Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 3

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Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

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Lecture 3

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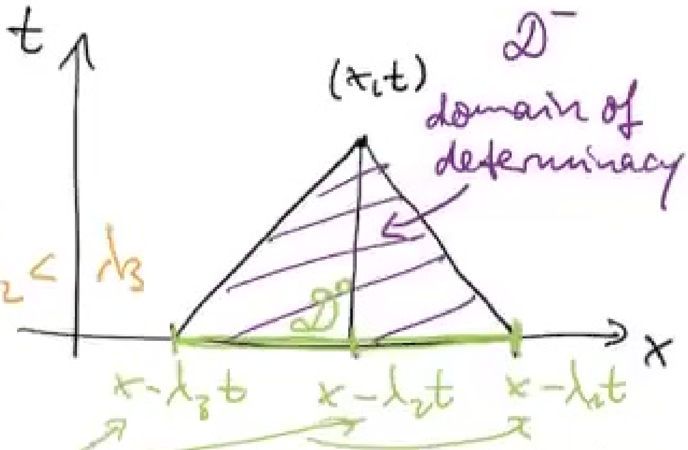
$m = 3$ system

$\lambda_1 < 0 < \lambda_2 < \lambda_3$



linear 1D system

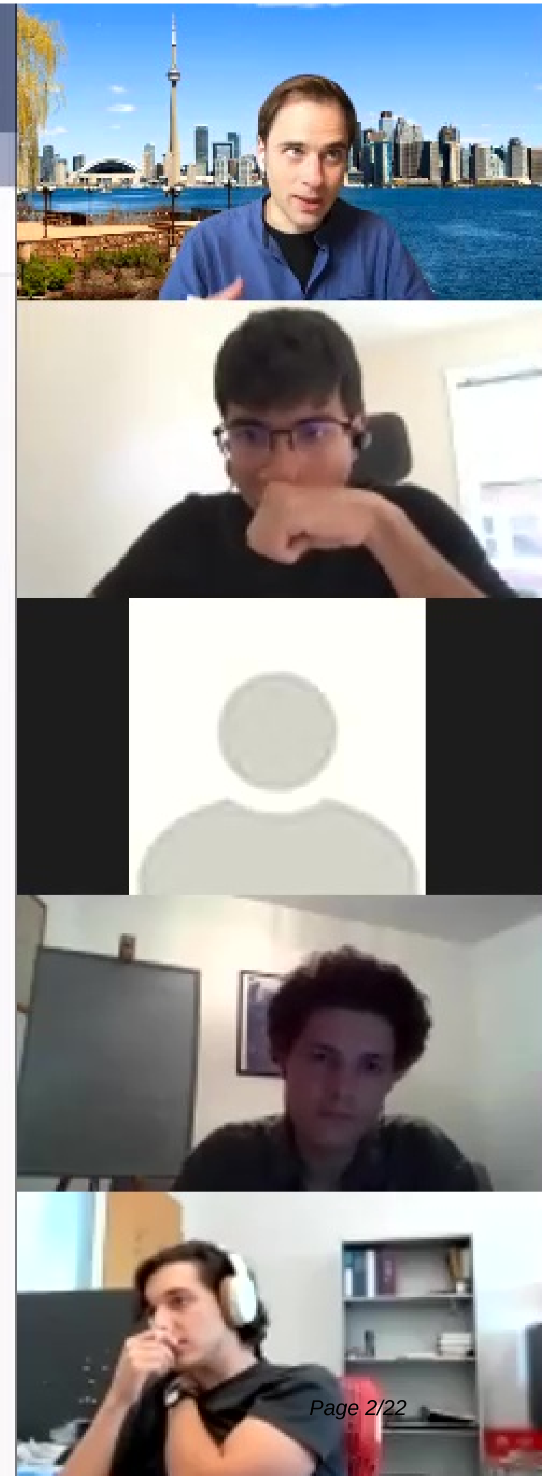
$U_t + A U_x = 0$

A const.

$\leadsto U(x,t) = \sum_{i=1}^{m} v_i^0 (x - \lambda_i t)\, r_i$

$$\mathcal{D}(x,t) = \{ x - \lambda_i t \mid i = 1, \dots, m \}$$

Def: Let $x_0 \in \mathbb{R}^n$. The range of influence $\mathcal{D}^+ \subseteq \mathbb{R}^n \times (0,\infty)$ is the set of points (x,t) in which the solution $U(x,t)$ is influenced by initial data $U^0(x_0)$ at the point



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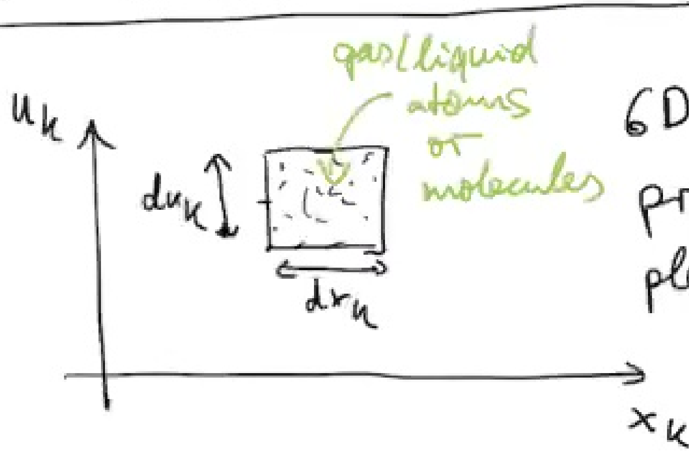
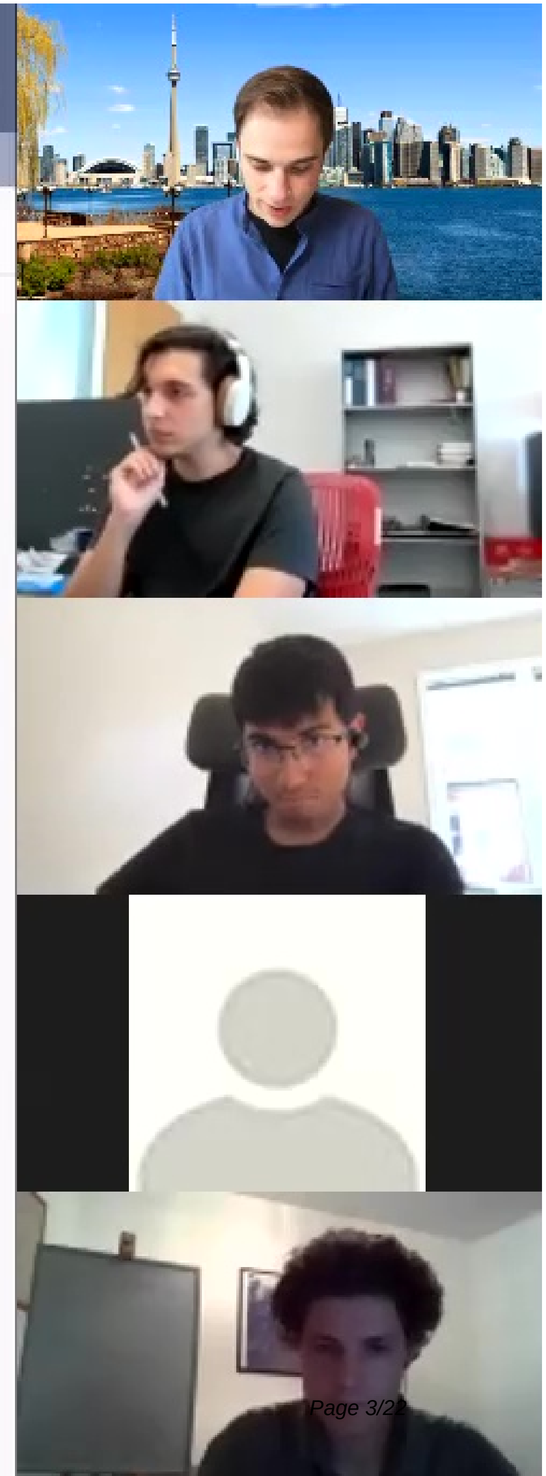
Chaper 2: Basic Equations

2.1 Continuous media: Boltzmann Eq.



6D phase space projected onto $x_k - u_k$ plane

In principle evolution of system fully



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$$\frac{d\vec{x}_i}{dt} = u_i \,, \quad \frac{d\vec{u}_i}{dt} = \vec{F}_i(x_j, u_j, t) \quad \forall \text{ particles } i,j$$

force on i

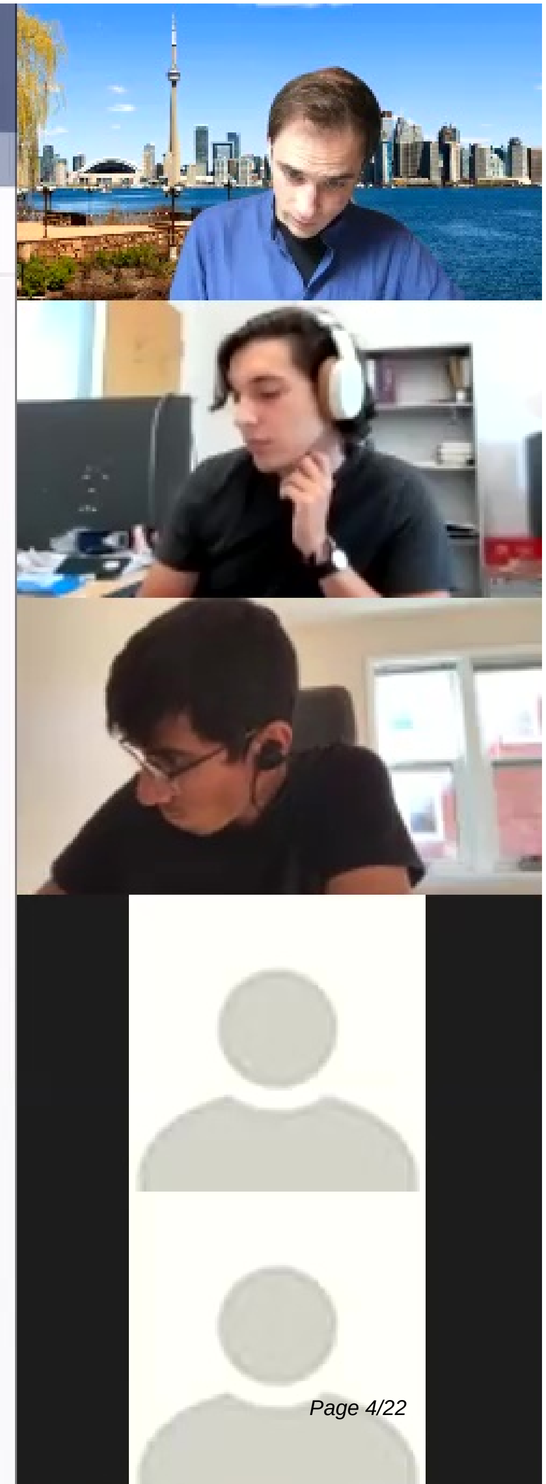
<u>But</u>: 1 mole $\hat{=}$ $N = 6 \times 10^{23}$ particles

→ computationally prohibitive

→ <u>statistical approach:</u>

$$dN = \underbrace{f(\vec{x}, \vec{u}, t)}_{\text{distribution function}} \, d\vec{x}\, d\vec{u}$$

number of particles in control volume $d\vec{x}\,d\vec{u}$

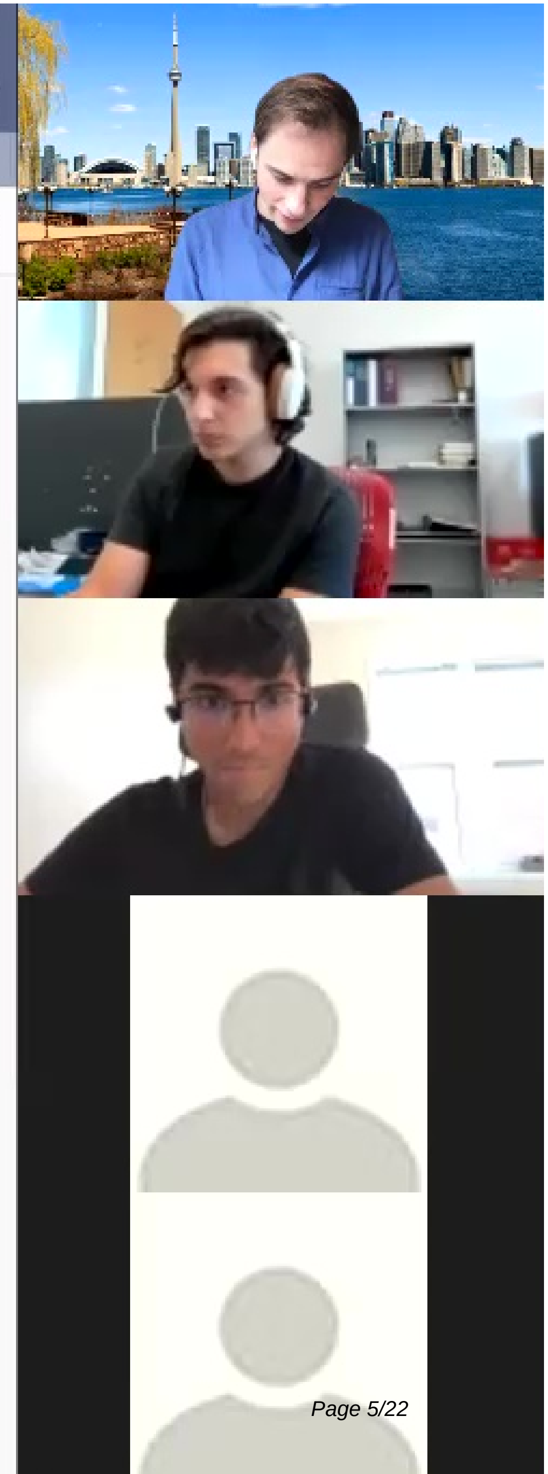


$$dN = \underbrace{f(\vec{x}, \vec{u}, t)}_{\text{distribution function}} \, d\vec{x}\, d\vec{u}$$

number of particles in control volume $d\vec{x}\, d\vec{u}$

Assume particles subject to external force per unit mass $\vec{F}$ ($\approx$ constant over typical interparticle separation)

$\rightsquigarrow$ particles number $f(\vec{x}, \vec{u}, t)\, d\vec{x}\, d\vec{u}$ conserved along their trajectories in phase space (Liouville's theorem) in absence of interactions:



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$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial u_i} = \left[\frac{\partial f}{\partial t}\right]_{coll}}$$

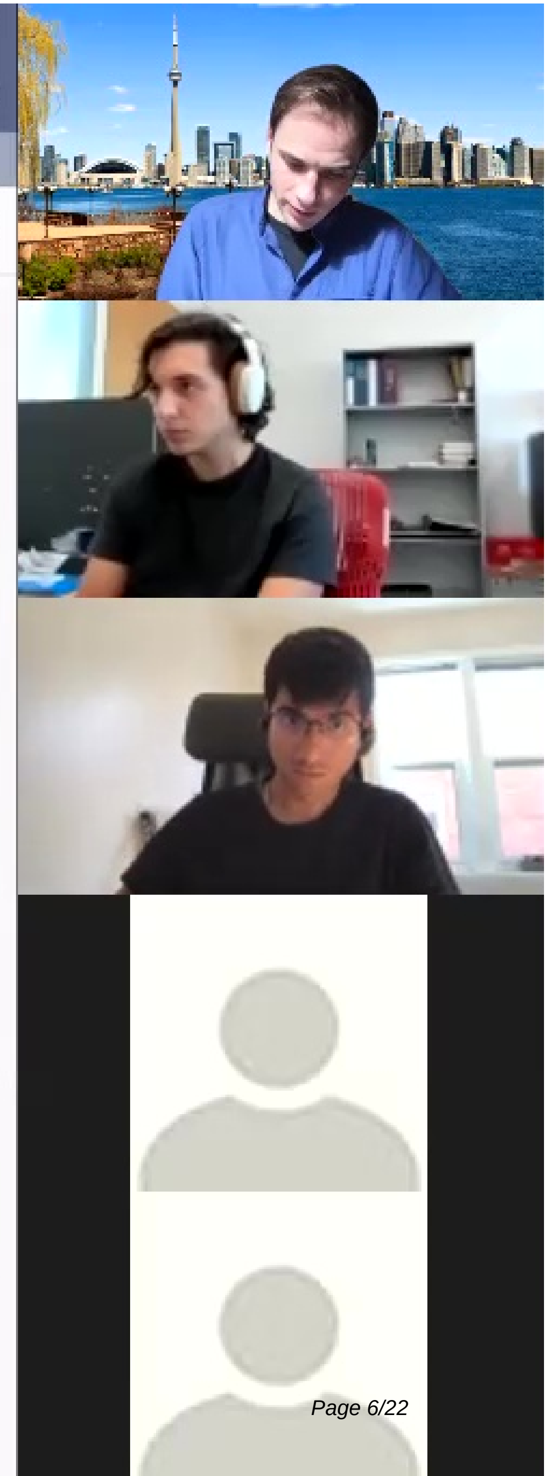
→ evolution of distribution function in 6D phase space

Boltzmann transport equations

Some definitions:

- $n(\vec{x}, t) \equiv \int \underbrace{f(\vec{x}, \vec{u}, t)\, d\vec{u}}_{\text{\# particles per unit volume in } \vec{u},\ \vec{u}+d\vec{u}}$

total number of particles per unit volume



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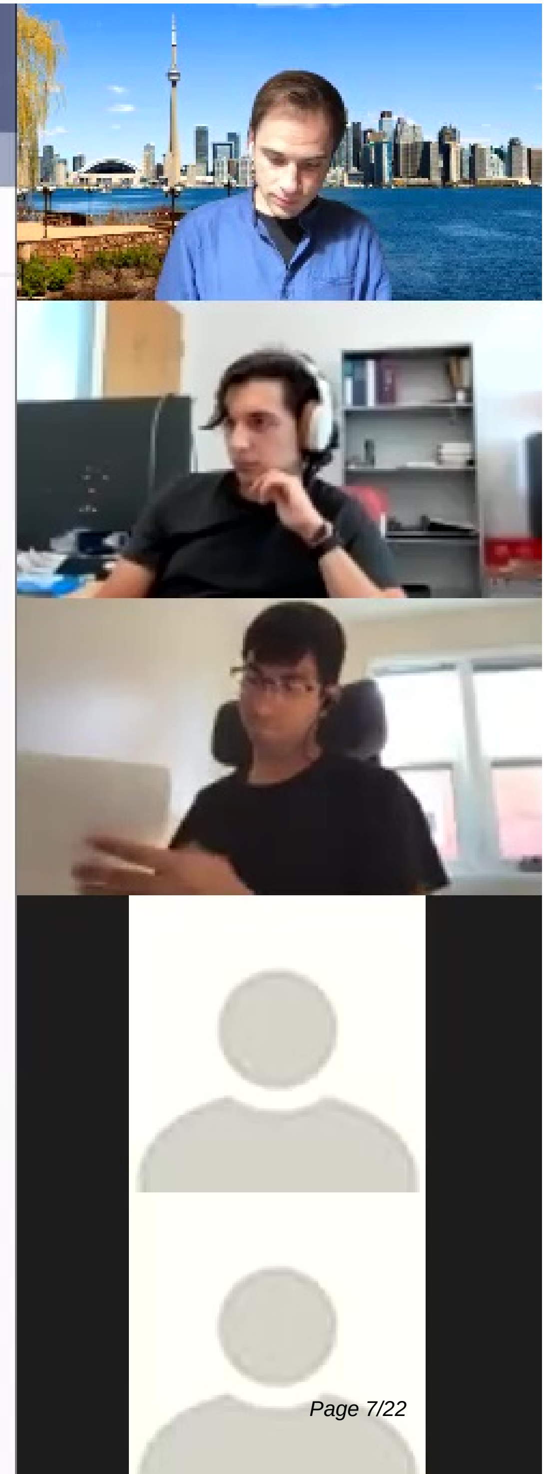
\# particles per unit volume in $\vec{u}, \vec{u}+d\vec{u}$

- $\ell \equiv n^{1/3}$ mean particle separation

Note: any physical length scale dx we are interested in must be $\gg \ell$ for statistical approach to be valid

- $$\rho(\vec{x},t) \equiv \underbrace{m}_{\text{mass of particle}} \int f(\vec{x},\vec{u},t)\, d\vec{u}$$ mass density



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mass of particle

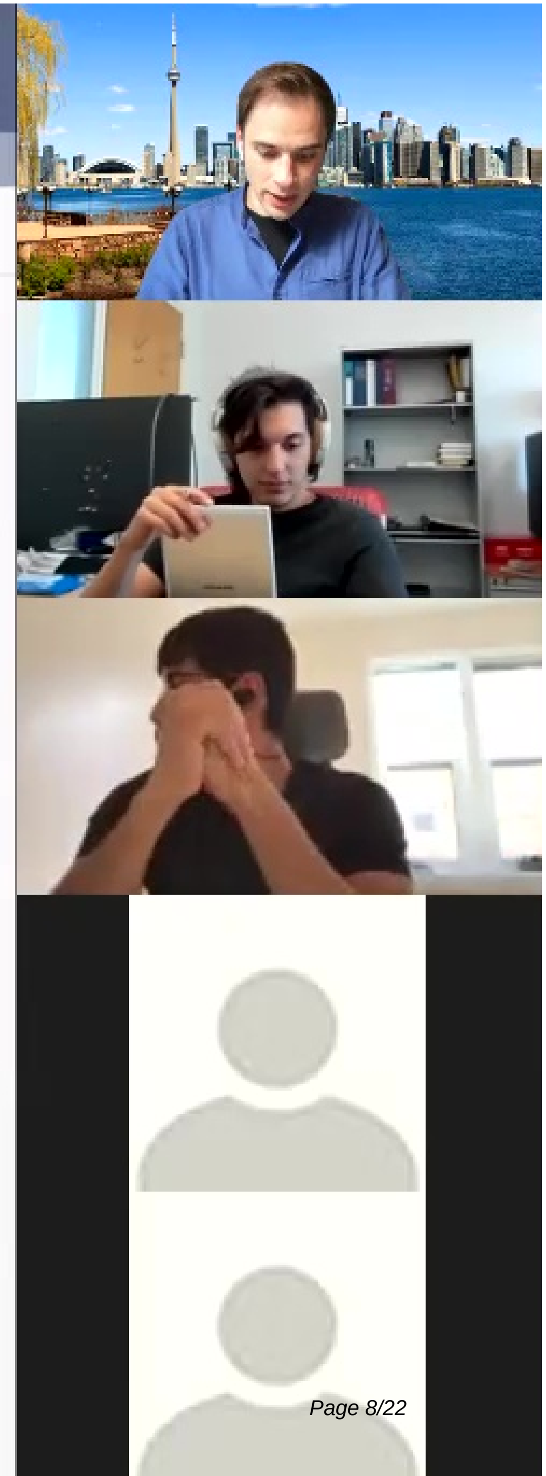
- $\vec{v}(\vec{x},t) = \frac{1}{\rho}\int \vec{u}\, m\, f(\vec{x},\vec{u},t)\, d\vec{u}$ bulk velocity

- $\rho\varepsilon(\vec{x},t) = \frac{1}{2}\int m\, \tilde{u}^2 f(\vec{x},\vec{u},t)\, d\vec{u}$ specific internal energy ε

$\vec{\tilde{u}} \equiv \vec{u} - \vec{v}$ peculiar velocity

<u>Special case:</u>

- elastic collisions (energy & momentum conserved)
- low density
- absence of external forces $\vec{F} = 0$



velocity distribution $\exp\left[-\frac{\dots}{2k_B T(\vec{x},t)}\right]$ ...

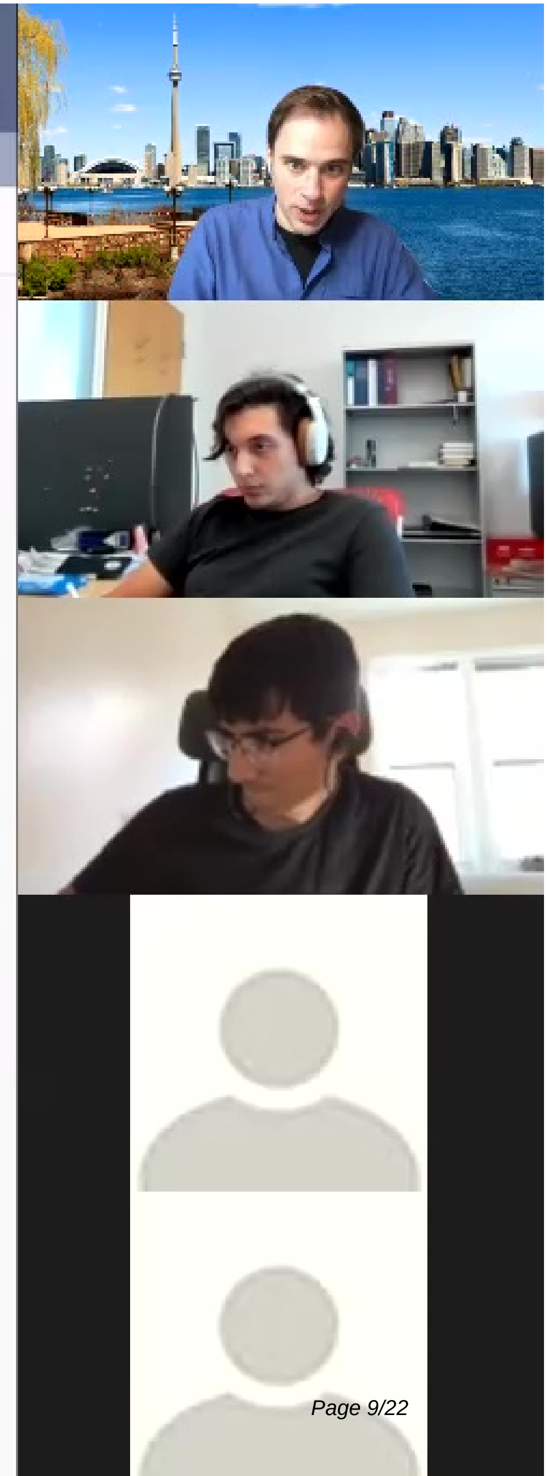
2.2 From Boltzmann to Euler

<u>Def:</u> k-th moment of the Boltzmann equation

$$\int U_k \left[\frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial u_i}\right] d\vec{u} = \int U_k \left[\frac{\partial f}{\partial t}\right]_{coll} d\vec{u}$$

$$U_k \equiv \vec{u}^{\,k}, \quad U_0 = 1, \quad U_1 = \vec{u}, \quad U_2 = \vec{u}^{\,2}$$

<u>General properties of collision term:</u>



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$$\int U_k \left[\frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial u_i}\right] d\vec{u} = \int U_k \left[\frac{\partial f}{\partial t}\right]_{coll} d\vec{u}$$

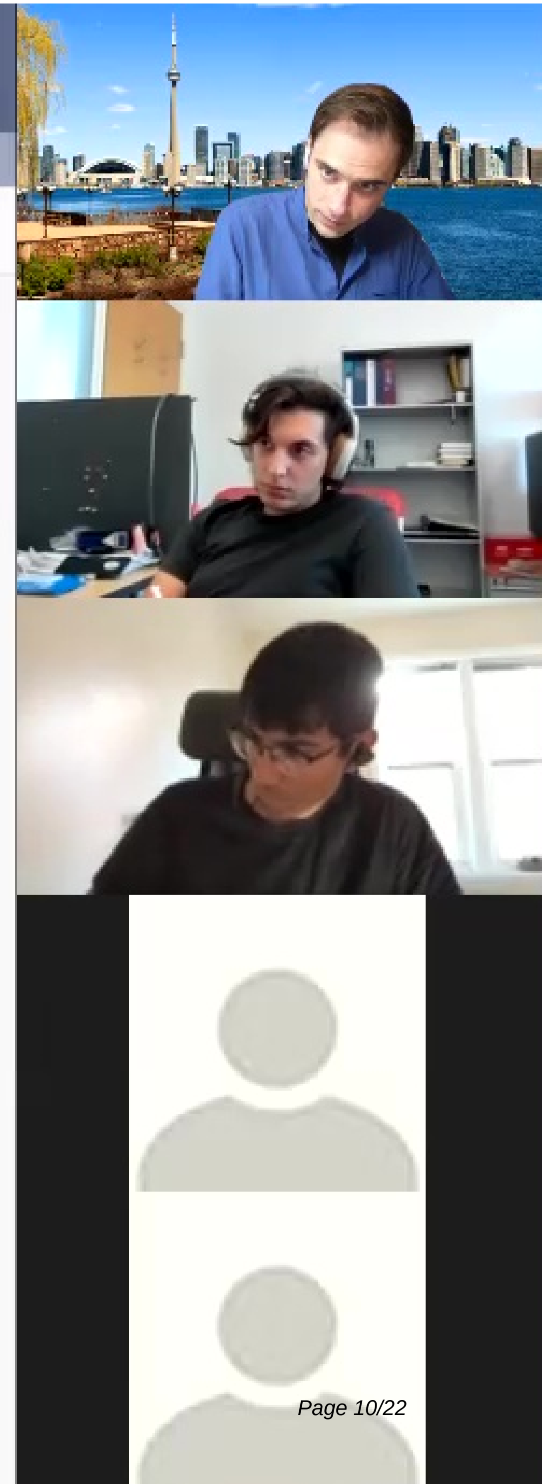
$$U_k = \vec{u}^{\,k}, \quad U_0 = 1, \quad U_1 = \vec{u}, \quad U_2 = \vec{u}^{\,2}$$

<u>General properties of collision term:</u>

If collisions are elastic and neither create nor destroy particles, then

$$\int \left[\frac{\partial f}{\partial t}\right]_{coll} d\vec{u} = 0 \qquad \text{number of particles is conserved}$$

$$\int \left[\frac{\partial f}{\partial t}\right]_{coll} u_i \, d\vec{u} = 0 \qquad \text{total momentum conservation}$$



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create nor destroy particles, then

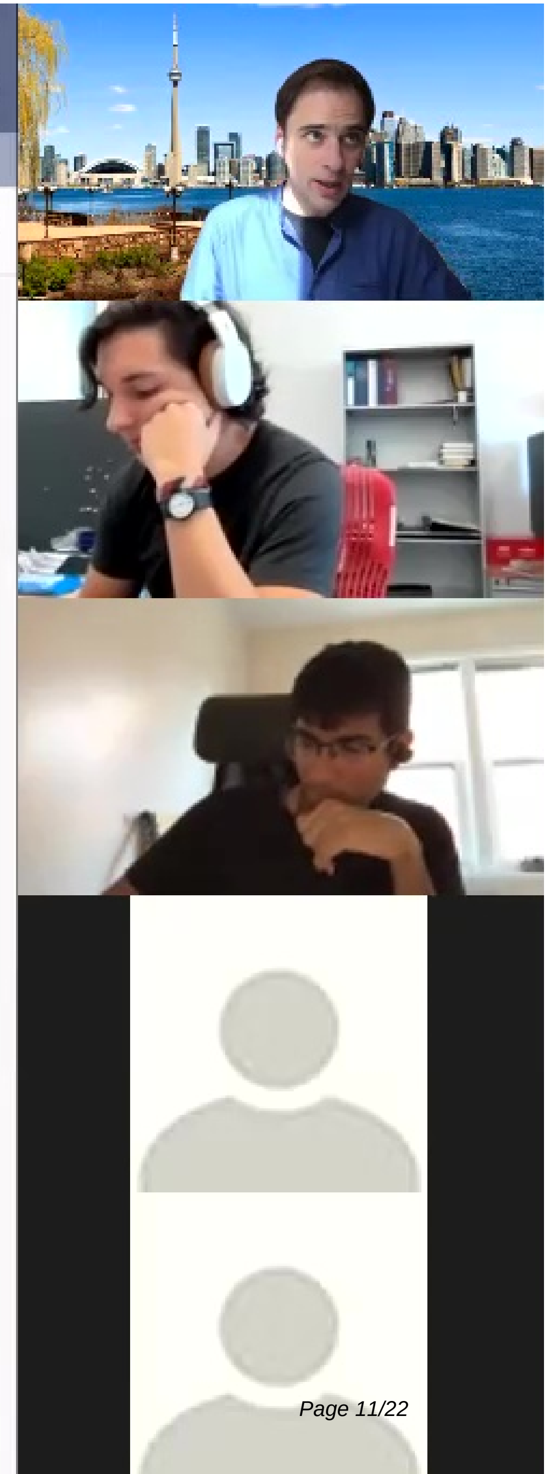
$$\int \left[\frac{\partial f}{\partial t}\right]_{coll} d\vec{u} = 0$$ number of particles is conserved

$$\int \left[\frac{\partial f}{\partial t}\right]_{coll} u_i \, d\vec{u} = 0$$ total momentum conservation

$$\int \left[\frac{\partial f}{\partial t}\right]_{coll} u^2 d\vec{u} = 0$$ total energy conserved

$$\lim_{u \to 0} \vec{u}^k f = 0$$

Also: $$\int 1 \cdot \frac{\partial f}{\partial u_i} d\vec{u} \overset{P.I.}{=} \int_{|u| = \pm\infty} f \, u_i \, d\Gamma - \int \frac{\partial (1)}{\partial u_i} f \, d\vec{u}$$



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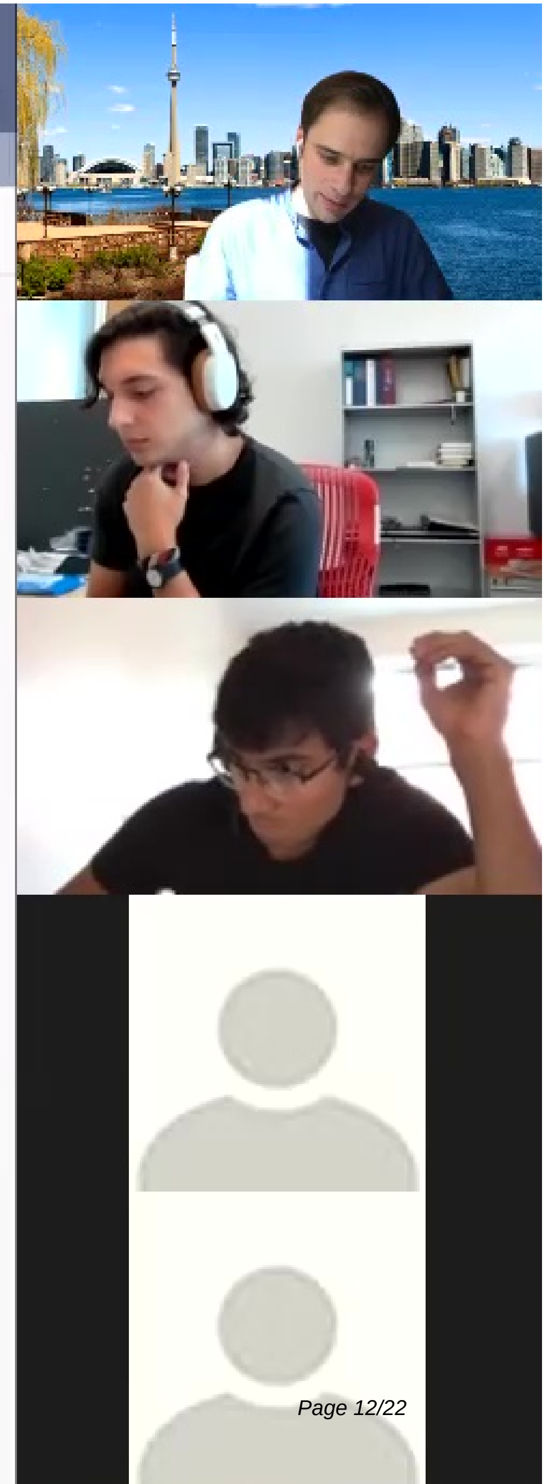
$$\lim_{u \to 0} \vec{\partial}^k f = 0$$

Also:

$$\boxed{\int 1 \cdot \frac{\partial f}{\partial u_i}\, d\vec{u} \overset{P.I.}{=} \int_{|u|=\pm\infty} f\, n_i\, d\Gamma - \int \underbrace{\frac{\partial(1)}{\partial u_i}}_{=0} f\, d\vec{u} = 0}$$

$$\boxed{\int u_j \frac{\partial f}{\partial u_i}\, d\vec{u} \overset{P.I.}{=} \mp\, \delta_{ij} \frac{\rho}{m}}$$

$$\frac{1}{2}\int u^2 \frac{\partial f}{\partial u_i}\, d\vec{u} = \frac{1}{2}\int_{|u|=\pm\infty} u^2 f\, n_i\, d\Gamma$$



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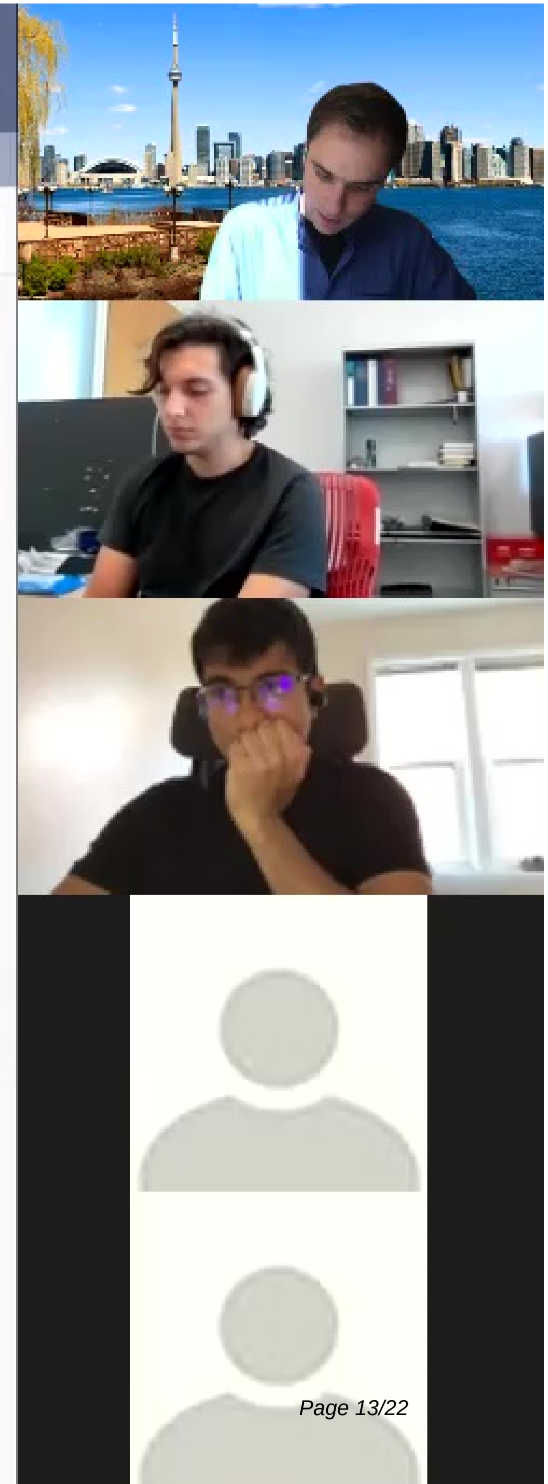
① From 0-moment of Boltzmann equation:

$$m\int \frac{\partial f}{\partial t}\, d\vec{u} + m\int u_i \frac{\partial f}{\partial x_i}\, d\vec{u} + m F_i \underbrace{\int \frac{\partial f}{\partial u_i}\, d\vec{u}}_{=0} = \int \left[\frac{\partial f}{\partial t}\right]_{coll} d\vec{u}$$

x_i, u_i independent

$$\Leftrightarrow \frac{\partial}{\partial t}\int m f\, d\vec{u} + \frac{\partial}{\partial x_i}\int u_i m f\, d\vec{u} = 0$$

$$\overset{\text{Def}}{\Leftrightarrow} \boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \vec{v}) = 0}$$ continuity equation



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$$\rightsquigarrow \int \frac{\partial \rho}{\partial t}\, dV + \underbrace{\int \nabla\cdot(\rho\vec{v})\, dV}_{\overset{\text{Gauss}}{\underset{\text{theorem}}{=}} \int_{\partial V} \rho\, v^i\, n_i\, d\Omega \;=\; 0}$$

$\vec{v} = 0$

$$\Rightarrow \frac{\partial}{\partial t}\int_V \rho\, dV = \frac{\partial M_V}{\partial t} = 0$$

mass conservation

② 1st moment of Boltzmann equat

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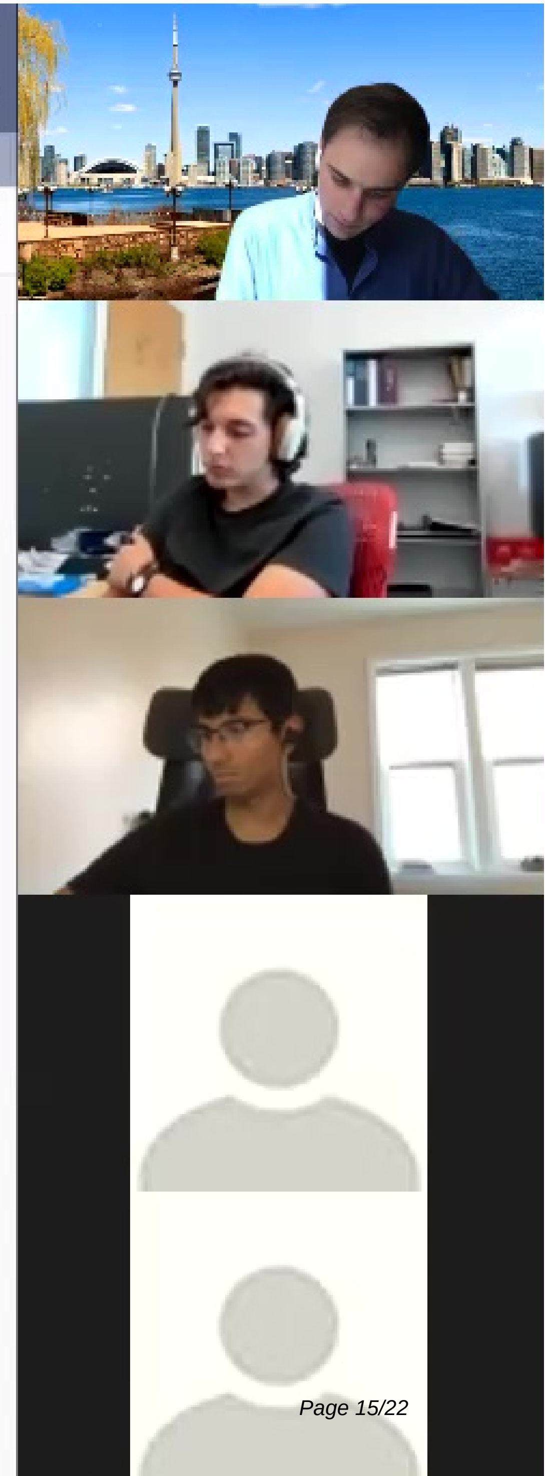
② 1st moment of Boltzmann equation:

$$m\int u_i \frac{\partial f}{\partial t}\, d\vec{u} + m\int u_i u_j \frac{\partial f}{\partial x_i}\, d\vec{u} + m F_j \underbrace{\int u_i \frac{\partial f}{\partial u_j}\, d\vec{u}}_{-\delta_{ij}\frac{\rho}{m}}$$

$$= \int u_i \left[\frac{\partial f}{\partial t}\right]_{coll} d\vec{u} = 0$$

$$\Leftrightarrow \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_i}\int m\, u_i u_j f\, d\vec{u} - \rho F_i = 0$$

$$\int m\, u_i u_j f\, d\vec{u} = \int m(\tilde{u}_i + v_i)(\tilde{u}_j + v_j) f\, d\vec{u}$$



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$$\int m\, u_i u_j\, f\, d\vec{u} = \int m\,(\tilde{u}_i + v_i)(\tilde{u}_j + v_j)\, f\, d\vec{u}$$

$$= \int m\, v_i v_j\, f\, d\vec{u} + \underbrace{\int m\, \tilde{u}_i \tilde{u}_j\, f\, d\vec{u}}$$

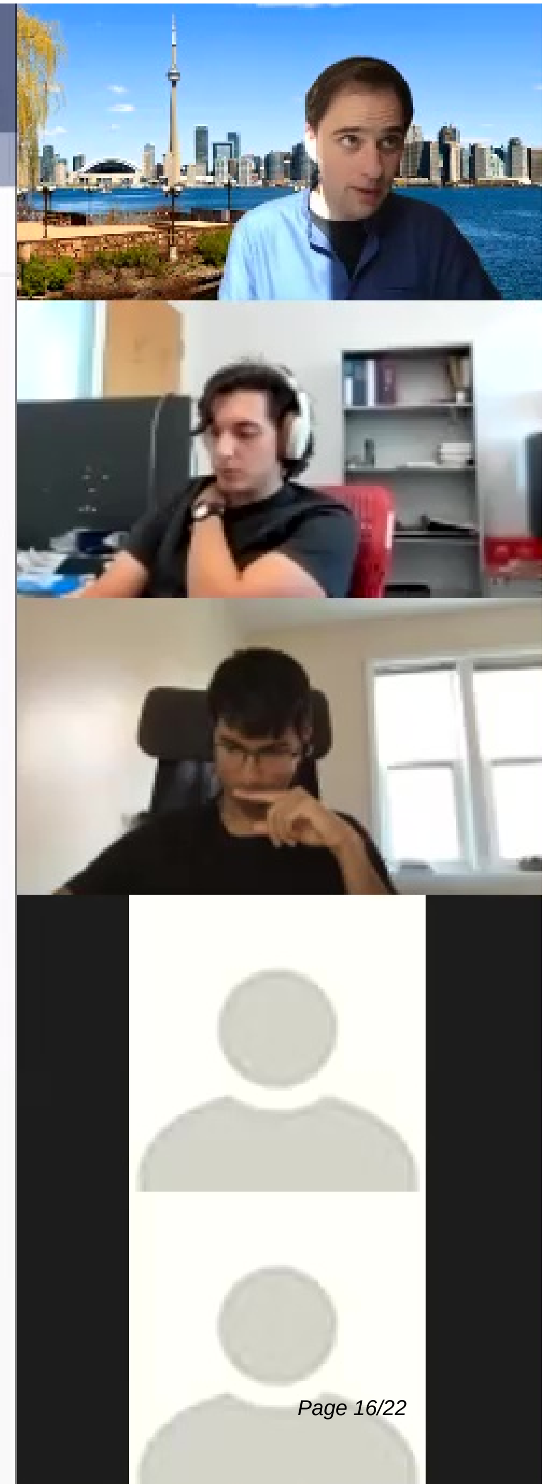
$$\equiv \rho\, v_i v_j + P_{ij}$$ pressure tensor

most cases

isotropic pressure $\rightsquigarrow$ $P_{ij} = P\,\delta_{ij}$

$$P \equiv \frac{1}{3} \int m\, \tilde{u}^2 f\, d\vec{u}$$

$$\left(P = \frac{2}{3}\rho\varepsilon\right)$$



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③ 2nd-moment of Boltzmann eqn:

(Exercise):

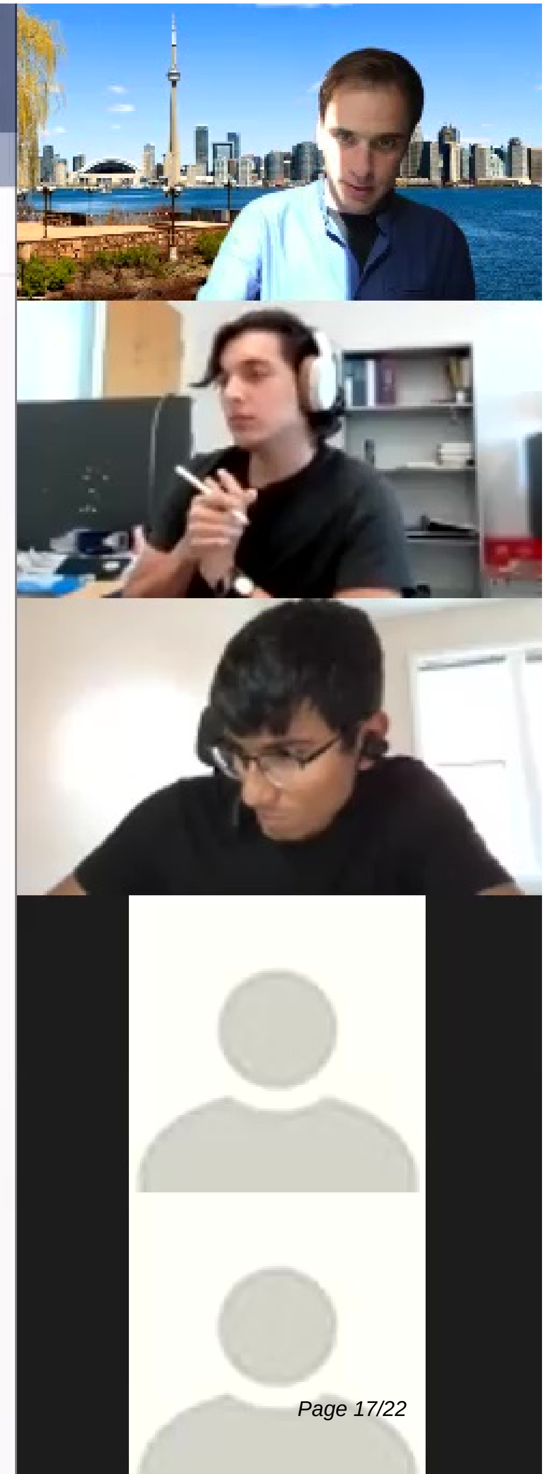
$$\frac{\partial}{\partial t}\left[\rho\left(\frac{v^2}{2}+\varepsilon\right)\right] + \frac{\partial}{\partial x_i}\left[\rho v_i\left(\frac{v^2}{2}+\varepsilon\right)\right] = -\frac{\partial h_i}{\partial x_i}$$

$$-\frac{\partial}{\partial x_i}\left(P v_i\right) + \rho v_i F_i$$

$$h_i \equiv \int \frac{m}{2} \tilde{u}_i \tilde{u}^2 f \, d\vec{u}$$ conduction heat flux

(can be neglected for astrophysical systems)

total

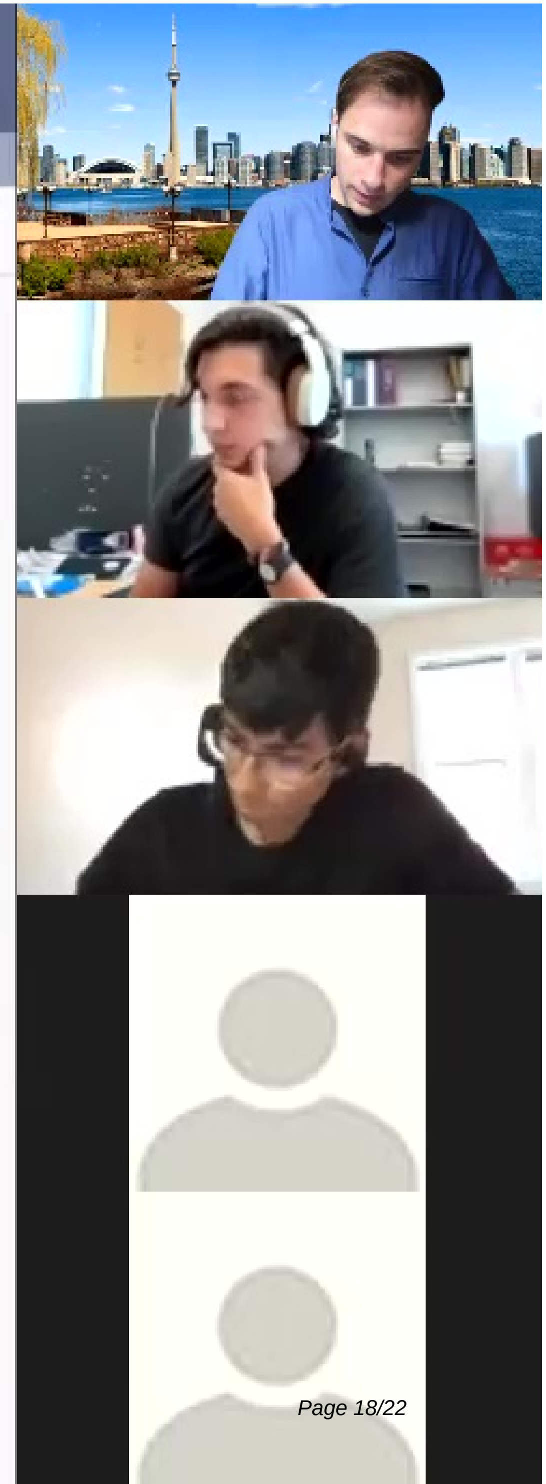


$$E = \frac{1}{2}\rho v^2 + \rho \varepsilon$$

<u>Remarks:</u> 1) (1) (2) (3) are known as the Euler equations. They can be rewritten as a <u>system of conservation laws</u>:

$$\boxed{\begin{aligned} u_t + f^i(u)_{x_i} &= S \\ \partial_t u + \frac{\partial}{\partial x_i} f^i(u) &= S \end{aligned}}$$

where: $u = \begin{pmatrix} \rho \\ \rho \vec{v} \\ \vdots \end{pmatrix}$ "conserved variables"



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$$\boxed{\partial_t u + \frac{\partial}{\partial x_i} f^i(u) = S}$$

where: $u = \begin{pmatrix} \rho \\ \rho \vec{v} \\ E \end{pmatrix}$ "conserved variables"

$$f^i = \begin{pmatrix} \rho v_i \\ \rho v_i v_j + \delta_{ij}\, p \\ v_i (E+p) \end{pmatrix}$$

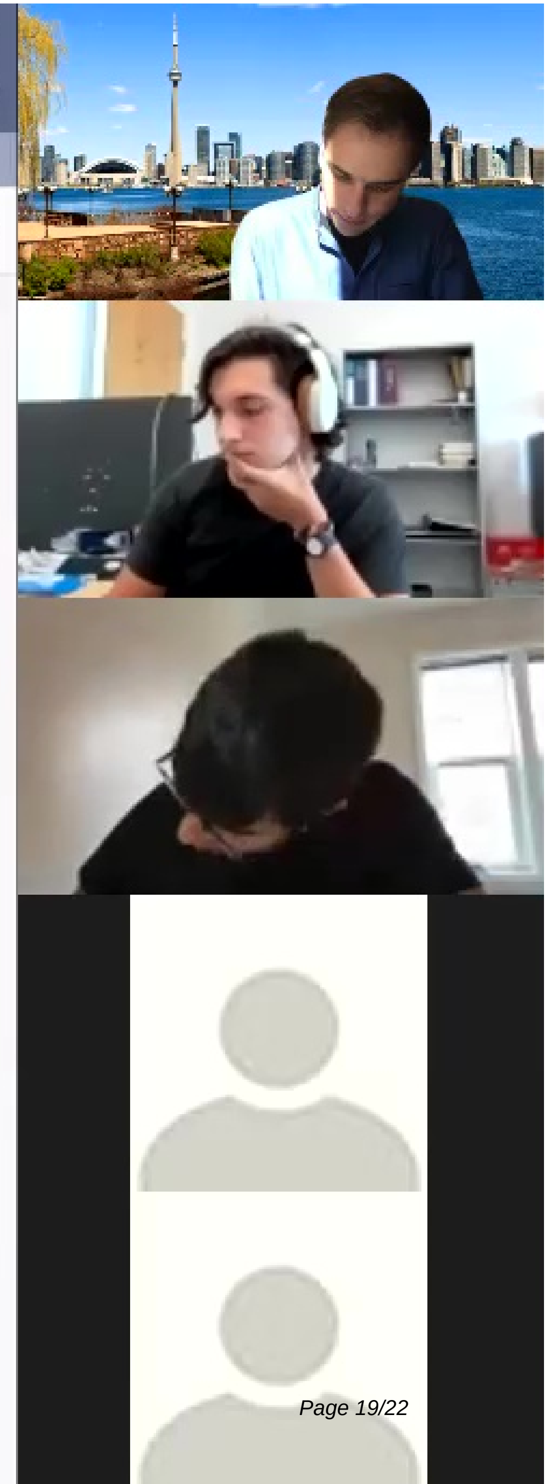
$j = 1, 2, 3$ (momentum components)

$i = 1, 2, 3$ (flux directions)

"fluxes"

$$S = \begin{pmatrix} 0 \\ \rho \vec{F} \\ \ldots \end{pmatrix}$$

"sources"



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$$f^i = \begin{pmatrix} \rho v_i \\ \rho v_i v_j + \delta_{ij}\, p \\ v_i (E+p) \end{pmatrix}$$

$j = 1, 2, 3$ (momentum components)

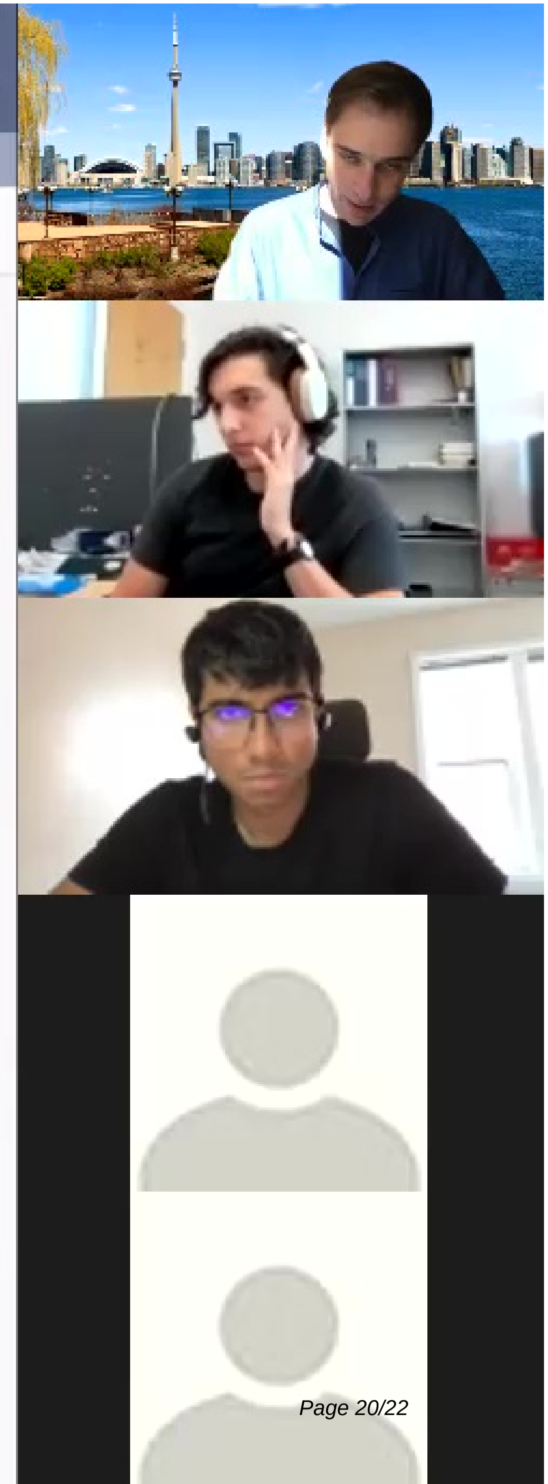
$i = 1, 2, 3$ (flux directions)

"fluxes"

$$S = \begin{pmatrix} 0 \\ \rho \vec{F} \\ \rho \vec{v} \cdot \vec{F} \end{pmatrix}$$

"sources"

2) <u>Equation of state:</u> The Euler equations are five equations for



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2) <u>Equation of state:</u> The Euler equations are five equations for six unknowns $\rho, \{v^i\}, P, e = \rho\varepsilon$

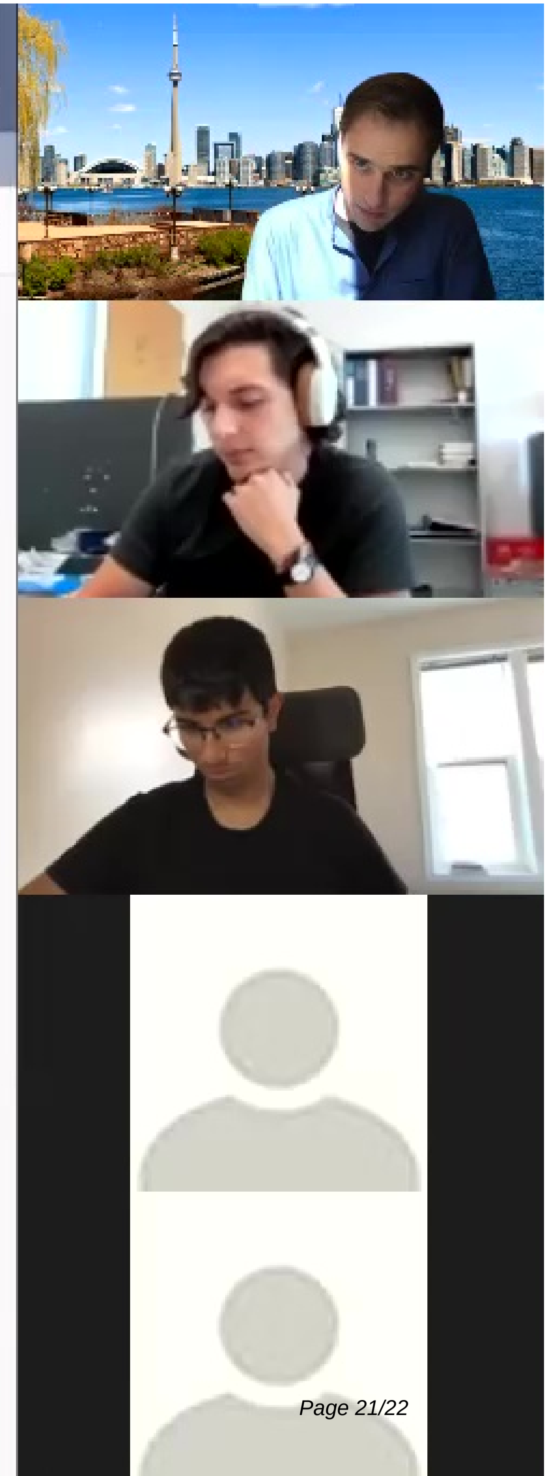
$\Rightarrow$ require relation $P = P(\rho, e)$ to close the system

$\longrightarrow$ "equation of state"

Ideal gas: $P = (\gamma - 1)\, e = (\gamma - 1)\, \rho \varepsilon$

$\gamma = \frac{c_p}{c_v}$ specific heats

(adiabatic constant)



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Ideal gas: $p = (\gamma-1)e = (\gamma-1)\rho\varepsilon$

$\gamma = \frac{c_p}{c_v}$ specific heats

(adiabatic constant)

In general: $p = p(\rho, e, X_i, \ldots)$ can be very complicated

→ different chemical elements, ionization states, complicated chemical reactions etc.