Title: Bootstrapping Matrix Quantum Mechanics

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Series: Quantum Fields and Strings

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Abstract: Abstract: Large N matrix quantum mechanics are central to holographic duality but not solvable in the most interesting cases. We show that the spectrum and simple expectation values in these theories can be obtained numerically via a `bootstrap' methodology. In this approach, operator expectation values are related by symmetries -- such as time translation and SU(N) gauge invariance -- and then bounded with certain positivity constraints. We first demonstrate how this method efficiently solves the conventional quantum anharmonic oscillator. We then reproduce the known solution of large N single matrix quantum mechanics. Finally, we present new results on the ground state of large N two matrix quantum mechanics.

Bootstrapping matrix quantum mechanics

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Quantum Fields and Strings

arXiv:2004.10212 [Han, Hartnoll, Kruthoff, 20']
arXiv:2002.08387 [Lin, 20']
arXiv:1906.08781 [Han, Hartnoll, 19']
arXiv:2006.06002 [Han, 20']

Motivation

- Why matrix quantum mechanics?
  - Emergence of spacetime
  - Quantum mechanical
  - Wide range of models
- What do we learn?
  - Holography and string theory
  - Black hole dynamics
  - ...
- But generally difficult!
  - Solvable for one matrix
  - No geometric locality built in
  - Gauge theory with fermions
- Single particle $\Rightarrow$ Single / Two matrices $\Rightarrow$?

Single particle example

How to solve the following Hamiltonian?

$$H = p^2 + x^2 + gx^4$$

Bootstrap! From symmetries I know that

$$\langle [H, O] \rangle = 0, \quad \forall O,$$

in any energy eigenstate or thermal state.

Ex 1: Take $O = x^2$:

$$\langle xp \rangle + \langle px \rangle = 0,$$

$$\langle xp \rangle - \langle px \rangle = i, \quad \Rightarrow \quad \langle xp \rangle = -\langle px \rangle = i/2.$$

Ex 2: Take $O = xp$ (Virial theorem):

$$\langle -2p^2 + 2x^2 + 4gx^4 \rangle = 0.$$

Single particle example

More systematically,

- Any expectation value $\langle x^r p^s \rangle$ can be reduced to functions of $\langle x^m \rangle$ or $\langle x^n p \rangle$.
- Take $O = x^r$,

$$\langle [H, O] \rangle = \langle [p^2, x^r] \rangle = 0$$

$$\langle p x^{r-1} + x p x^{r-2} + \ldots + x^{r-1} p \rangle = 0$$

  $\langle x^n p \rangle$ can be reduced to $\langle x^m \rangle$ too!
- There is a recurrence relation on $\langle x^m \rangle$.

Overall, knowledge of $\langle x^2 \rangle$ and $E = \langle H \rangle$ (or equivalently $\langle x^4 \rangle$) yields values for all operators.

Single particle example

How to bound $\langle x^2 \rangle$ and $E = \langle H \rangle$ then?

Positivity requirement:

$$\langle O^\dagger O \rangle \geq 0, \quad \forall O.$$

If we choose $O = \sum c_i O_i$,

$$\sum c_i^* c_j \langle O_i^\dagger O_j \rangle \geq 0, \quad \forall c_i,$$

that is, the matrix $\mathcal{M}_{ij} = \langle O_i^\dagger O_j \rangle$ must be positive semidefinite.

$\langle x^4 \rangle \geq \langle x^2 \rangle^2$

Ex:

	1	x^2	x	x^3	x^5
1	$\langle 1 \rangle$	$\langle x^2 \rangle$	0	0	0
x^2	$\langle x^2 \rangle$	$\langle x^4 \rangle$	0	0	0
x	0	0	$\langle x^2 \rangle$	$\langle x^4 \rangle$	$\langle x^6 \rangle$
x^3	0	0	$\langle x^4 \rangle$	$\langle x^6 \rangle$	$\langle x^8 \rangle$
x^5	0	0	$\langle x^6 \rangle$	$\langle x^8 \rangle$	$\langle x^{10} \rangle$

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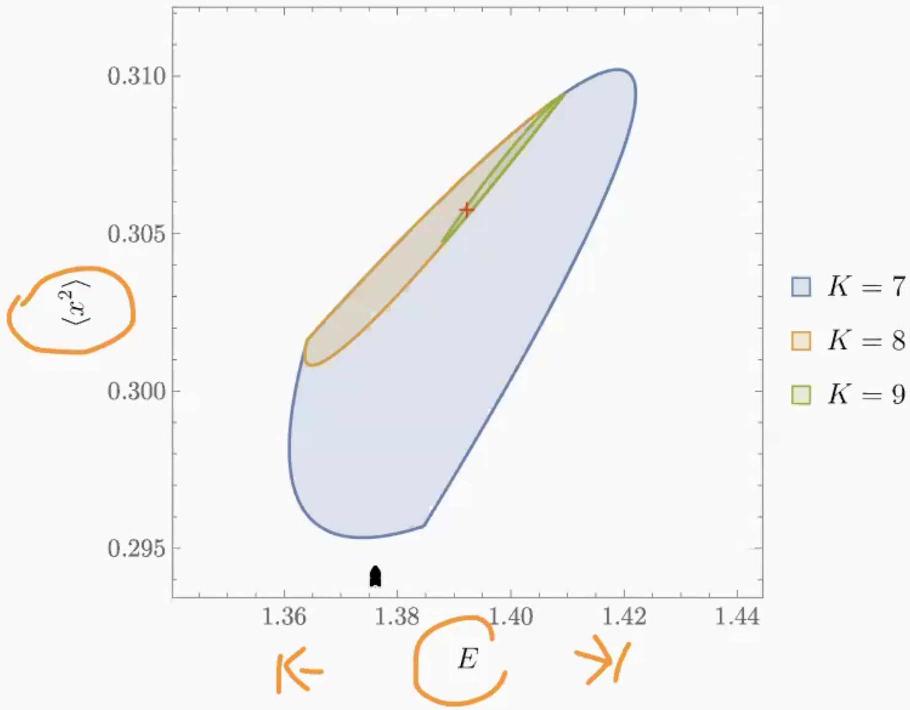
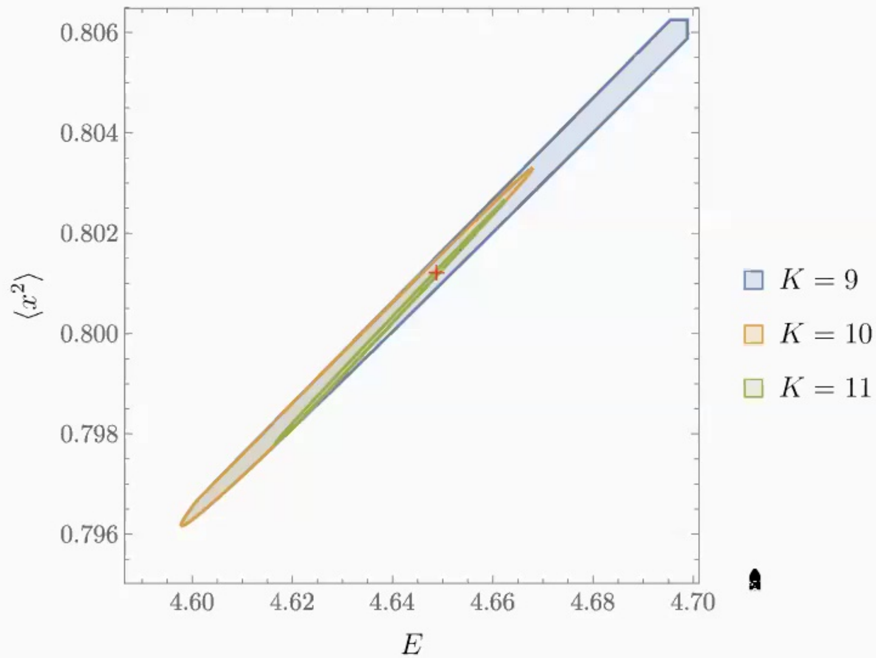
Single particle example



- Shrinking islands corresponding to the spectrum
- Higher energies need more constraints

Single particle example



- Shrinking islands corresponding to the spectrum
- Higher energies need more constraints

Single matrix example

- What is the matrix model?

$$H = \mathrm{tr}(P^2 + X^2 + g^2 X^4)$$

Here X and P are N-by-N hermitian matrices with commutator $[P_{ij}, X_{kl}] = \delta_{il}\delta_{jk}$.

- Where is the emergent geometry?

$$Z = \int dX\, e^{-\mathrm{tr}(X^2+X^4)} = \int dx_i \Delta(x_i) e^{-\sum(x_i^2+x_i^4)}$$
$$= \int dx_i e^{-\sum(x_i^2+x_i^4)+\sum_{i\neq j}\log|x_i-x_j|}$$

The Jacobian $\Delta(x_i) = \prod_{i<j}(x_i - x_j)^2$. Large N is a large number of eigenvalue "particles" with repulsive interaction in an external potential.

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Single matrix example

The goal is to bound possible values of observable expectations:

$$\langle \operatorname{tr} X \rangle, \langle \operatorname{tr} XP \rangle, \langle \operatorname{tr} XXXX \rangle \dots$$

There are some linear relations between them due to symmetries:

$$\langle [H, O] \rangle = 0, \quad (\operatorname{tr} GO = 0).$$

Virial theorem:

- $H = \operatorname{tr} P^2 + \operatorname{tr} X^2 + \operatorname{tr} X^4$
- $O = \operatorname{tr} XP$

$$\langle -2 \operatorname{tr} P^2 + 2 \operatorname{tr} X^2 + 4 \operatorname{tr} X^4 \rangle = 0$$

Odd polynomials vanish due to a $\mathbf{Z}_2$ symmetry.

Single matrix example

Cyclicity of the trace and large N further impose some quadratic equalities:

$$\operatorname{tr} PX^3 - \operatorname{tr} X^3 P = -i(\operatorname{tr} I \operatorname{tr} X^2 + \operatorname{tr} X \operatorname{tr} X + \operatorname{tr} X^2 \operatorname{tr} I),$$

so

$$\langle \operatorname{tr} PX^3 \rangle - \langle \operatorname{tr} X^3 P \rangle = -2iN \langle \operatorname{tr} X^2 \rangle - i \langle \operatorname{tr} X \rangle \langle \operatorname{tr} X \rangle.$$

Single matrix example

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It is also possible to not factorize and bootstrap for finite N.

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Single matrix example

Positivity requirement:

$$\langle \operatorname{tr} O^\dagger O \rangle \geq 0, \quad \forall O.$$

If we choose $O = \sum c_i O_i$,

$$\sum_{ij} c_i^* c_j \langle \operatorname{tr} O_i^\dagger O_j \rangle \geq 0, \quad \forall c_i,$$

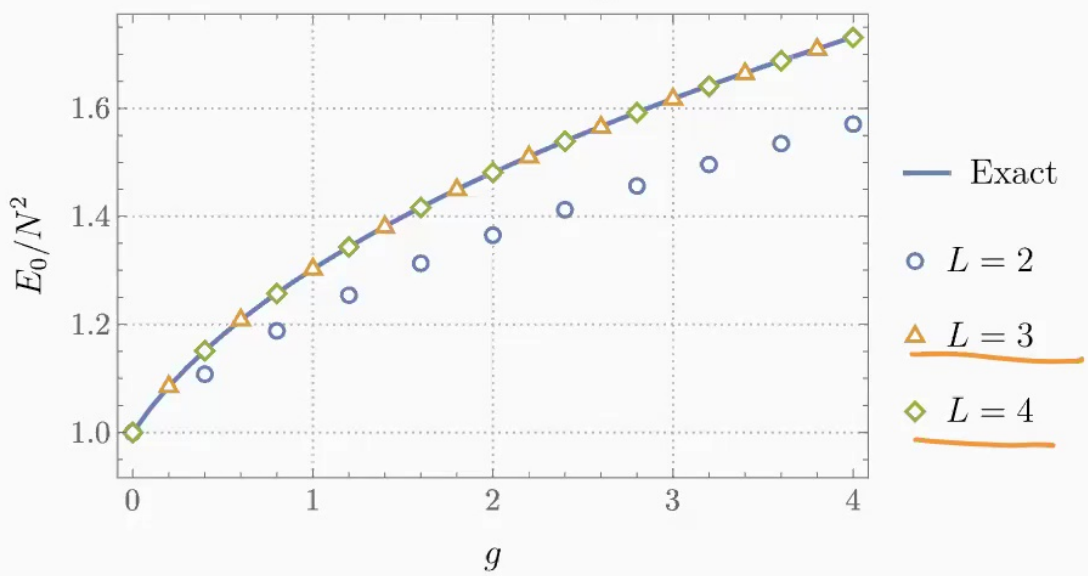
that is, the matrix $\mathcal{M}_{ij} = \langle \operatorname{tr} O_i^\dagger O_j \rangle$ must be positive semidefinite.

Example:

	I	X^2	X	P
I	$\langle \operatorname{tr} I \rangle$	$\langle \operatorname{tr} X^2 \rangle$	0	0
X^2	$\langle \operatorname{tr} X^2 \rangle$	$\langle \operatorname{tr} X^4 \rangle$	0	0
X	0	0	$\langle \operatorname{tr} X^2 \rangle$	$\langle \operatorname{tr} XP \rangle$
P	0	0	$\langle \operatorname{tr} PX \rangle$	$\langle \operatorname{tr} P^2 \rangle$

Single matrix example

$$H = \operatorname{tr} P^2 + \operatorname{tr} X^2 + g N^{-1} \operatorname{tr} X^4$$



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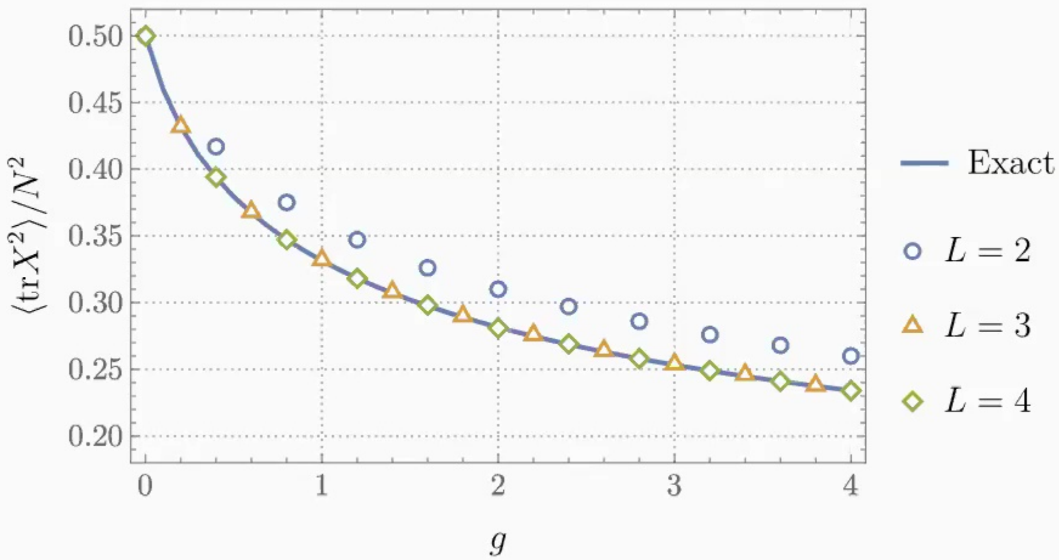
Single matrix example

$$H = \mathrm{tr}\, P^2 + \mathrm{tr}\, X^2 + gN^{-1}\, \mathrm{tr}\, X^4$$



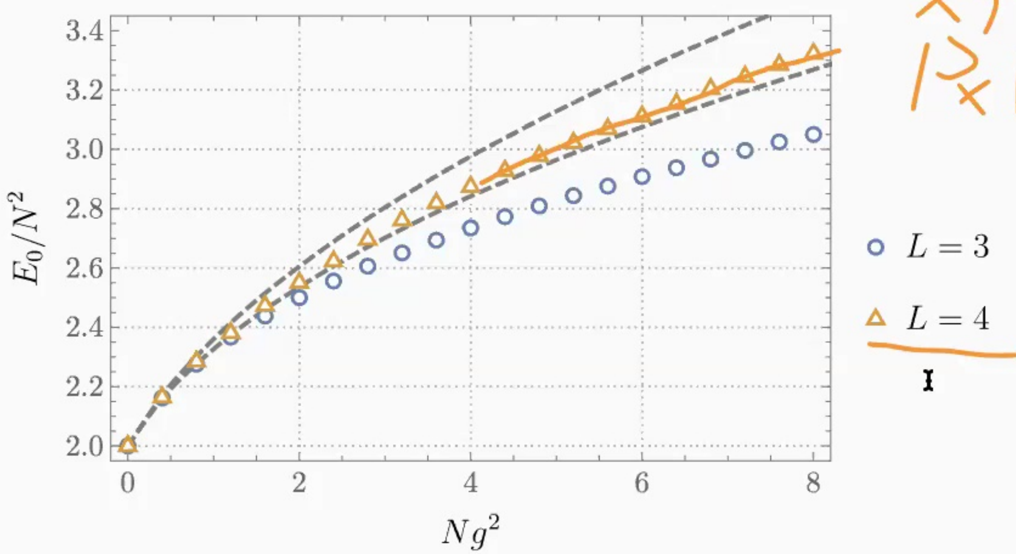
Two matrices

$$H = \text{tr}\left(P_X^2 + P_Y^2 + m^2(X^2 + Y^2) - g^2[X,Y]^2\right)$$

The $L = 4$ energy fits well to the scaling

$$E_0/N^2 \approx 1.40\,(Ng^2)^{1/3} + 1.01\,m^2/(Ng^2)^{1/3},$$

and the leading order coefficient agrees with MC result.



Bootstrap Recipe

1. Find a subset of operators of which expectation values are interesting $(\operatorname{tr} X^2, \operatorname{tr} X^4, \ldots)$
2. List all the equations between them from first principles (or something else that we know), for example from symmetries
3. Also add the semidefinite constraints $\langle O^{\dagger} O \rangle \geq 0$
4. Minimize $\langle H \rangle$ subject to the constraints in 2 and 3, if you want to know about ground states
5. That gives you a lower bound on ground state energy! If an upper bound is also know, minimize/maximize observables within this energy range and bounds on other observables are obtained.

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Hubbard model in 2d

$n = 1$	$U = 2$	$U = 4$	$U = 6$	$U = 8$
$E_{\text{lb}}\|_{K=7}$	-1.221	-0.913	-0.705	-0.565
$E_{\text{lb}}\|_{K=\infty}$	–	–	$-0.66(2)$	$-0.54(2)$
E_{DMET}	$-1.1764(3)$	$-0.8604(3)$	$-0.6562(5)$	$-0.5234(10)$
E_{DMRG}	$-1.176(1)$	$-0.8605(5)$	$-0.6565(1)$	$-0.5241(1)$
$n = 0.875$	$U = 2$	$U = 4$	$U = 6$	$U = 8$
$E_{\text{lb}}\|_{K=7}$	-1.316	-1.103	-0.963	-0.867
$E_{\text{lb}}\|_{K=\infty}$	–	–	$-0.86(5)$	$-0.77(3)$
E_{DMET}	$-1.2721(6)$	$-1.031(3)$	$-0.863(13)$	$-0.749(7)$

Table 2: Bootstrap lower bounds E_{lb} of two-dimensional Hubbard model ground state energies (per site) E_0, at fillings $n = 1$ and $n = 0.875$. Solutions from DMET and DMRG are shown for comparison.

Bounds on observables

A rigorous bound on ground state observable O is also possible by minimizing / maximizing $\langle O \rangle$ while imposing previous constraints along with

$$\langle H \rangle \leq E_{\text{ub}}.$$

An upper bound E_{ub} on ground state energy can be from, for example, a variational method.

$n = 1$	$U = 2$	$U = 4$	$U = 6$	$U = 8$
$d_{\text{lb}}\|_{K=7}$	0.160	0.106	0.071	0.049
$d_{\text{ub}}\|_{K=7}$	0.224	0.169	0.117	0.079
d_{DMET}	0.1913(4)	0.1261(1)	0.08095(4)	0.05398(7)
d_{DMRG}	0.188(1)	0.126(1)	0.0809(3)	0.0539(1)

Table 3: Bootstrap bounds of ground state double occupancy (per site) $D = n_{x\uparrow} n_{x\downarrow}$, for the two-dimensional Hubbard model at half filling.

Bounds on observables

These bounds are tight in the limit of an infinite number of constraints.

Formally we were looking for a linear functional of operators. Any linear functional can be written as $\operatorname{tr} MO$ for some operator M. Constraints $\operatorname{tr} MO^\dagger = (\operatorname{tr} MO)^*$ restrict M to be hermitian, and $\operatorname{tr} O^\dagger O \geq 0$ require M to be positive. Similarly $\operatorname{tr} MI = 1$ requires that M traces to one.

So if we impose these constraints for all operators O, and then minimize $\operatorname{tr} MH$, the minimal value (only a lower bound if only some O are considered) will be the true ground state energy.

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Conclusion

- Lower bound on ground state energy can be obtained from symmetry, positivity and locality constraints.
- The bound is systematically improved by considering more operators and converges to the exact value.
- The lower bound is complementary to the variational upper bound and provides rigorous bounds on other observables.
- Geometric symmetries and supersymmetry should be powerful in bootstrapping realistic multi-matrix models.
- Other applications?
- Thank you!

SO $\mathrm{tr} X^i X^i$

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