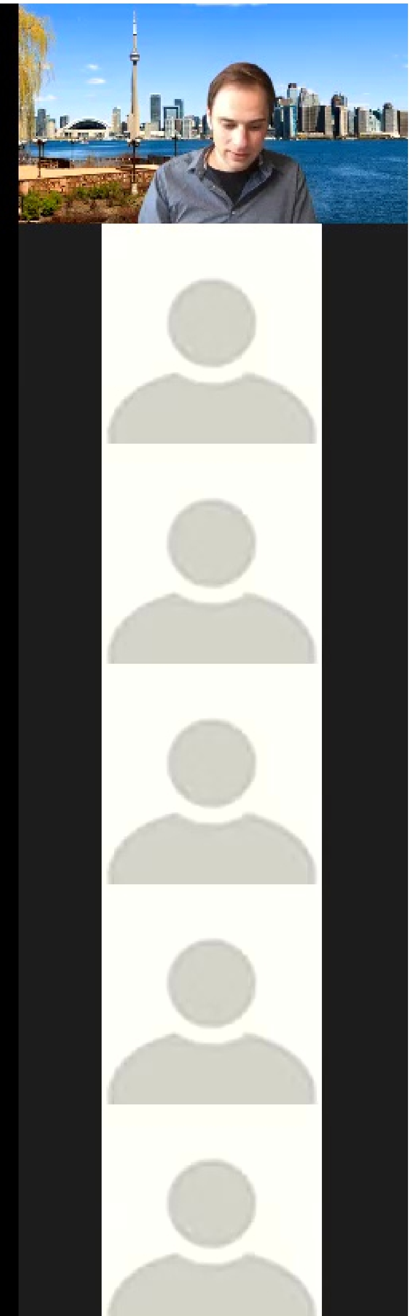
Example: $z = p$

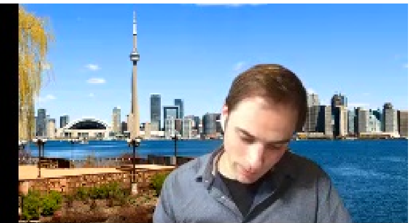
$$\rightsquigarrow \quad v^i = \frac{S^i}{\tau + p}, \quad W = \sqrt{1 - v^i v_i}$$

$$\rho = \frac{D}{W}$$

$$\varepsilon = \frac{\tau - DW + p(1 - W^2)}{DW}$$

See Siegel et al. ApJ 859, 71 (2018) for a discussion of recovery schemes





DW

See Siegel et al. ApJ 859, 71 (2018)
for a discussion of recovery schemes

5) <u>Hyperbolicity:</u> the eqns in conservative form are strictly hyperbolic if the

we are not in vacuum and the EOS is <u>causal</u>,

$$0 < c_s < c$$

sound speed

see Anile 1990, Anton et al. 2006 ApJ 637, 296

6) <u>Special-relativistic limit:</u>

$$\alpha \to 1, \quad \beta^i \to 0$$

9:41 AM Tue Jan 9

Lecture 6

Lecture 6 | Chap_2_Equations

657,791

6) <u>Special-relativistic limit:</u>

$$\alpha \to 1, \quad \beta^i \to 0, \quad \gamma_{ij} \to \eta_{ij}$$

$$(\tilde{v}^i \to v^i)$$

$$\rightsquigarrow \quad u = (D, S_i, \tau)$$

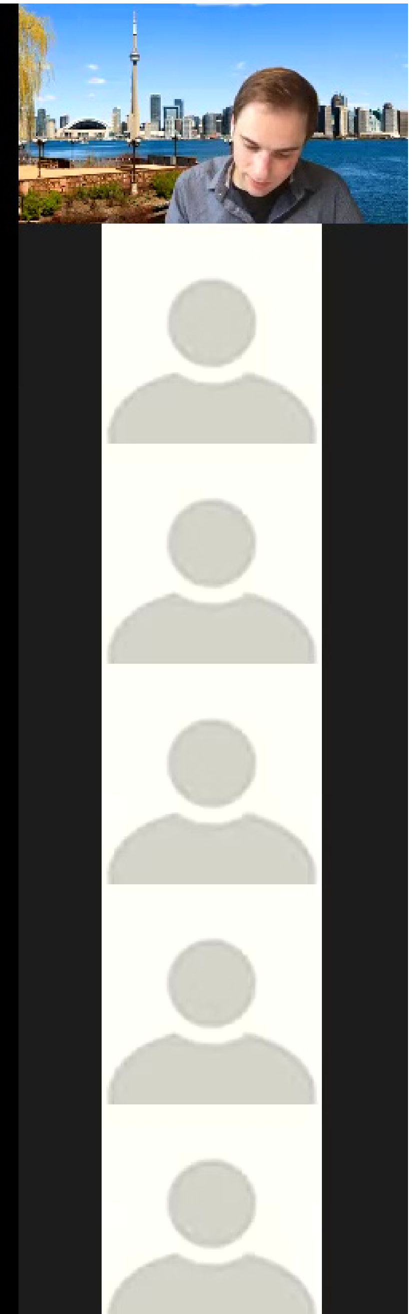
$$f^i(u) = (D v^i, \; S_j v^i + p\delta^i{}_j, \; \tau v^i + p v^i)$$

$$S(u) = 0$$

7) <u>Newtonian limit:</u>

$$\frac{v}{c} \ll 1, \quad W \to 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2$$

$$\frac{p}{\rho c^2} \ll 1, \quad h \to 1$$



9:41 AM Tue Jan 9

Lecture 6

Lecture 6 Chap_2_Equations

$$S(u) = 0$$

7) Newtonian limit:

$$\frac{v}{c} \ll 1, \quad W \to 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2$$

$$\frac{p}{\rho c^2} \ll 1, \quad h \to 1 \quad \left(1 + \varepsilon + \cancel{\frac{p}{\rho}}\right)$$

$$\tau \to E = \frac{1}{2}\rho v^2 + \rho\varepsilon$$

$$\rightsquigarrow \quad u = (\rho, \rho v_j, E)$$

$$f^i(u) = (\rho v^i, \rho v^i v_j + p\delta^i{}_j, (E+p)v^i)$$

$$S(u) = 0$$

(cf. Sec. 2.2)

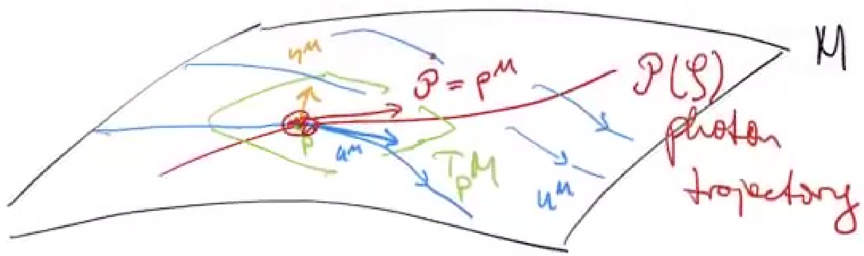
9:41 AM Tue Jan 9

Lecture 6

Lecture 6 | Chap_2_Equations

2.7 Relativistic radiation transfer

Reading: Thorne 1981 MNRAS 194, 439



$$p^\mu p_\mu = 0$$

$$p^\mu = -p^\nu u_\nu \left(u^\mu + n^\mu \right)$$

$$\ell \equiv \int (-p^\mu \cdot u_\mu)\, d\varsigma \,, \qquad d\ell = (-p^\mu u_\mu)\, d\varsigma$$

$$f(x^\mu, p^\mu)$$

photon trajectory

u^μ T_pM u^μ

$$\boxed{p^\mu p_\mu = 0}$$

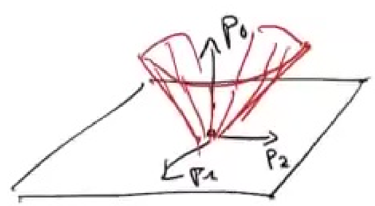
$$p^\mu = -p^\nu u_\nu (u^\mu + n^\mu)$$

$$\ell \equiv \int (-p^\mu \cdot u_\mu) d\varsigma, \quad d\ell = (-p^\mu u_\mu) d\varsigma$$

$f(x^\mu, p^\mu)$

x^μ → point in spacetime p

p^μ → 4-momentum on light cone in T_pM



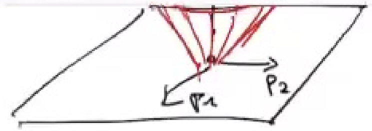
actually: $f = f(t, x^i, p^i)$ as p^0 determined by $p_\mu p^\mu = 0$

$$\frac{df}{d\ell} =$$

$f(x^\mu, p^\mu)$

x^μ: point in spacetime $\mathcal{P}$

p^μ: 4-momentum on light cone in $T_{\mathcal{P}} M$

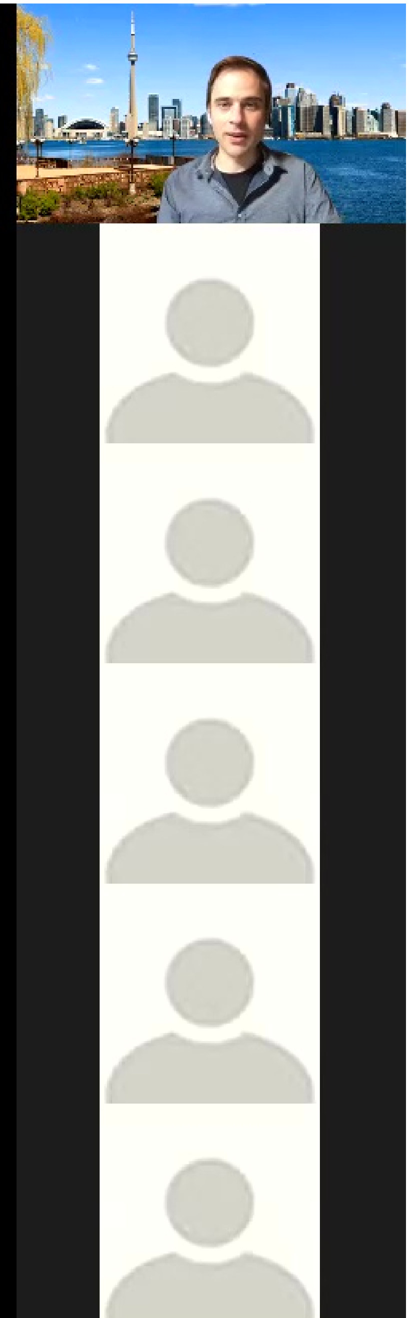


actually: $f = f(t, x^i, p^i)$ as p^0 determined by $p_\mu p^\mu = 0$

$$\boxed{\frac{df}{d\ell} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\ell} + \frac{\partial f}{\partial p^i}\frac{dp^i}{d\ell} = S(x^\mu, p^\mu, f)}$$

Thorne 1981: moment scheme

Shibata 2011:



9:41 AM Tue Jan 9

Lecture 6

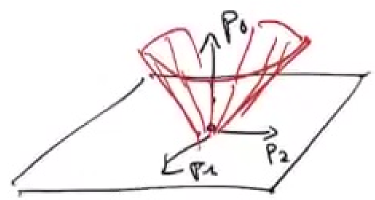
Lecture 6 | Chap_2_Equations

$$\ell \equiv \int (-p^\mu \cdot u_\mu)\, d\varsigma, \qquad d\ell = (-p^\mu u_\mu)\, d\varsigma$$

$f(x^\mu, p^\mu)$

x^μ: point in spacetime $\mathcal{P}$

p^μ: 4-momentum on light cone in $T_{\mathcal{P}} M$



actually: $f = f(t, x^i, p^i)$ as p^0 determined by $p_\mu p^\mu = 0$

$$\boxed{\frac{df}{d\ell} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\ell} + \frac{\partial f}{\partial p^i}\frac{dp^i}{d\ell} = S(x^\mu, p^\mu, f)}$$

Thorne 1981: moment scheme

Shibata 2011: Prog. Theo. Phys. 125, 1255