Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 2

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Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

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1.2 First-order PDEs

Def (classification):

$$\left(\begin{array}{l} x \in \mathbb{R}^n,\ \Omega_1, \Omega_2 \subseteq \mathbb{R}^n \\ L : \mathcal{F}_1(\Omega_1) \longrightarrow \mathcal{F}_2(\Omega_2) \\ \quad\ u \longmapsto Lu \end{array}\right)$$

$C^2(\Omega_1)$ $\quad$ $C^0(\Omega_2)$

$$u_{x_i} \equiv \partial_i u = \frac{\partial u}{\partial x_i}$$

(i) non-linear:

$$Lu = L(x, u, \{u_{x_i}\})$$

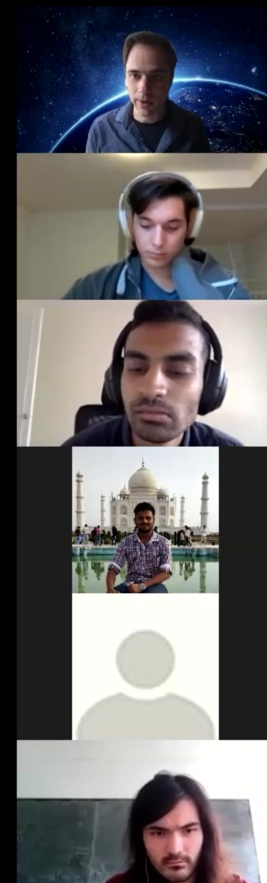
Example: $u_{x_1} - u_{x_2}^2 = 0$

(ii) quasi-linear:

$$Lu = \sum_{k=1}^{n} a_k(x, u)\, u_{x_k} + b(x, u) = 0$$

Example: $u_{x_1} + u\, u_{x_2} = 0$ $\quad$ inviscid

$\partial_t u + u\, \partial_x u = 0$ $\quad$ Burger's equation



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$$Lu = L(x, u, \{u_{x_i}\})$$

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Example: $u_{x_1} - u_{x_2}^2 = 0$

(ii) quasi-linear:

$$Lu = \sum_{k=1}^{n} a_k(x,u)\, u_{x_k} + b(x,u) = 0$$

Example:
$$u_{x_1} + u\, u_{x_2} = 0$$
$$\partial_t u + u\, \partial_x u = 0$$
inviscid Burger's equation

(iii) semi-linear:

$$Lu = \sum_{k=1}^{n} a_k(x)\, u_{x_k} + b(x,u) = 0$$

Example: $u_{x_1} = u$

(iv) linear:

$$Lu = \sum_{k=1}^{n} a_k(x)\, u_{x_k} + b(x) = 0$$

Example: $u_{x_1} + a\, u_{x_2} = 0$ linear advection

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equation

<u>Def:</u> let $U = (u_1, \dots, u_m) : I \subseteq \mathbb{R} \times \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$x, \xi \in \Omega$, $A_i(t,x,U)$ $m \times m$ matrices $\forall\, i \in \{1, \dots, n\}$

and

$$\mathcal{A}(t,x,U;\xi) = \sum_{i=1}^{n} A_i(t,x,U)\,\xi_i$$

The <u>quasi-linear system</u>

$$(*) \qquad U_t + \sum_{i=1}^{n} A_i(t,x,U)\,U_{x_i} + B(t,x,U) = 0$$

is called <u>hyperbolic</u> if $\mathcal{A}(t,x,U;\xi)$ is diagonalizable for $x, \xi \in \Omega$, $t \in I$, $u \in \mathbb{R}^m$.

In particular, $(*)$ is hyperbolic

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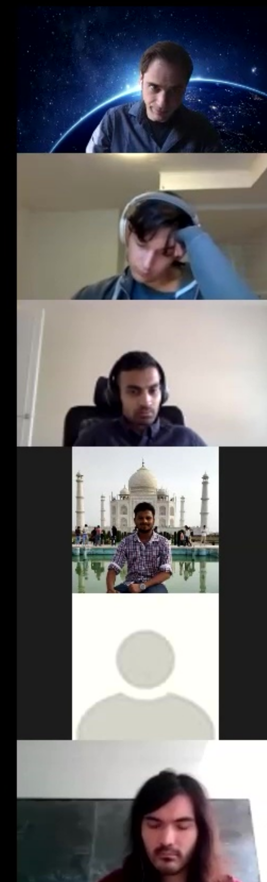

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$\forall\, x, \xi \in \Omega,\ t \in I$ $\mathcal{A}$ has m real eigenvalues

$$\lambda_1(t,x,u;\xi) \le \dots \le \lambda_m(t,x,u;\xi)$$

and corresponding eigenvectors

$$\{ r_i(x,t,u;\xi) \}_{i=1,\dots,m}$$

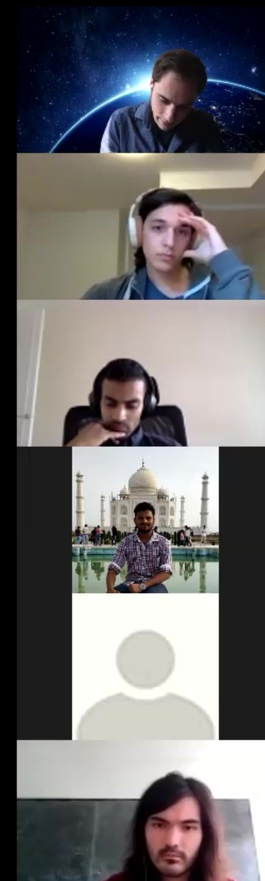
which form a basis of $\mathbb{R}^m$. Two special cases:

1) $A_i(t,x,u)$ symmetric $\forall\, i \in \{1,\dots,n\}$

$\Rightarrow$ $\mathcal{A}$ symmetric

$\Rightarrow$ $\mathcal{A}$ diagonalizable

"(*)



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2) $A(t,x,u;\xi)$ has m real, distinct eigenvalues

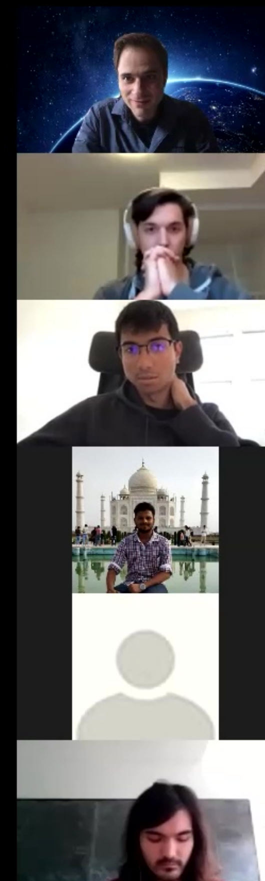
$$\lambda_1(t,x,u;\xi) < \lambda_2(\,) \dots < \lambda_m(\dots)$$

$\forall\, x,\xi \in \Omega,\ t \in I$

$\Rightarrow A$ diagonalizable

"(*) strictly hyperbolic"

Examples:



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$$\lambda_1(t,x,u;\xi) < \lambda_2(\ ) \dots < \lambda_m(\dots)$$

$$\forall\ x,\xi \in \Omega,\ t \in I$$

$\Rightarrow A$ diagonalizable

"(✱) strictly hyperbolic"

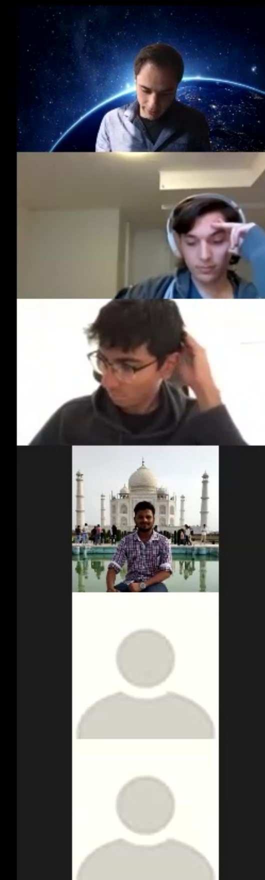
Examples:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{strictly hyperbolic}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \text{symmetric hyperbolic}$$

Remarks: 1) The system (✱) is elliptic if none of the eigenvalues is real.

2) "hyperbolic": 2nd order hyperbolic PDEs can be recast as systems of 1st order hyperbolic PDEs



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hyperbolic PDEs.

Def (Conservation laws): Conservation laws are systems of 1st order PDEs that can be written as

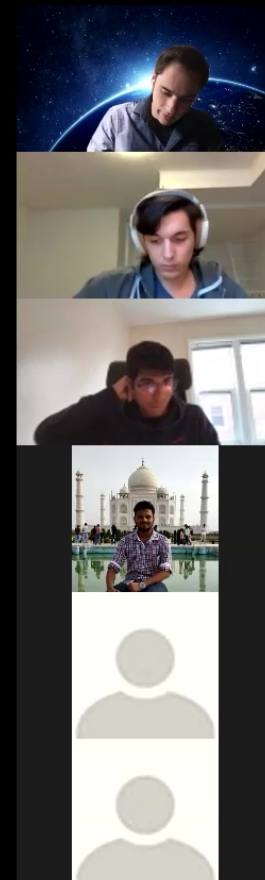
$$u_t + \sum_{i=1}^{n} \underbrace{F^i(u)_{x_i}}_{= \partial_{x_i} F^i(u)} = S(t, x, u)$$

$u = (u_1, \ldots, u_m)$ "conserved variables"

$F^i(u) = (f_1^i(u), \ldots, f_m^i(u))$, $i = 1, \ldots, n$

"vector of fluxes"

$S(t, x, u)$: "sources terms"



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"vector of fluxes"

$S(t,x,u)$: "sources terms"

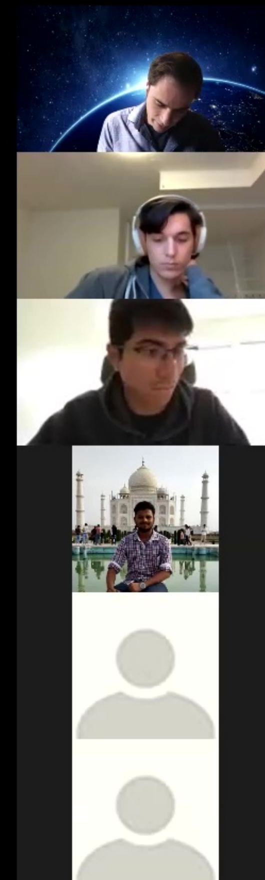
The Jacobians of the flux functions $F^i(u)$ is

$$A_i(u) = \frac{\partial F^i}{\partial u} = \begin{pmatrix} \frac{\partial f_1^i}{\partial u_1} & \cdots & \frac{\partial f_1^i}{\partial u_m} \\ \vdots & & \\ \frac{\partial f_m^i}{\partial u_1} & \cdots & \frac{\partial f_m^i}{\partial u_m} \end{pmatrix}$$

$S \neq 0$: often called "balance law"

Remarks: 1) conservation laws (CL) can be as system of quasi-linear PDEs of 1st order by applying the chain rule:

$\partial_{x_i} F$



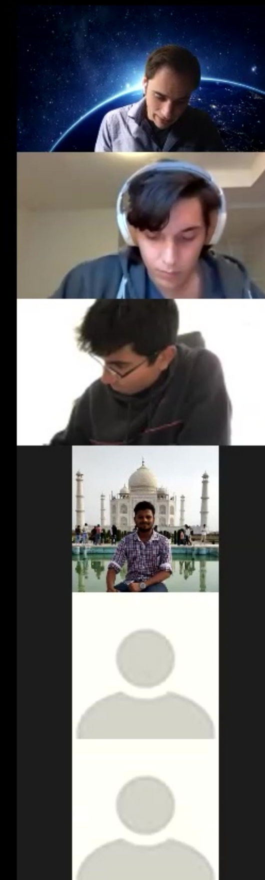
Remarks: 1) conservation laws (CL) can be as system of quasi-linear PDEs of 1st order by applying the chain rule:

$$\partial_{x_i} F^i(u) = \frac{\partial F^i}{\partial u} \frac{\partial u}{\partial x_i} = A_i(u)\, u_{x_i}$$

2) S: arise due to

- physical sources or sinks of otherwise conserved quantities
- due to changes of coordinates "geometric source terms"

3)



3) Motivation: consider integral form

$$\frac{d}{dt}\int U \, dV = -\int \partial_{x_i} F^i(U)\, dV \qquad V\text{: control volume}$$

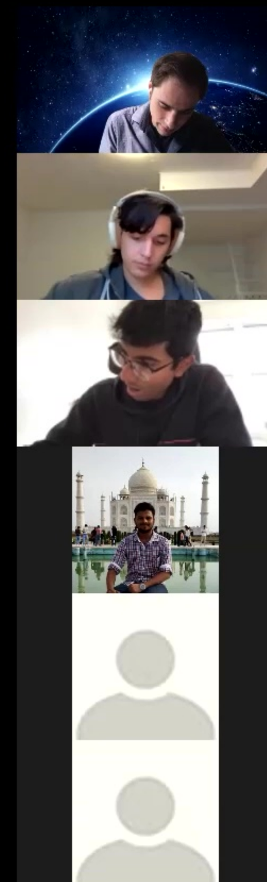
$$\overset{\text{divergence theorem}}{=} -\oint_{\partial V} F^i(U)\, n_i \, dS$$

n_i: unit normal pointing outward

"time-rate change of U inside V depends only on the total flux through the surface S"

$\rightsquigarrow$ "$\underline{U \text{ is conserved}}$"

Example



$\rightsquigarrow$ "u is conserved"

Examples:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 , \quad f(u) = a u$$

$\rightsquigarrow$ linear advection equation

$$f(u) = \frac{1}{2} u^2 \Rightarrow \frac{\partial f}{\partial x} = u u_x$$

$\rightsquigarrow$ inviscid Burger's equation

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1.3 Some properties of 1st order hyperbolic systems

1.3.1 Characteristics

1) Linear hyperbolic systems

Consider hyperbolic system of the form

$$U_t + AU_x = 0 \qquad n=1$$

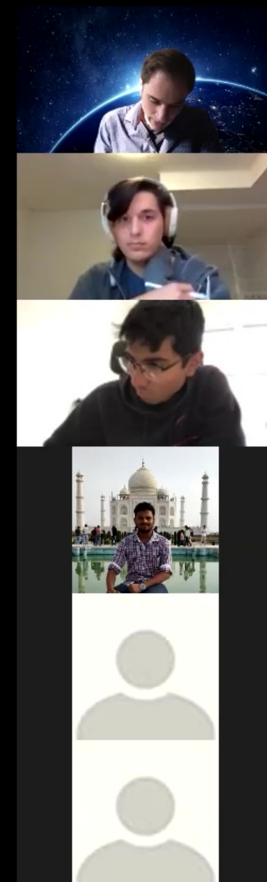
A: const. coefficients

hyperbolicity $\Rightarrow$ $\exists\, Q$ with $A = Q \Lambda Q^{-1}$

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$$

λ_i: eigenvalues of A

Q



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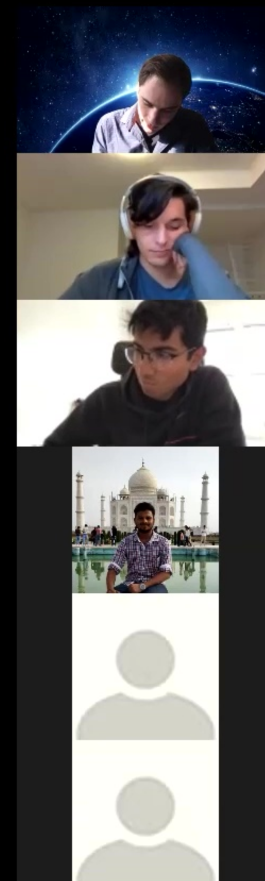
eigenvalues of A

$$Q = (r_1, \dots, r_m)$$

columns

$$\rightsquigarrow \quad u_t + Q \Lambda Q^{-1} u_x = 0$$



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hyperbolicity $\Rightarrow \exists\, Q$ with $A = Q \Lambda Q^{-1}$

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$$

$\uparrow$ eigenvalues of A

$$Q = (r_1, \dots, r_m)$$

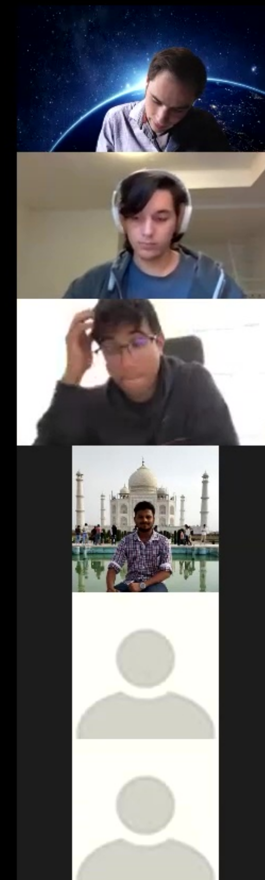
$\uparrow$ columns

$$\rightsquigarrow \quad u_t + Q \Lambda Q^{-1} u_x = 0 \qquad \Big| \times Q^{-1}$$

define $V = Q^{-1} u$ "characteristic variables"

$$\rightsquigarrow \quad \boxed{V_t + \Lambda V_x = 0}$$

deco



define $V = Q^{-1} U$ "characteristic variables"

$\rightsquigarrow \boxed{V_t + \Lambda V_x = 0}$ decoupled system

<u>characteristics</u>: curves $\gamma(t)$ along which the PDE becomes an ODE

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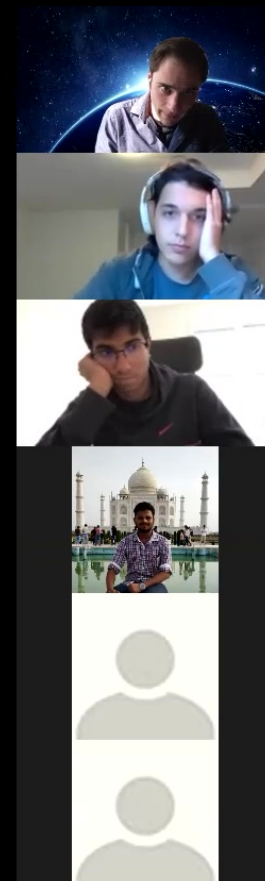

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variables"

$$\leadsto \boxed{V_t + \Lambda V_x = 0}$$

decoupled system

Characteristics: curves $\gamma(t)$ along which the PDE becomes an ODE

Consider $\gamma(t) = x$, $v_i = v_i(\underbrace{\gamma(t)}_{x}, t)$

$$\leadsto \frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + \frac{dx}{dt}\frac{\partial v_i}{\partial x} = \frac{\partial v_i}{\partial t} + \gamma'(t)\frac{\partial v_i}{\partial x}$$

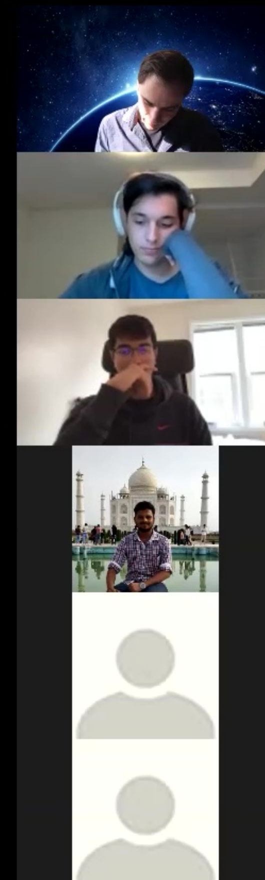
choose $\gamma'(t) = \lambda_i$ "characteristic speeds"

$$\Rightarrow \frac{dv_i}{dt} = 0$$ $\leadsto$ v_i are constant along characteristics

Therefore: given $v_i(x,0) = v_i^0(x)$ at $t=0$

$$v_i(x,t) = v_i^0(x_0) = v_i^0(x - \lambda_i t)$$

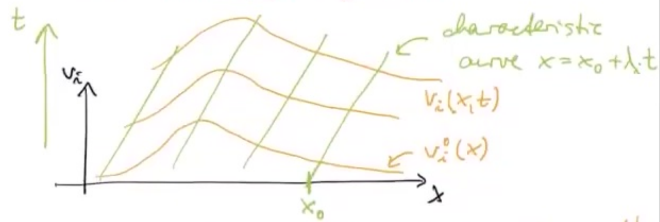
where $x(t) = x_0 + \lambda_i t$ is the characteristic



$\gamma(t)$ that passes through x at t.

$\rightsquigarrow$ v_i at (x,t) is entirely determined by initial data v_i^0 at x_0

$\rightsquigarrow$ "advection equation"



$\longrightarrow$ advection of v_i^0 with speed λ_i

<u>General initial value problem:</u>

Given $\quad U_t + A U_x = 0 \qquad \text{(I)}$

$\qquad\qquad U^0 = (u_1^0, \ldots, u_m^0)$

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$\longrightarrow$ advection of v_i^0 with speed λ_i

General initial value problem:

Given $\quad U_t + AU_x = 0 \qquad \text{(I)}$

$\qquad\qquad U^0 = (u_1^0, \dots, u_m^0)$

$\downarrow$ introduce characteristic variables $V = Q^{-1}U$

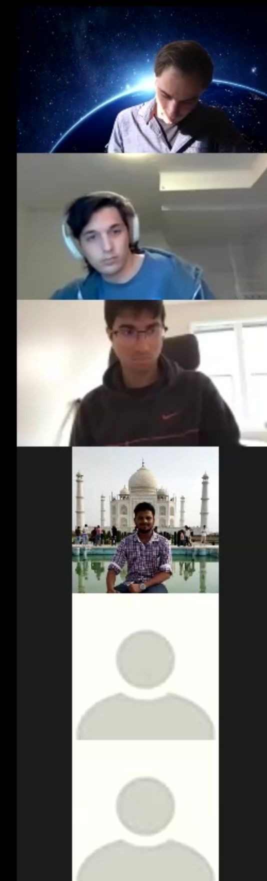
$$\begin{cases} V_t + \Lambda V_x = 0 \\ V^0 = Q^{-1}U^0 = (v_1^0, \dots, v_m^0) \end{cases}$$

$\downarrow$

$$V = \{v_i(x,t)\} = \{v_i^0(x - \lambda_i t)\}, \quad i = 1, \dots, m$$

$\downarrow$ transform back

$$U(x,t) = Q\,V(x,t)$$



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$$\downarrow$$

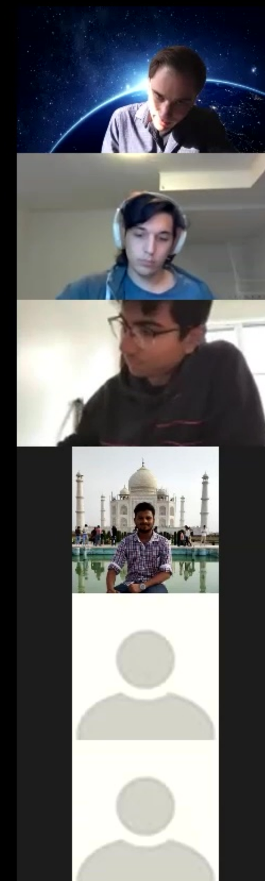
$$V = \{ v_i(x,t) \} = \{ v_i^0(x - \lambda_i t) \}, \quad i = 1, \dots, m$$

$$\downarrow \text{ transform back}$$

$$\boxed{\begin{aligned} U(x,t) &= Q\, V(x,t) \\ &= \sum_{i=1}^{m} v_i(x,t)\, r_i \\ &= \sum_{i=1}^{m} v_i^0(x - \lambda_i t)\, r_i \end{aligned}}$$

solution to (I)

Remarks: • $U(x,t)$ is superposition of eigenvectors i.e. waves propagating at



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$$= \sum_{i=1} v_i^0 (x - \lambda_i t)\, r_i$$

<u>Remarks:</u>

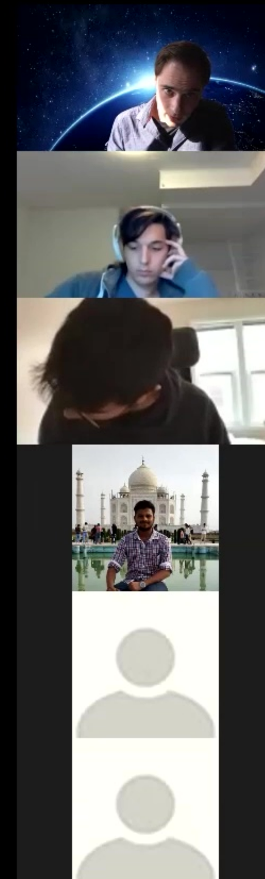
- $U(x,t)$ is superposition of eigenvectors i.e. waves propagating at finite speed $\lambda_i \neq 0$
- $U(x,t)$ entirely determined by initial data v_i^0 at points $x_0^i = x - \lambda_i t$

2) <u>Systems of CLs</u>

Consider hyperbolic system

$$U_t + F(U)_x = 0$$

$\rightsquigarrow$ can locally transform into decoupled system



$$U_t + F(U)_x = 0$$

⇝ can locally transform into decoupled system

hyperbolicity ⇒ $Q = Q(x,t,U)$, $A \equiv \frac{\partial F}{\partial U} = Q \Delta Q^{-1}$

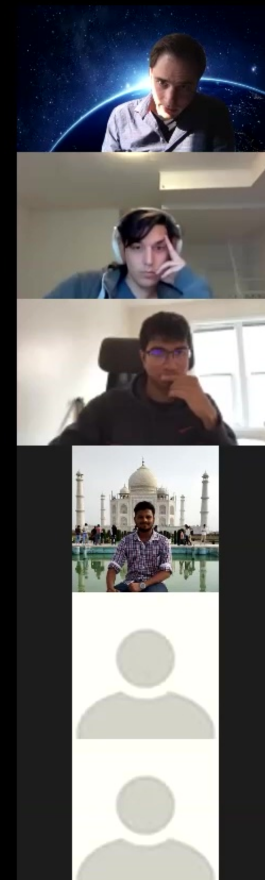
diagonal matrix

<u>locally</u>: $V_t + \Delta V_x = 0$

meaning of V depends on U

Δ: characteristics depend on state vector U

⇝ the state/solution is "<u>self-propagating</u>"



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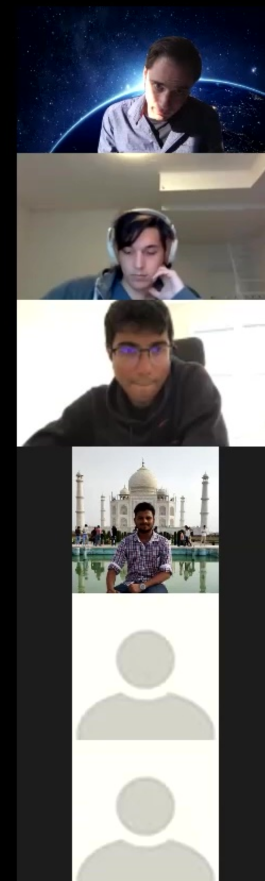
1.3.2 Domain of dependence & range of influence

Def: The domain of dependence $\mathcal{D}(x,t) \subseteq \mathbb{R}^n$ is the domain of the initial data $u^0(x)$ that entirely determines the solution $u(x,t)$ of a hyperbolic system

$$u_t + \sum_{i=1}^{n} A_i u_{x_i} = 0$$

Theorem: Let u be solution



that entirely determines the solution $U(x,t)$ of a hyperbolic system

$$U_t + \sum_{i=1}^{n} A_i U_{x_i} = 0$$

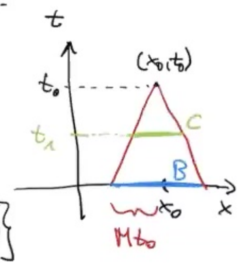
<u>Theorem:</u> Let U be solution of symmetric hyperbolic system

$$U_t + \sum_{i=1}^{n} \underbrace{A_i}_{\text{const. coeff.}} U_{x_i} = 0$$

Consider (x_0, t_0), $0 \le t_1 \le t_0$ and

$B \equiv \{x \in \mathbb{R}^n \mid |x - x_0| \le M t_0\}$

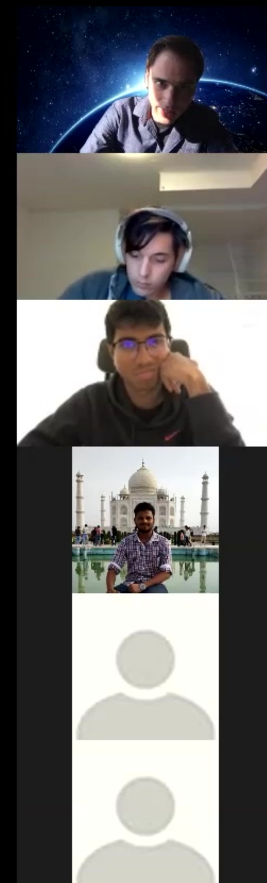
$C \equiv \{x \in \mathbb{R}^n \mid |x - x_0| \le M(t_0 - t_1)\}$



with

$$M = \max\{\lambda_i\}$$

λ_i are eigenvalues of $A(\xi) = \sum_{i=1}^{n} A_i \xi_i$.



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$C \equiv \{ x \in \mathbb{R}^n \mid |x - x_0| \leq M(t_0 - t_1) \}$ Mt_0

with

$$M = \max \{ \lambda_i \}$$

λ_i are eigenvalues of $\mathcal{A}(\xi) = \sum_{i=1}^{n} A_i \xi_i$.

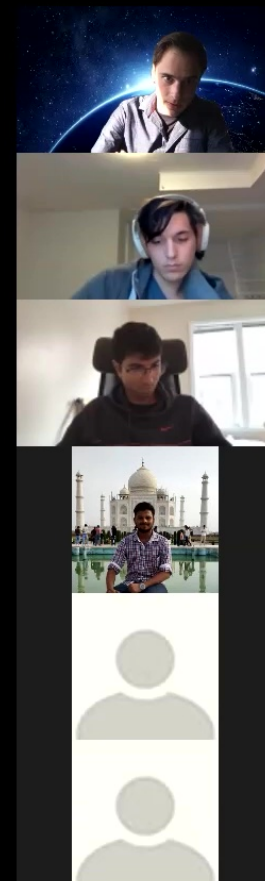
If $|u| \equiv 0$ on B_1 then $|u| \equiv C$.

<u>Remarks:</u> 1) $\mathcal{D}$ is determined by fastest and slowest characteristic speed

2) $\mathcal{D}$ is bounded (λ_i are finite)

Boundedness also applies to non-linear systems

"Information propagates at finite speed"



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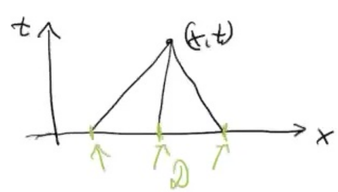
Boundedness also applies to non-linear systems

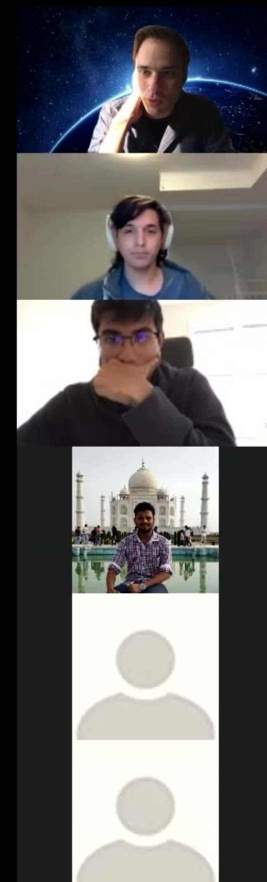
"Information propagates at finite speed"

Example: 1D: $u_t + A u_x = 0$

$$u(x,t) = \sum_{i=1}^{m} v_i^0(x-\lambda_i t)\, r_i$$

$$\rightarrow \mathcal{D}(x,t) = \{ x - \lambda_i t \mid i = 1, \dots, m \}$$





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