

Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 1

Speakers: Daniel Siegel

Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

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Lecture course: Computational Fluid Dynamics



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Lecture I: Preliminaries

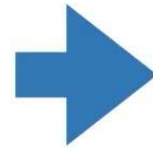
Astrophysical 'fluids': ubiquitous in the Universe on all scales



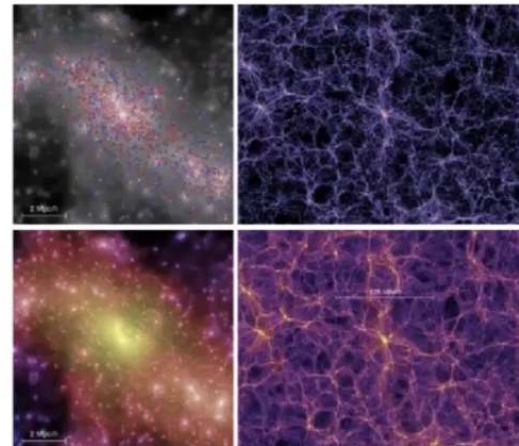
Dust rings of Saturn



Credit: NASA/JPL-Caltech/Space Science Institute



Matter distribution of the Universe

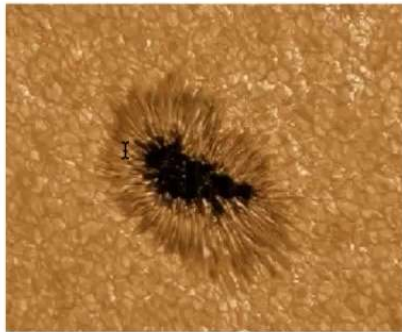


Credit: Springel et al. 2005, Nature, 435, 629

Astrophysical 'fluids': a few more examples...

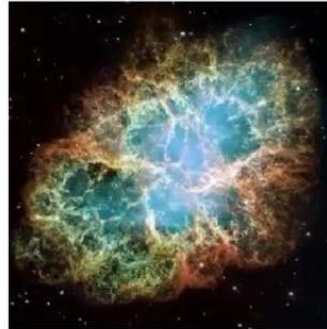


Solar & stellar astrophysics



Credit: Leibniz Institute for Solar Physics (KIS)

Supernova remnants



Credit: NASA, STScI, ESA

Interstellar medium



Credit: NASA

Tidal disruption events



Credit: NASA / CXC / M. Weiss

Neutron-star mergers



Credit: Robin Dienel, Carnegie Institution for Science

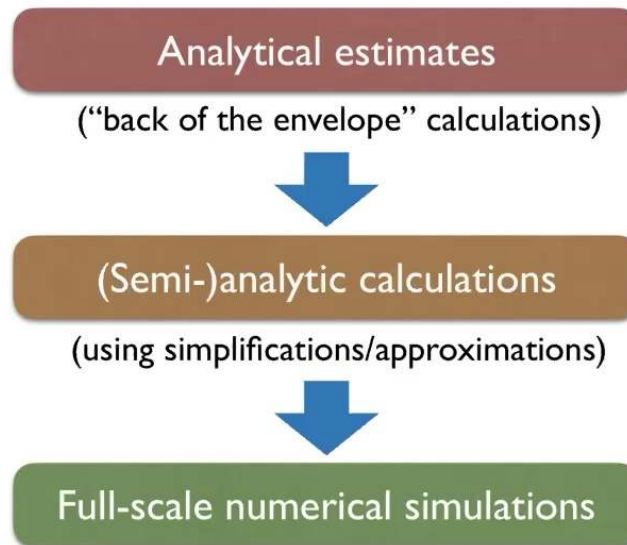
Astrophysical fluid dynamics



- Astrophysical fluid flows central to understanding our universe
 - equations of gas/plasma dynamics (**hydrodynamics**)
(+ **gravity** and **microphysics**: chemistry, nuclear & atomic physics, EM fields, etc.)
- Receive information about the universe based on EM radiation that is reprocessed as it travels toward us
 - equations of **radiation transport**
- We now also receive information via gravitational waves and high-energy particles
 - **multi-messenger astrophysics**

Interpretation of observational data

Theoretical interpretation of observational data (stages of refinement):



Credit: NASA

experiment
'substitute'

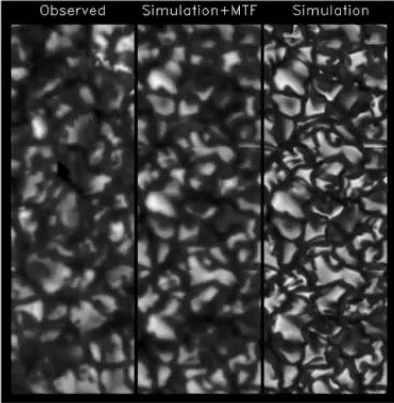


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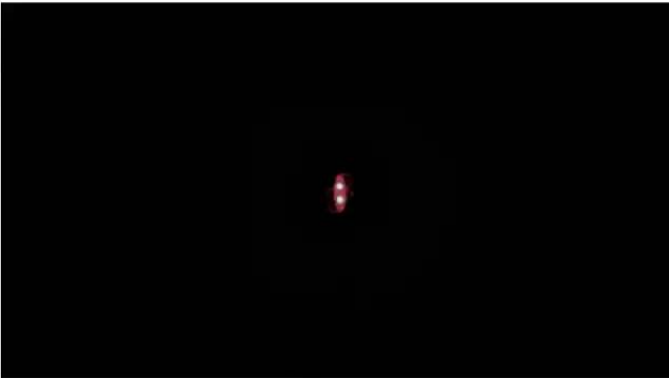
Simulations can achieve amazing things...



Solar granulation
(radiation hydrodynamics)



Credit: Stein+ 1998, ApJ 499, 914



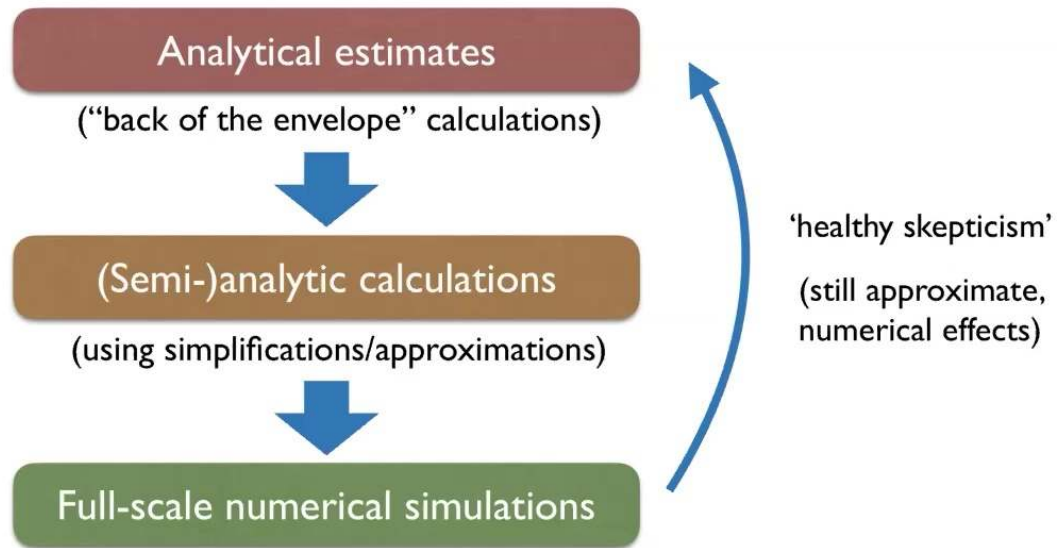
Movie credit: Ciolfi+, Siegel 2017, PRD 95, 063016



Credit: Springel et al. 2005, Nature, 435, 629

Interpretation of observational data

Theoretical interpretation of observational data (stages of refinement):



Credit: NASA

experiment
'substitute'



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Expectations of expectations

reduce complexity and provide stability to social systems (Luhmann)

...so let's talk about expectations:

Course contents and 'mechanics'

- For course outline and policies, please see:
<https://www.physics.uoguelph.ca/course-outlines/special-topics-astrophysics-numerical-hydrodynamics-phys7900>
- Course website (repository with course material):
<https://github.com/dmsiegel/computational-fluid-dynamics-course.git>
- 'Interpolation' between applied mathematics, physics, astrophysics
'blackboard' lectures, *hand-written* lecture notes
- Homework (30% final grade):
 - Problem sets* with analytical problems
 - Project component* with hands-on problems to implement
 - finite difference schemes and explore their properties
 - an 'exact' Riemann solver for the Euler Equation
 - a Godunov-type scheme for Newtonian hydrodynamics
- Exam (70% final grade): date & time TBD



Course contents and 'mechanics'

- Contents

1. Basic Notions of Partial Differential Equations
2. Basic Equations of Computational Fluid Dynamics
3. Finite Difference Methods for PDEs
4. Properties of conservation laws (theoretical background)
5. Riemann problem for the Euler Equations
6. Numerical schemes for conservation laws
7. Approximate Riemann solvers
8. Source terms and higher dimensions
9. Outlook: a primer on discontinuous Galerkin methods

**Very general, directly relevant
for Einstein's equations as
well!**



Course contents and 'mechanics'



- References

Numerical Methods:

- E. Toro: *Riemann Solvers and Numerical Methods for Fluid Dynamics* (Springer, 3rd edition, 2009)
- R. Leveque: *Finite Volume Methods for Hyperbolic Problems* (Cambridge Univ. Press, Cambridge Texts in Applied Mathematics, 2002)

More mathematically inclined literature:

- D. Kröner: *Numerical Schemes for Conservation Laws* (Wiley, 1997)
- L. Evans: *Partial Differential Equations* (Graduate Studies in Mathematics, American Mathematical Society, 2nd edition, 2010)

Other useful literature:

- A. Anile: *Relativistic fluids and magneto-fluids* (Cambridge Univ. Press, 1990)
- P. Bodenheimer, G. Laughlin, M. Rozyczka, H. Yorke: *Numerical Methods in Astrophysics* (Taylor & Francis, 2007)
- R. Leveque: *Finite Difference Methods for Ordinary and Partial Differential Equations* (SIAM, 2007)

Course contents and 'mechanics'

- First 'Homework':

- 1) please install *python* on your laptop/computer
- 2) please familiarize yourself with GitHub and create a first git repository.
This will be useful for working on the hands-on assignments.

- Office hours: virtually, on demand



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Chap_1_PDEs Lecture 1

Chapter 1: Basic notions of PDE

1.1. PDES of 2nd order

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$

$L: \mathcal{F}_1(\Omega_1) \rightarrow \mathcal{F}_2(\Omega_2)$ differential operator

\uparrow \uparrow
 e.g. $C^2(\Omega_1)$ e.g. $C^0(\Omega_2)$

$u \in \mathcal{F}_1(\Omega_1) = \Omega_1 \rightarrow \Omega_2$

Notation: $u_{x_i} \equiv \partial_{x_i} u = \frac{\partial u}{\partial x_i}$, $u_{x_i x_j} \equiv \partial_{x_i} \partial_{x_j} u = \frac{\partial^2 u}{\partial x_i \partial x_j}$

Def (classifier):

(i) non-linear PDEs

$Lu = 0$, $L = L(x, u, \{u_{x_i}\}, \{u_{x_i x_j}\})$

and $\frac{\partial L}{\partial u_{x_i x_j}} \neq 0$ for at least



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Chap_1_PDEs

(i) non-linear PDEs

$$Lu = 0, \quad L = L(x, u, \{u_{x_i}\}, \{u_{x_i x_j}\})$$

and $\frac{\partial L}{\partial u_{x_i x_j}} \neq 0$ for at least one pair indices

Example: $\Omega_1 = \Omega_2 = \mathbb{R}^2$

$$Lu = u_{x_1 x_2} u_{x_2 x_2} - u_{x_1 x_1}^2 = 0$$

(ii) Quasi-linear PDEs

$$Lu = \sum_{i,j=1}^n a_{ij}$$


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Chap_1_PDEs Lecture 1

$$Lu = 0, \quad L = L(x, u, \{u_{x_i}\}, \{u_{x_i x_j}\})$$

and $\frac{\partial L}{\partial u_{x_i x_j}} \neq 0$ for at least one pair indices

Example: $\Omega_1 = \Omega_2 = \mathbb{R}^2$

$$Lu = u_{x_1 x_2} u_{x_2 x_1} - u_{x_1 x_1 x_2}^2 = 0$$

(ii) Quasi-linear PDEs

$$Lu \equiv \sum_{i,j=1}^n a_{ij}(x, u, \{u_{x_i}\}) u_{x_i x_j} + b(x, u, \{u_{x_i}\}) = 0$$

Example: $\Omega_1 = \Omega_2 = \mathbb{R}^n$

$$\sum_{i,j=1}^n \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{1 + |Du|^2} \right) u_{x_i x_j} = 0 \quad \text{minimal surfaces}$$

(iii) Semi



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Chap. PDEs

$$Lu = 0, \quad L = L(x, u, \{u_{x_i}\}, \{u_{x_i x_j}\})$$

and $\frac{\partial L}{\partial u_{x_i x_j}} \neq 0$ for at least one pair indices

Example: $\Omega_1 = \Omega_2 = \mathbb{R}^2$

$$Lu = u_{x_1 x_2} u_{x_2 x_2} - \overset{z}{u_{x_1 x_2}} = 0$$

(ii) Quasi-linear PDEs

$$Lu = \sum_{i,j=1}^n a_{ij}(x, u, \{u_{x_i}\}) u_{x_i x_j} + b(x, u, \{u_{x_i}\}) = 0$$

Example: $\Omega_1 = \Omega_2 = \mathbb{R}^n$

$$\sum_{i,j=1}^n \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{1 + |u|_2^2} \right) u_{x_i x_j} = 0 \quad \underline{\text{minimal surfaces}}$$

(iii) Semi-linear PDE



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(iii) Semi-linear PDE

$$Lu = \sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} + b(x, u, \{u_{x_i}\}) = 0$$

Example: $-\Delta u = |\nabla u|^2 u$

(iv) linear P



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Example: $-\Delta u = |v| - u$

(iv) Linear PDEs

$$Lu \equiv \sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^n a_i(x) u_{x_i} + a(x)u + f(x) = 0$$

Example: $\Delta u = 0$

Remark: if we are looking



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Remark: if we are looking for a solution
 $u \in C^2(\mathbb{R}^n)$
 $a_{ij} = a_{ji}$
($A = (a_{ij})$ is symmetric \Rightarrow real eigenvalues)

Def: let $x_0 \in \mathbb{R}^n$



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Def: Let $x_0 \in \mathbb{R}^n$, $\lambda_1, \dots, \lambda_n$ the (real) eigenvalues of $A = (a_{ij}(x_0))_{i,j=1,\dots,n}$ and

$$t \equiv \#\{\lambda_i < 0\}$$

$$d_0 \equiv \#\{\lambda_i = 0\}$$

At $x = x_0$

$$Lu = \sum_{i,j=1}^n a_{ij}(x) a_{x_i} x_j + \sum_{i=1}^n a_i(x) u_{x_i} + a(x)u + f(x) = 0$$

is called

elliptic $\Leftrightarrow d_0 = 0$ and $(t=0, t=n)$

hyperbolic $\Leftrightarrow d_0 = 0$ and $(t=1$ or $t=n-1)$

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$$Lu = \sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^n a_i(x) u_{x_i} + a(x)u + f(x) = 0$$

is called

elliptic $\Leftrightarrow d_0 = 0$ and $(t=0, t=n)$

hyperbolic $\Leftrightarrow d_0 = 0$ and $(t=1$ or $t=n-1)$

parabolic $\Leftrightarrow d_0 > 0$

Remarks: 1) classification depends on the location

2)

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Remarks: 1) classification depends on the location

2) classification only depends on 2nd order terms (principal part)

Geometric interpretation:

Diagonalize $A = CDC^*$, $D = \text{diag}(\lambda_1, \dots, \lambda_n)$

$$C^*C = CC^* = \mathbb{1}$$

and consider the quadratic form

$$Q(\xi) = \xi^T A \xi = \sum_{i,j} a_{ij} \xi_i \xi_j \quad (= CDC^* \xi \xi^T)$$

$$= \xi^T \underbrace{C^* C}_{\mathbb{1}} \underbrace{C D C^*}_A \xi = DC^* \xi$$


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Chap_1_PDEs Lecture 1

2) classification only depends on 2nd order terms (principal part)

Geometric interpretation:

Diagonalize $A = CDC^*$, $D = \text{diag}(\lambda_1, \dots, \lambda_n)$
 $C^*C = CC^* = \mathbb{1}$

and consider the quadratic form

$$Q(\xi) = \xi^T A \xi = \sum_{i,j} a_{ij} \xi_i \xi_j \quad (= CDC^* \xi \xi)$$

$$= \xi^T \underbrace{C^* C}_{\mathbb{1}} \underbrace{A}_{\text{diag}(\lambda_i)} C \xi = DC^* \xi \cdot C^* \xi$$

$$= \sum_{i=1}^n \lambda_i \eta_i^2 \equiv \tilde{Q}(\eta), \quad \eta \equiv C^* \xi$$

Consider level sets $N = \{\eta \in \mathbb{R}^n \mid \tilde{Q}(\eta) = c\}$

Example: $n=2 \Rightarrow N = \{\eta \in \mathbb{R}^2 \mid \lambda_1 \eta_1^2 + \lambda_2 \eta_2^2 = c\}$

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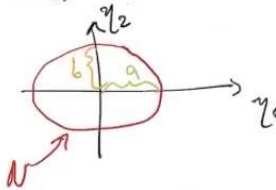
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$$= \sum_{i=1}^n \lambda_i \eta_i = Q(\eta), \quad \eta \in \mathbb{C}^n$$

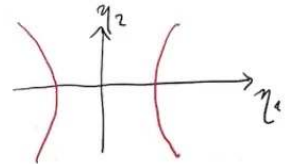
Consider level sets $N = \{\eta \in \mathbb{R}^n \mid Q(\eta) = \text{const}\}$

Example: $n=2 \Rightarrow N = \{\eta \in \mathbb{R}^2 \mid \lambda_1 \eta_1^2 + \lambda_2 \eta_2^2 = \text{const}\}$

elliptic case: $\lambda_1, \lambda_2 > 0$

$$\frac{\eta_1^2}{\left(\frac{\text{const}}{\lambda_1}\right)} + \frac{\eta_2^2}{\left(\frac{\text{const}}{\lambda_2}\right)} \equiv \frac{\eta_1^2}{a^2} + \frac{\eta_2^2}{b^2} = 1$$


hyperbolic case: $\lambda_1 > 0, \lambda_2 < 0$

$$\frac{\eta_1^2}{a^2} - \frac{\eta_2^2}{b^2} = 1$$


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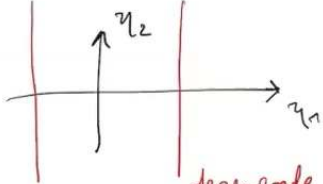
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parabolic case: $\lambda_1 > 0, \lambda_2 = 0$

$\lambda_1 \eta_1^2 = \text{const}$

$(\lambda \eta_1^2 + \eta_2 = \text{const.})$



degenerate parabola

Typical examples for principal types:

assume $a_{ij} = \text{const}$ and consider

$$Lu = \sum_{i,j=1}^n a_{ij} u_{x_i x_j} \quad A = CD$$

$$= \sum_{i,j=1}^n \sum_{k=1}^n C_{ik} \lambda_k C_{kj}^* u_{x_i x_j}$$


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Chap. J. PDEs Lecture 1

assume $a_{ij} = \text{const}$ and consider

$$Lu = \sum_{i,j=1}^n a_{ij} u_{x_i x_j} \quad A = CDC^T$$

$$= \sum_{i,j=1}^n \sum_{k=1}^n C_{ik} \lambda_k C_{kj}^T u_{x_i x_j}$$

$$= \sum_{k=1}^n \lambda_k \left(\sum_{i,j=1}^n u_{x_i x_j} C_{ik} C_{jk} \right)$$

define $v(y) \equiv u(Cy), \quad x = Cy$

$$v_{y_i y_i}(y) = \sum_{k=1}^n \dots$$

$$v_{y_i y_k} = \dots$$

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Chao_J_PDEs Lecture 1

assume $a_{ij} = \text{const}$ and consider

$$Lu = \sum_{i,j=1}^n a_{ij} u_{x_i x_j} \quad A = CDC^T$$

$$= \sum_{i,j=1}^n \sum_{k=1}^n C_{ik} \lambda_k C_{kj}^T u_{x_i x_j}$$

$$= \sum_{k=1}^n \lambda_k \left(\sum_{i,j=1}^n u_{x_i x_j} C_{ik} C_{jk} \right)$$

define $v(y) \equiv u(Cy)$, $x = Cy$

$$v_{\gamma_i \gamma_j}(y) = \sum_{k=1}^n \dots$$

$$v_{\gamma_k \gamma_k} = \dots$$

$$= \sum_{k=1}^n \lambda_k v_{\gamma_k \gamma_k}(y)$$


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Chap_3_PDEs Lecture 1

$$= \sum_{k=1}^n \lambda_k \left(\sum_{i,j=1}^n u_{x_i x_j} C_{ik} C_{jk} \right)$$

define $v(y) \equiv u(Cy)$, $x = Cy$

$$v_{y_i}(y) = \sum_{k=1}^n \dots$$

$$v_{y_k y_k} = \dots$$

$$= \sum_{k=1}^n \lambda_k v_{y_k y_k}(y)$$

elliptic case: $\lambda_1, \dots, \lambda_n > 0$, $y = \sqrt{D} \tilde{y}$

set $\tilde{v}(\tilde{y}) \equiv v(\sqrt{\lambda_1} \tilde{y}_1, \dots, \sqrt{\lambda_n} \tilde{y}_n)$, $y_i = \sqrt{\lambda_i} \tilde{y}_i$

and $\sum_{k=1}^n \tilde{v}_{\tilde{y}_k \tilde{y}_k}(\tilde{y}) = \sum_{k=1}^n \lambda_k v_{y_k y_k}(\sqrt{\lambda_1} \tilde{y}_1, \dots, \sqrt{\lambda_n} \tilde{y}_n) = 0$

$\Leftrightarrow \Delta \tilde{v} = 0$



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$\Leftrightarrow \Delta \tilde{v} = 0$ Poisson equation

hyperbolic case: $\lambda_1 < 0, \lambda_j > 0$

$\Rightarrow \square u = u_{tt} - \Delta u = 0$ linear wave equation

parabolic case: $\lambda_1 = 0$

$\Delta u = 0$
 \uparrow
 \mathbb{R}^{n-1}



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Chao_J_PDEs Lecture 1

$\Leftrightarrow \boxed{\Delta \tilde{v} = 0}$ Poisson equation

hyperbolic case: $\lambda_1 < 0, \lambda_j > 0$

$\Rightarrow \boxed{\square u = u_{tt} - \Delta u = 0}$ linear wave equation

parabolic case: $\lambda_1 = 0$

$\boxed{u_t - \Delta u = 0}$ heat conduction equation (diffusion equation)

\uparrow
 \mathbb{R}^{n-1}
 $x \in \{x_1, \dots, x_n\}$

