Title: On the geometry of nodal domains for random eigenfunctions on compact surfaces

Speakers: Suresh Eswarathasan

Series: Mathematical Physics

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Abstract: A classical result of R. Courant gives an upper bound for the count of nodal domains (connected components of the complement of where a function vanishes) for Dirichlet eigenfunctions on compact planar domains. This can be generalized to Laplace-Beltrami eigenfunctions on compact surfaces without boundary. When considering random linear combinations of eigenfunctions, one can make this count more precise and pose statistical questions on the geometries appearing amongst the nodal domains: what percentage have one hole? ten holes? what percentage have their boundary being tangent 100 times to a fixed non-zero vector field? The first 20-25 minutes will give a survey on some fundamental results of Nazarov-Sodin, Sarnak-Wigman, and Gayet-Welschinger before presenting some joint works with I. Wigman (King's College London) and Matthew de Courcy-Ireland (Ã‰cole Polytechnique FÃ©dÃ©rale de Lausanne) answering these questions in the last 25-30 minutes.

Geometry of nodal domains of eigenfunctions

2 pages

Geometry of nodal domains of eigenfunctions

§ Introduction

- (M, g) compact Riemannian mfld w/o boundary
- $-\Delta_g$ (positive) Laplace Beltrami operator

2 pages

§1.1 Physical Interpretations

Main interpretations:

① $-\Delta \varphi_\lambda = \lambda^2 \varphi_\lambda$ gives stationary states for a (free) quantum particle on M.

② (Born) $\int_E |\varphi_\lambda(x)|^2 \, dV$

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2 pages

on M.

(2) (Born) $\int_E |\varphi_\lambda(x)|^2 \, dV_g = \mathbb{P}(\text{particle} \in E)$ $\quad E \subsetneq M$

On (M, g), $\mathrm{Spec}(-\Delta_g) = \{\lambda_j^2\}_{j=0}^{\infty}$

w/ $\lambda_j^2 \nearrow \infty$

2 pages

E $\qquad E \subsetneq M$

$$\mathbb{P}(\text{particle} \in E)$$

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Big Q (following Einstein '17):
if the classical system on M is chaotic, what are the patterns exhibited by $\mathrm{spec}(-\Delta_g)$

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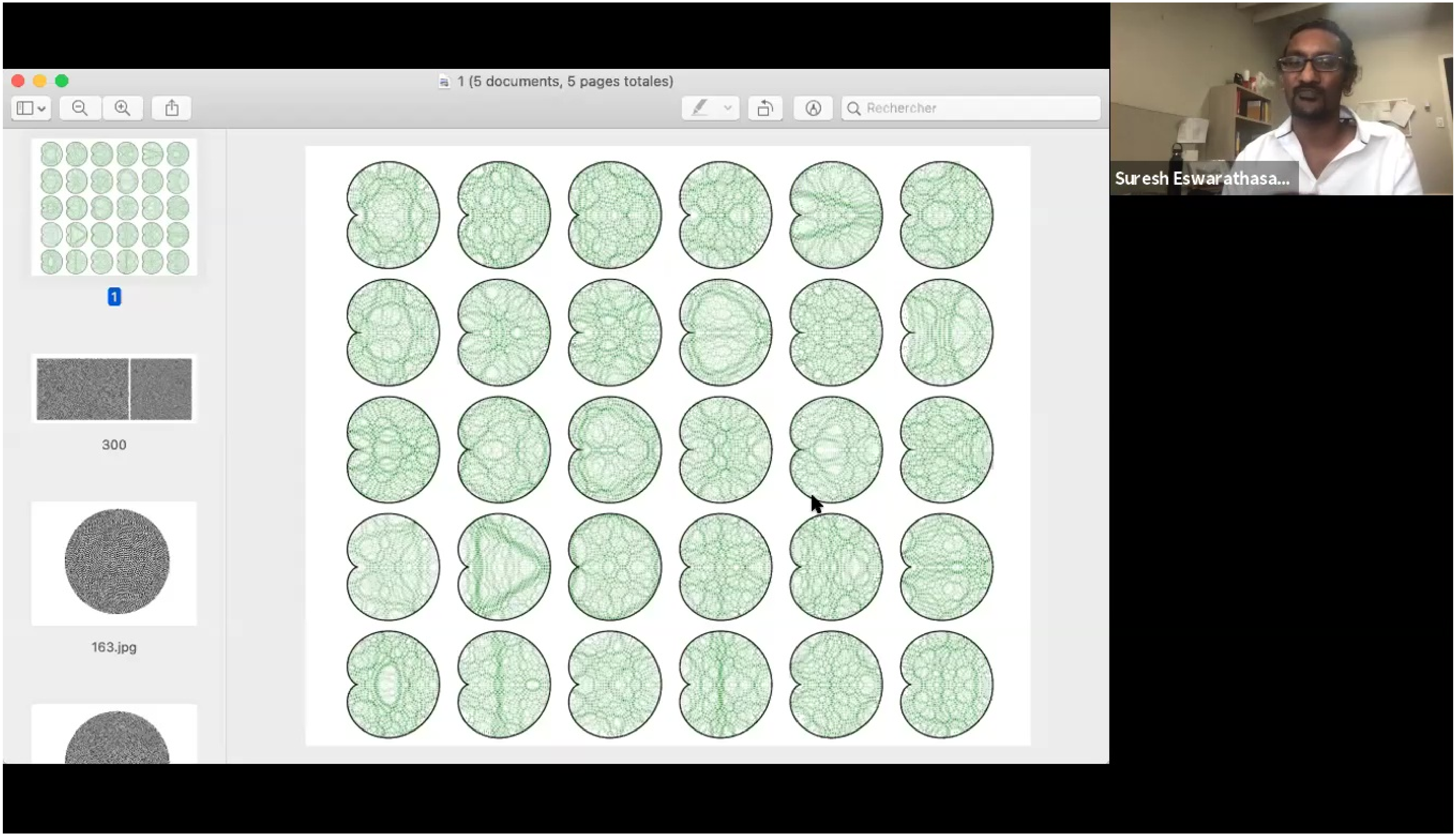
if the classical system on M is chaotic, what are the patterns exhibited by $\operatorname{Spec}(-\Delta_g)$ and $\{\varphi_{\lambda_j}\}_{j=0}^{\infty}$?? Using Bohr's principle, behaviors for λ_j^2 large ??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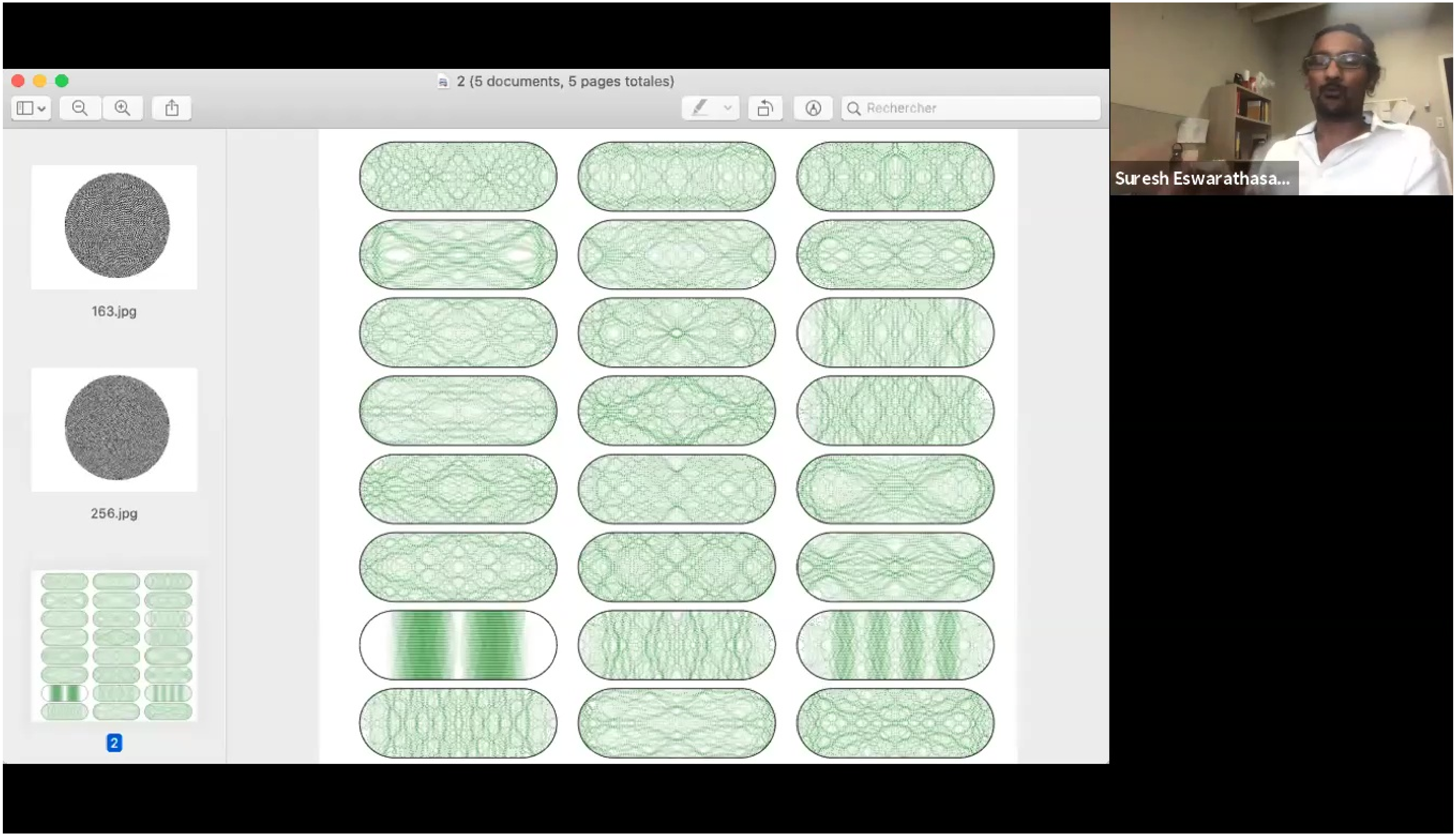
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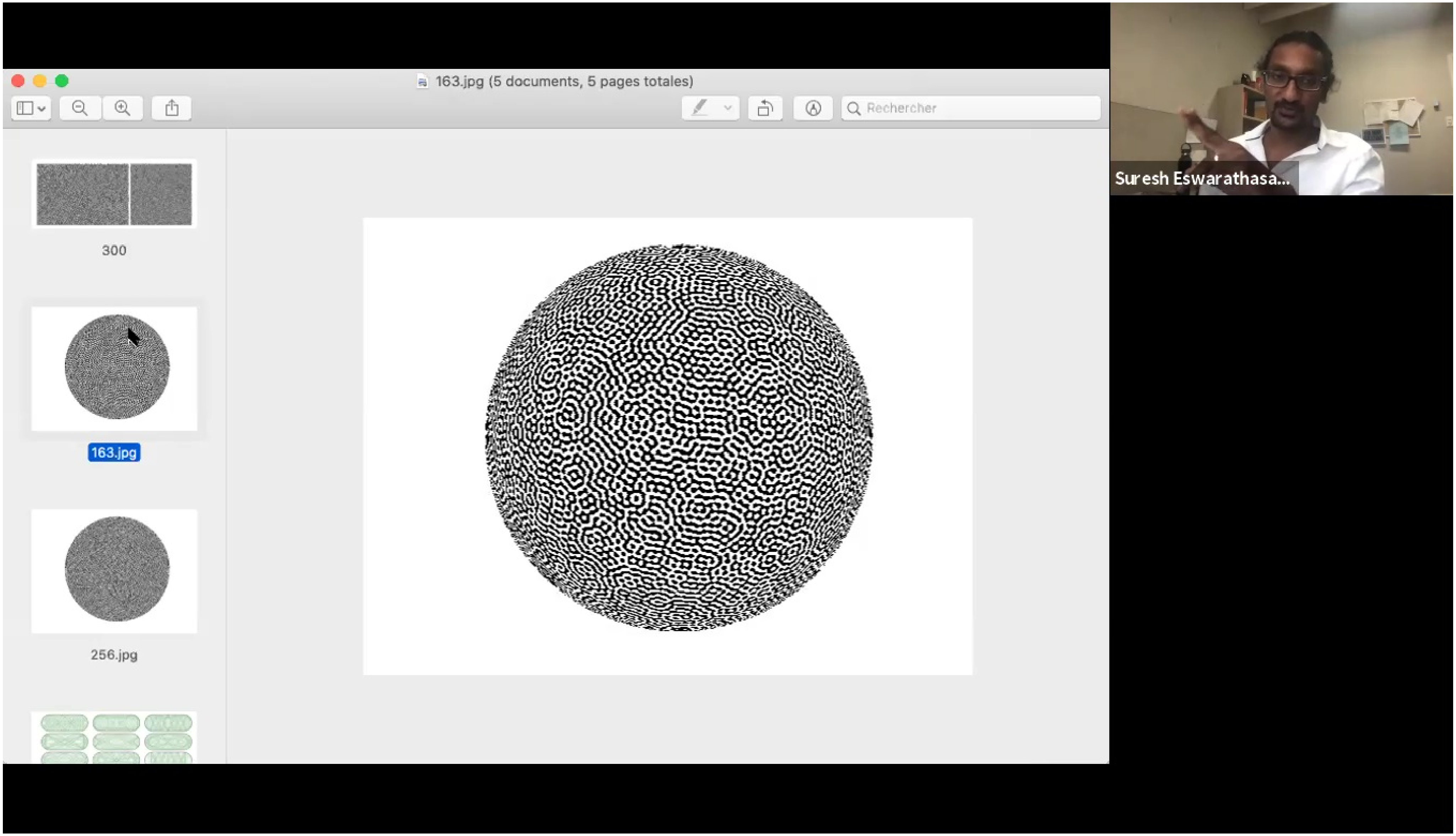
Chaotic, what are the patterns exhibited by $\mathrm{spec}(-\Delta_g)$ and $\{\varphi_{\lambda_j}\}_{j=0}^{\infty}$?? Using Bohr's principle, behaviors for λ_j^2 large ??

Berry '77: high-energy eigenfunctions on chaotic (M,g) $\overset{\text{statistically}}{=}$ random combos of plane waves / spherical harmonics.

2 pages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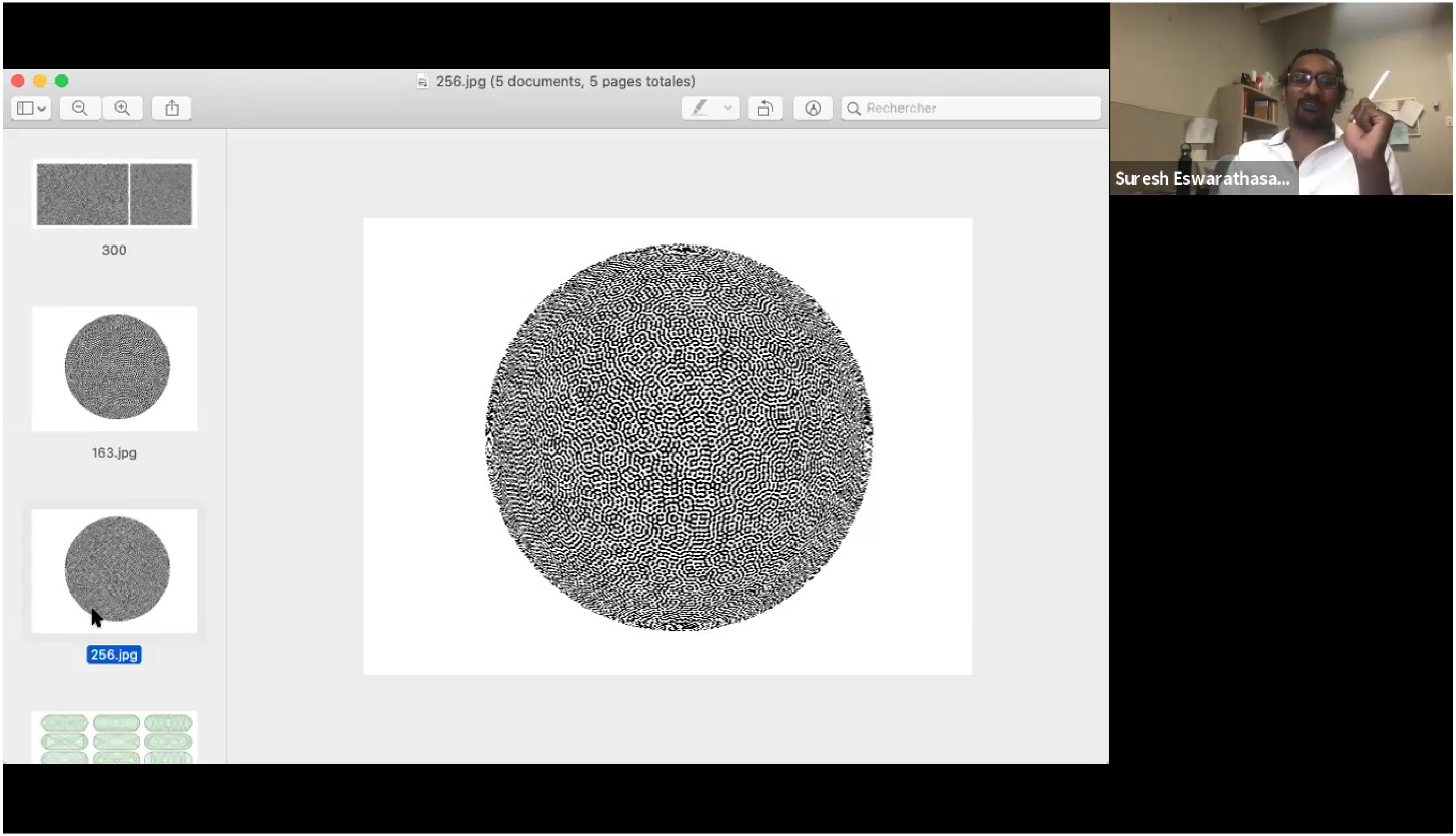
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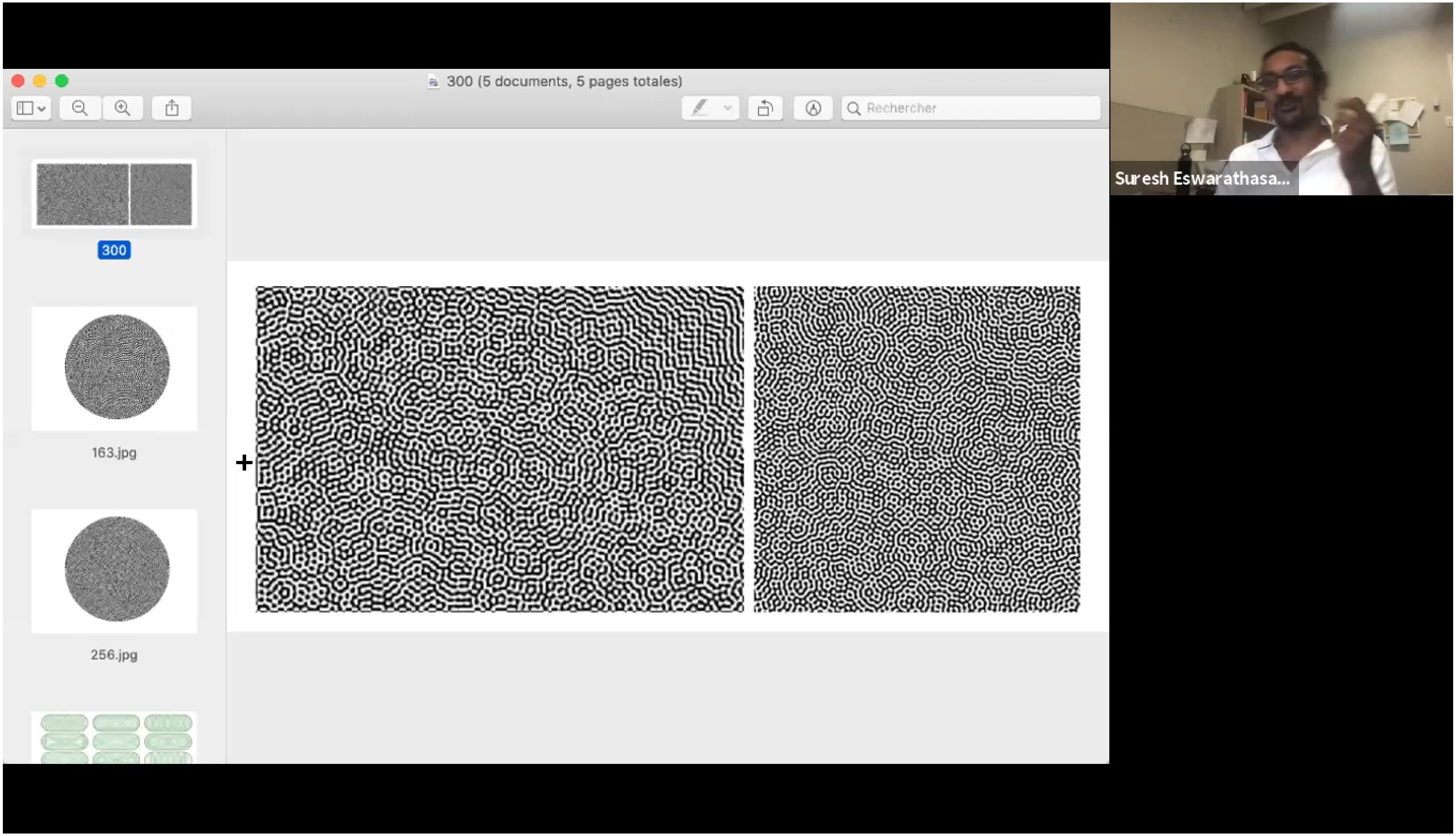
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Berry '77 : high-energy eigenfunctions on chaotic (M,g) $\underset{\text{statistically}}{=}$ random combos of plane waves / spherical harmonics.

Model on (S^2, g_{round}) : ① $\mathrm{spec}(-\Delta_g) = \{\ell(\ell+1)\}_{\ell=0}^{\infty}$

② $$\varphi_\ell := \frac{1}{\sqrt{2\ell+1}} \sum_{m=-\ell}^{\ell} a_m e^{im\theta} P_\ell^m(\cos\phi)$$

2 pages

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Model on (S^2, g_{round}) : ☺ $\operatorname{spec}(-\Delta_g) = \{l(l+1)\}_{l=0}^{\infty}$

satisfy eigenvalue eq.

☺ $$q_l := \frac{1}{\sqrt{2l+1}} \sum_{m=-l}^{l} a_m \left(e^{im\theta} P_l^m(\cos\phi)\right)$$

$$q_\ell := \frac{1}{\sqrt{2\ell+1}} \sum_{m=-\ell}^{\ell} a_m e^{im\theta} P_\ell^m(\cos\phi)$$

a_m $N(0,1)$ iid

Big (general) Q for today: Can we further describe $N(q_\ell) =$

$$\underbrace{\{x \in S^2 \mid q_\ell(x) = 0\}^c}_{\text{NODAL DOMAINS}} \text{ for } \ell \text{ large??}$$

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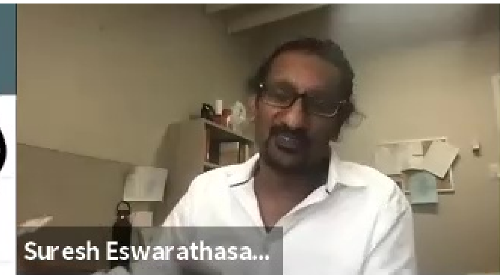

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Courant '53 / Cheng '76 :

$$\# N(q_\ell) \leq (\ell+1)^2$$

Nazarov – Sodin '09, '15 : $\mathbb{E}_\ell[\# N(q_\ell)]$

$= c_0 \ell^2 + o(\ell^2)$ and c_0 is universal (holds for more general surfaces)

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- inspired by BS02
- analogous results by Gayet - Welschinger

3 pages

— analogous results by Gayet – Welschinger

§ 2 Topology of domains

Specific Q for today: What's the "typical" topology of a connected

- 3 -

component $\in N(q_\lambda)$??

Consider $\frac{1}{\# N(q_\lambda)} \sum_{\substack{\text{comp} \in \\ N(q_\lambda)}} \delta_{\{\#\text{ of holes comp has}\}} = \mu$

component $\in N(q_\lambda)$ ::

Consider $\displaystyle \frac{1}{\# N(q_\lambda)} \sum_{\substack{\text{comp} \in \\ N(q_\lambda)}} \delta_{\{\#\text{ of holes comp has}\}}$

$:= \mu(q_\lambda)$, random measure

Sarnak-Wigman '18 :

measure

Sarnak–Wigman '18 : $\mu(q_\ell) \xrightarrow[\ell \to \infty]{} \mu_2$

a universal, deterministic, probability measure on $\mathbb{Z}_{\geq 0}$ and $\mathrm{supp}(\mu_2) = \mathbb{Z}_{\geq 0}$

Barnett '18 : $\mu_{2,1}(0) \approx .919$

4 pages

a universal, deterministic, probability measure on $\mathbb{Z}_{\geq 0}$ and $\operatorname{supp}(\mu_2) = \mathbb{Z}_{\geq 0}$

Barnett '18: $\mu_{2,1}(0) \approx .919$, $\mu_{2,1}(1) \approx .051$

4 pages

$\mu_{2,1}(1) \approx .051$

§3 Answers

de Courcy-Ireland — E. '20:

$$\mu_{2,1}(0) > 10^{-5}, \quad \mu_{2,1}(1) > 10^{-247}$$

- 4 -

goal: Show $\exists\, C > 0$, $\forall\, n \geq n_0$,

$$\mu_{2,1}(n) \leq C\, n^{-187/91}$$

6 pages

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"critical percolation exp for lattices

* Barnett's numeric confirm the goal

Confirm the goal

New binning of topologies: fix $\underline{V} \neq 0$ $\in \mathcal{J}(S^2)$ tangent to $\underline{V}$ K times??

E–Wigman '19 (TAMS): $\exists\, \mu_{2,1}^{\underline{V}}$

$$= \sum_{j \in 2\mathbb{Z}_{>0}} \mu_{2,1}^{\underline{V}}(j)$$

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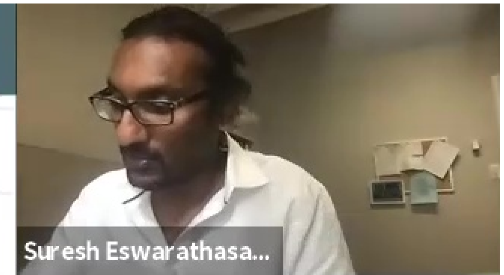


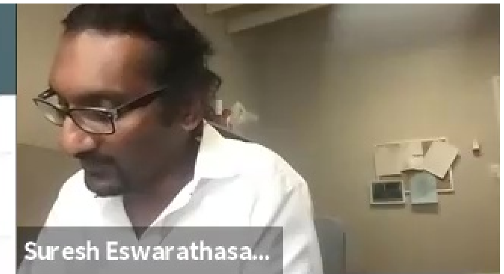
$\mathbb{E}$ - Wigman '19 (TAMS): $\exists\, \mu_{2,1}$

$$= \sum_{j \in 2\mathbb{Z}_{>0}} \mu^{\nabla}_{2,1}(j).$$ In other words,

$$\mathbb{E}\left[\# N^{K}_{\nabla}(\varphi_\ell)\right] = \begin{cases} C^{\nabla}_{0,10}\, \ell^2 + \ldots, & K \text{ even} \\ o(\ell^2), & K \text{ odd} \end{cases}$$

- 5 -

5 pages



$$= \sum_{j \in 2\mathbb{Z}_{>0}} \mu_{2,1}^{\nabla}(j).$$

In other words,

$$\mathbb{E}[\# N_{\nabla}^{K}(q_\lambda)] = \begin{cases} C_{0,\infty}^{\nabla}\lambda^2 + \dots, & K \text{ even} \\ o(\lambda^2), & K \text{ odd} \end{cases}$$

- 5 -

7 pages

$$\boxed{\text{Geometry of nodal domains of eigenfunctions}}$$

① $-\Delta_g \varphi_\lambda = \lambda^2 \varphi_\lambda$ gives stationary states for a (free) quantum particle on M.

② (Born) $\int_E |\varphi_\lambda(x)|^2 \, dV_g =$ $\quad E \subsetneq M$

$\mathbb{P}(\text{particle} \in E)$

- 1 -

$E \qquad\qquad\qquad E \subsetneq M$

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On (M, g), $\mathrm{Spec}(-\Delta_g) = \{\lambda_j^2\}_{j=0}^{\infty}$

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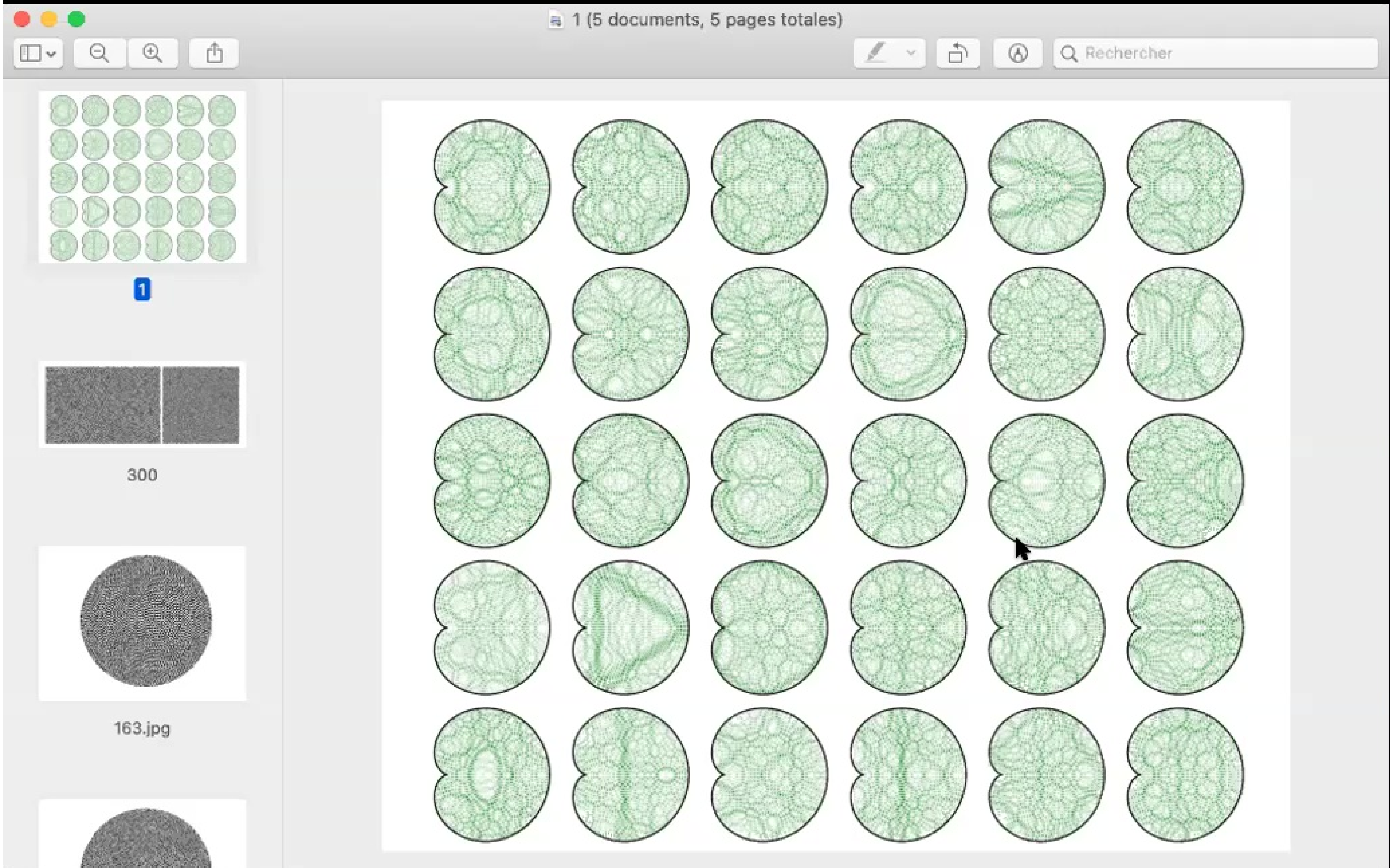
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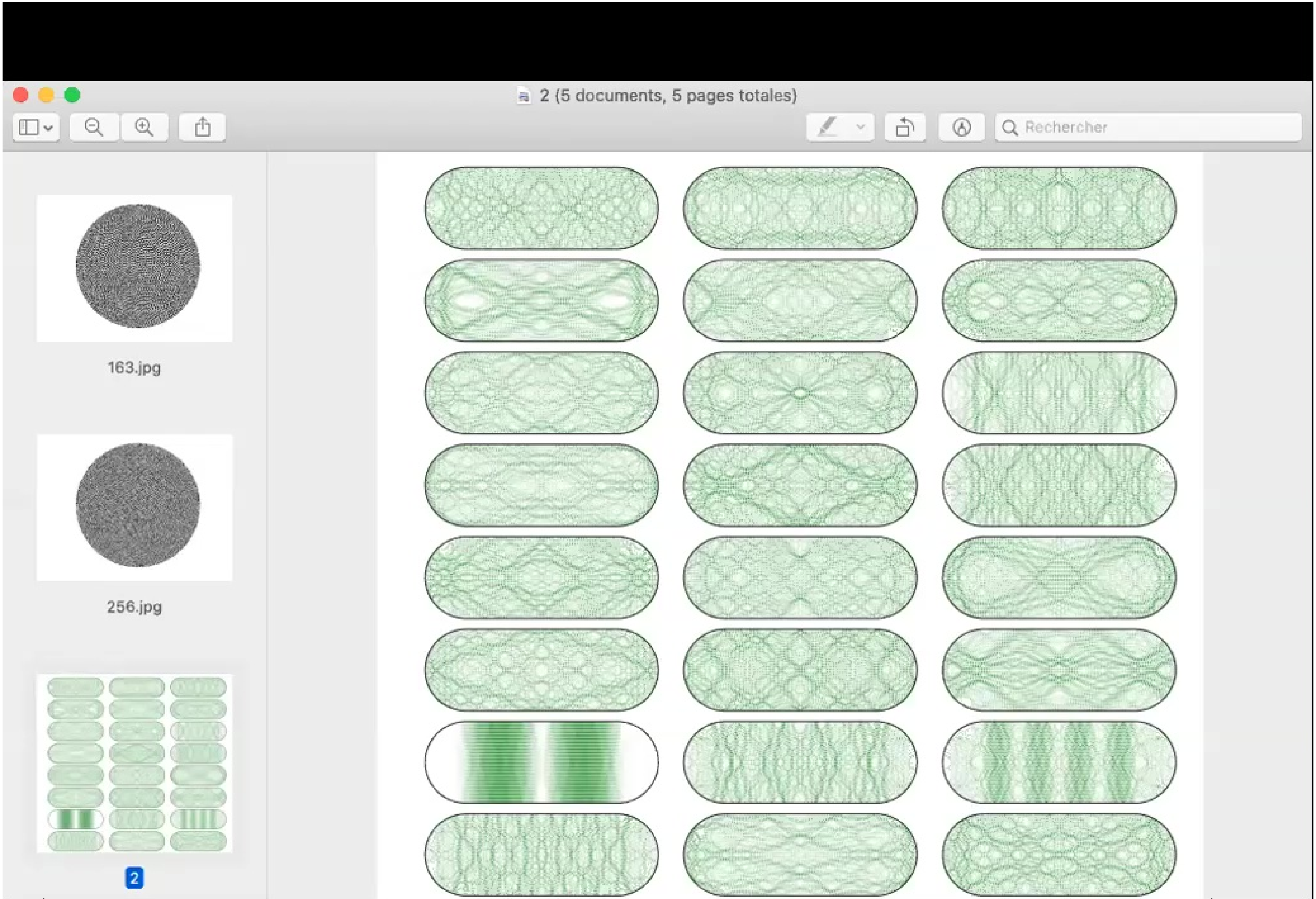
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Chaotic, what are the patterns exhibited by spec$(-\Delta_g)$ and $\{\varphi_{\lambda_j}\}_{j=0}^{\infty}$?? Using Bohr's principle, behaviors for λ_j^2 large ??

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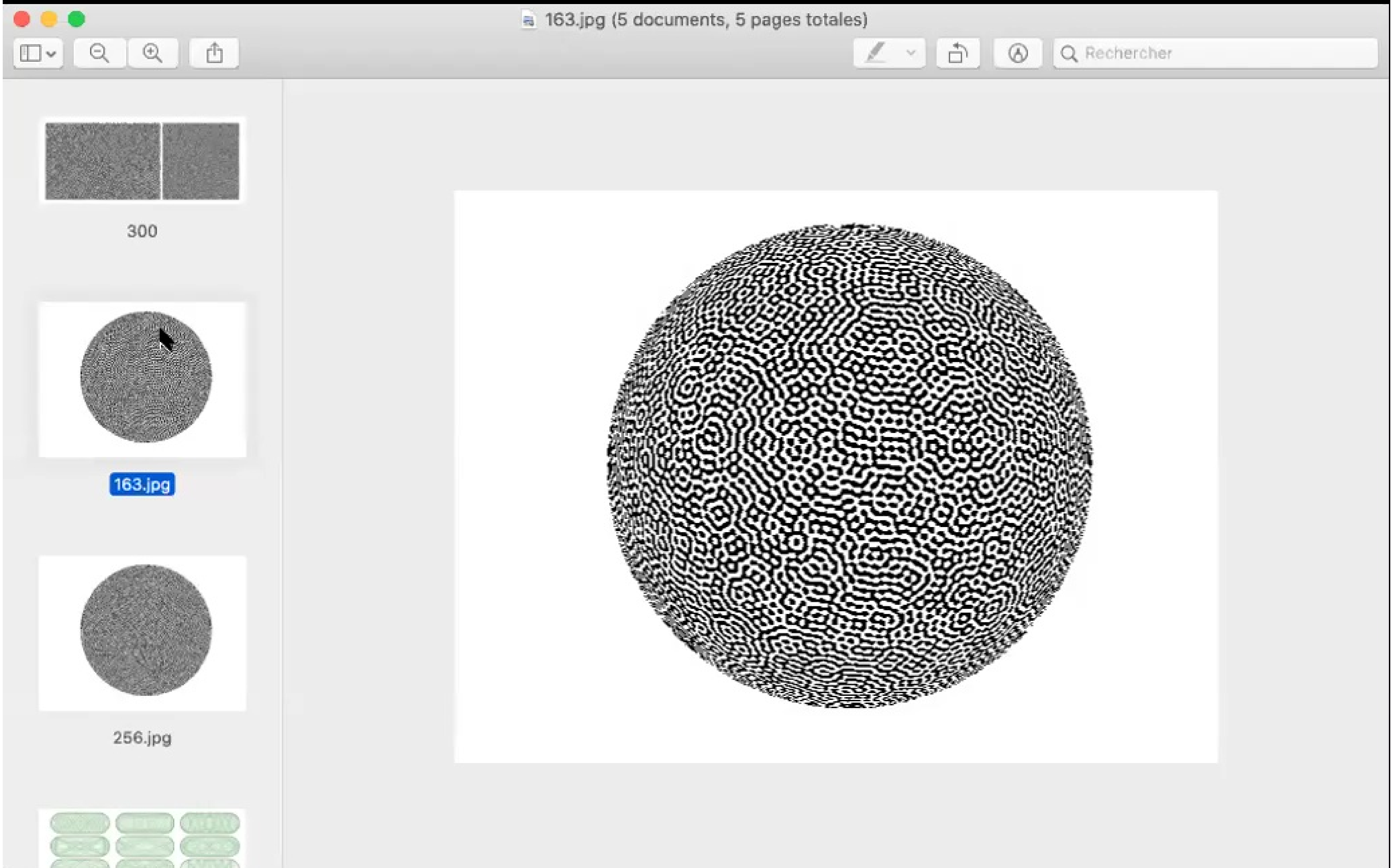
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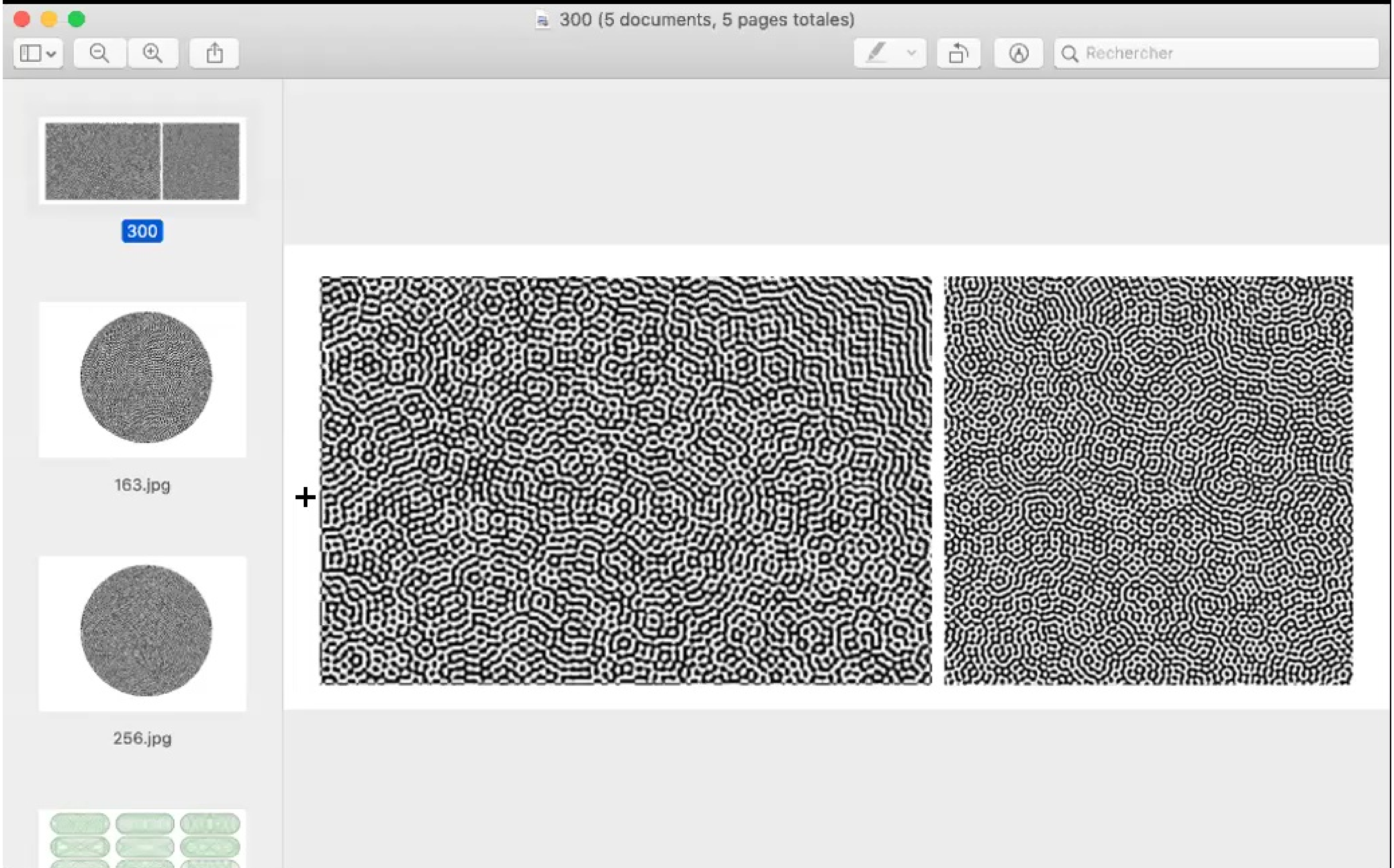
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Bohr's principle, deviations for λ large.

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Combos of plane waves / spherical harmonics.

Model on (S^2, g_{round}) : ⊙ $\mathrm{spec}(-\Delta_g) = \{\ell(\ell+1)\}_{\ell=0}^{\infty}$

satisfy eigenvalue eq^n

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$a_m \quad N(0,1)$

$\{\ldots\}_{l=0}$ satisfy eigenvalue eq^n

$\odot \quad q_l := \frac{1}{\sqrt{2l+1}} \sum_{m=-l}^{l} a_m \left(e^{im\theta} P_l^m(\cos\phi) \right)$

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Big (general) Q for today :

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<u>Big</u> (<u>general</u>) <u>Q for today</u>: Can we further describe $N(q_\ell) = $.

- 2 -

$$\underbrace{\{x \in S^2 \mid q_\ell(x) = 0\}^c}_{\text{NODAL DOMAINS}} \quad \text{for } \ell \text{ large??}$$

$$\underbrace{\{x \in S^2 \mid \varphi_l(x) = 0\}^c}_{\text{NODAL DOMAINS}} \quad \text{for } l \text{ large ??}$$

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$= c_0 \lambda^2 + o(\lambda^2)$ and c_0 is universal (holds for more general surfaces) universal (holds for more general surfaces)

- inspired by BSO2
- analogous results by Gayet - Welschinger

— analogous results by Viro-Welschinger

§ 2 Topology of domains

Specific Q for

- 3 -

— analogous results by Gayet – Welschinger

§ 2 Topology of domains

Specific Q for today : What's the "typical" topology of a Connected Component $\in \mathcal{N}(\varphi_L)$??

- 3 -

§ 2 Topology of domains

Specific Q for today: What's the "typical" topology of a connected component $\in N(q_L)$??

Consider
$$\frac{1}{\# N(q_L)} \sum_{\substack{\text{comp} \in \\ N(q_L)}} \delta_{\{\#\text{ of holes comp has}\}}$$
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component $\in N(q_n)$ ...

Consider $\displaystyle \frac{1}{\# N(q_n)} \sum_{\text{Comp} \in N(q_n)} \delta_{\{\#\text{ of holes comp has}\}}$

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$\# N(q_\ell)$ comp ϵ ... nds $\}$

$N(q_\ell)$

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§3 Answers

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goal : show $\exists\, C > 0$, $\forall\, n \geq n_0$,

$$\mu_{2,1}(n) \leq C\, n^{-187/91}$$

"critical

per

$p_{2,1}(n) \leq C\, n^{[illegible]/91}$ " Critical percolation exp for lattices

* Barnett's numeric confirm the goal

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Confirm the goal

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E - Wigman'19 (TAMS) :

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In other words,

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