Title: Protected spin characters, link invariants, and q-nonabelianization

Speakers: Fei Yan

Series: Mathematical Physics

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Abstract: In this talk I will describe a new link "invariant" (with certain wall-crossing properties) for links L in a three-manifold M, where M takes the form of a surface times the real line. This link "invariant" is constructed via a map, called the q-nonabelianization map, from the gl(N) skein algebra of M to the gl(1) skein algebra of a covering three-manifold M'. In the special case of M=R^3, this map computes well-known link invariants in a new way. As a physical application, the q-nonabelianization map computes protected spin character counting BPS ground states with spin for line defects in 4d N=2 theories of class-S. I will also mention possible extension to more general three-manifolds, as well as further physical applications to class-S theories. This talk is based on joint work with Andrew Neitzke, and ongoing work with Gregory Moore and Andrew Neitzke.

Protected spin characters, link invariants, and q-nonabelianization

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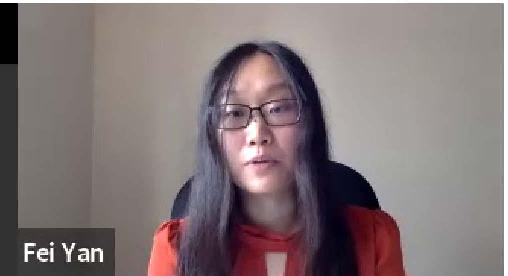
Mathematical Physics Seminar
Perimeter Institute

September 17th, 2020



Outlook

I will describe the construction of a new link "invariant" (with wall-crossing behaviors) for links L in a three-manifold M ($M = \text{surface} \times \mathbb{R}$).

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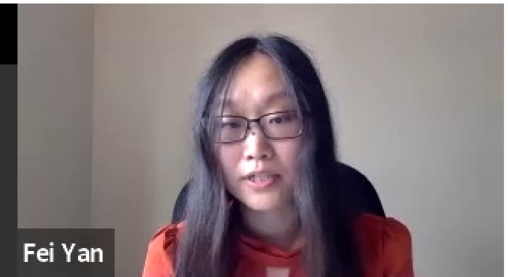
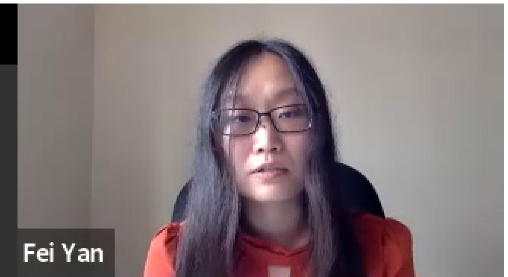
- This construction computes protected spin characters counting BPS ground states with spin for line defects in 4d $N = 2$ theories of class S.
- When $M = \mathbb{R}^3$, this construction computes well-known link invariants in a new way.

- *q-nonabelianization for line defects*
  arXiv:2002.08382
  To appear in JHEP
- 20XX.XXXXX
  21XX.XXXXX



Outlook

I will also describe further possible connections to exact WKB analysis in 4d $N = 2$ theories.



Line defects in 4d $N = 2$ theories

Consider 4d $N = 2$ supersymmetric field theory, with the insertion of a supersymmetric line defect extending along time direction, sitting at the origin of spacial $\mathbb{R}^3$. (susy Wilson-'t Hooft lines and generalizations)
[Kapustin],[Kapustin-Saulina],[Drukker-Morrison-Okuda],[Drukker-Gaiotto-Gomis], [Drukker-Gomis-Okuda-Teschner],[Gaiotto-Moore-Neitzke],[Córdova-Neitzke], [Aharony-Seiberg-Tachikawa],[Moore-Royston-van den Bleeken], . . .

Going to a point in the Coulomb branch, the bulk has IR effective description in terms of $U(1)^r$ gauge theory.

Question: What does a supersymmetric line defect look like in the IR?

A superposition of supersymmetric line defects in the abelian theory, with coefficients in this superposition given by framed protected spin characters. [Gaiotto-Moore-Neitzke]

Framed protected spin characters and the UV-IR map

The framed protected spin character (PSC) is a supersymmetric index counting the supersymmetric ground states of the bulk-defect system, with IR electromagnetic (and flavor) charge γ.

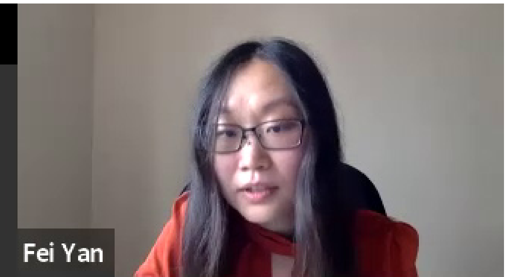
$$\overline{\underline{\Omega}}(\mathbb{L}, \gamma) := \mathrm{Tr}_{\mathcal{H}_{\mathbb{L},\gamma}} (-q)^{2J_3} q^{2I_3} \in \mathbb{Z}[q, q^{-1}]$$

J_3: $SU(2)_P$ spacial rotation, I_3: $SU(2)_R$ R-symmetry.

The UV-IR map for line defects:

$$\mathbb{L} \rightsquigarrow F(\mathbb{L}) := \sum_{\gamma} \overline{\underline{\Omega}}(\mathbb{L}, \gamma) X_{\gamma}$$

X_γ represent IR Wilson-'t Hooft lines with charge γ.



Framed protected spin characters and the UV-IR map

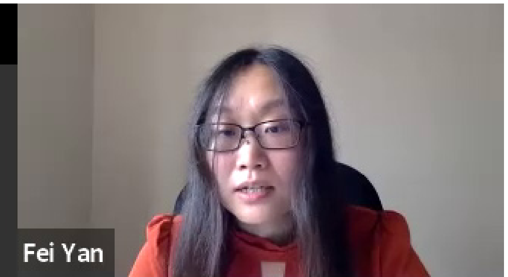
Example: [Gaiotto-Moore-Neitzke],[Córdova-Neitzke]
$N = 2$ pure $SU(2)$ SYM at a point in the weak-coupling region of its Coulomb branch, denote IR charge as (γ_e, γ_m).

- Wilson line $\mathbb{L}_{\underline{2}}$ in the fundamental representation:

$$F(\mathbb{L}_{\underline{2}}) = X_{(1,0)} + X_{(-1,0)} + X_{(0,1)}$$

- Wilson line $\mathbb{L}_{\underline{3}}$ in the adjoint representation:

$$\begin{aligned} F(\mathbb{L}_{\underline{3}}) =& X_{(2,0)} + X_{(0,0)} + X_{(-2,0)} + X_{(0,2)} \\ &+ (-q - q^{-1}) X_{(-1,1)} + (-q - q^{-1}) X_{(1,1)} \end{aligned}$$

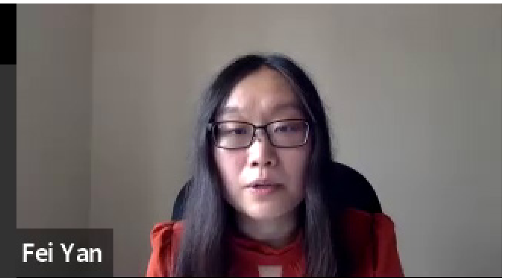


How could one compute PSC?

There has been extensive work on computation of PSC:

- For Lagrangian theories: supersymmetric localization, semiclassical computation
- For theories of quiver type: quiver quantum mechanics
- For theories of class S: spectral networks
- ...

[Gaiotto-Moore-Neitzke],[Córdova-Neitzke],[Moore-Royston-van den Bleeken] [Coman-Gabella-Teschner],[Ito-Okuda-Taki],[Gabella],[Galakhov-Longhi-Moore],...



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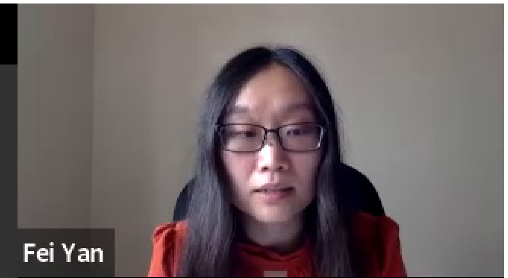
Today, I will describe a geometrical way to compute PSC for line defects in theories of class S.



Line defects in class S theories

Consider 6d $(2,0)$ theory of type $\mathfrak{gl}(N)$ on $C \times \mathbb{R}^{3,1}$ (C: Riemann surface) with certain twisting, compactify on $C \rightsquigarrow$ 4d N=2 theory of class S.
Supersymmetric surface defect in 6d wrapping $\ell \times \mathbb{R}^{0,1} \subset C \times \mathbb{R}^{3,1}$
$\rightsquigarrow$ 1/2-BPS supersymmetric line defect in 4d. [Gaiotto-Moore-Neitzke]
(We only use 6d surface defects in fundamental representation of $\mathfrak{gl}(N)$.)

A point in the Coulomb branch corresponds to a N-fold branched covering $\widetilde{C} \to C$, where $\widetilde{C} \subset T^*C$ is the Seiberg-Witten curve. In the IR the bulk is approximated by 6d $(2,0)$ theory of type $\mathfrak{gl}(1)$ on $\widetilde{C} \times \mathbb{R}^{3,1}$. IR line defects come from 6d surface defects wrapping $\widetilde{\ell} \times \mathbb{R}^{0,1} \subset \widetilde{C} \times \mathbb{R}^{3,1}$.

In the $q \to 1$ limit, the UV-IR map is roughly a map taking $\ell \subset C$ to $\widetilde{\ell} \subset \widetilde{C}$. [Gaiotto-Moore-Neitzke]
What about the case for a generic q?

Line defects OPE and skein algebras

When q is generic, supersymmetric line defects have to sit along a fixed axis in $\mathbb{R}^3 \rightsquigarrow$ non-commutative associative OPE $*$ [Gaiotto-Moore-Neitzke],[Yagi],...
In the UV, the OPE is in general complicated.
In the IR, the OPE is given by quantum torus:

$$X_{\gamma_1} * X_{\gamma_2} = (-q)^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1 + \gamma_2}.$$

$\langle , \rangle$ is the Dirac-Schwinger-Zwanziger pairing.

In class-S theories, line defects OPE are described via skein algebras.
[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],[Gaiotto-Moore-Neitzke],[Witten]
[Drukker-Gomis-Okuda-Teschner],[Tachikawa-Watanabe],[Coman-Gabella-Teschner],...
Hint: the UV-IR map is a map between the UV and IR skein algebras.

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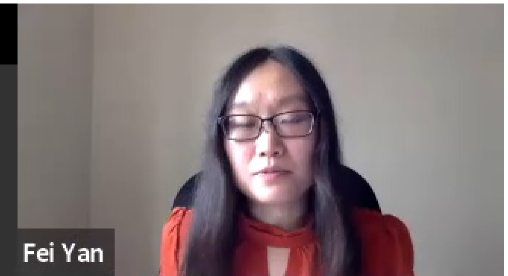


A new kind of line defects in class-S theories

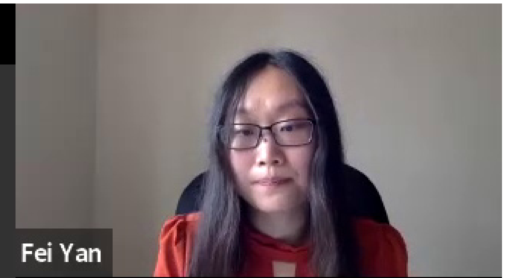
Consider an oriented link $L \subset M = C \times \mathbb{R}_h$, we put a supersymmetric surface defect in the 6d $(2,0)$ $\mathfrak{gl}(N)$ theory on

$$L \times \mathbb{R}^{0,1} \subset M \times \mathbb{R}^{2,1}$$

[Witten],[Gaiotto-Witten],[Ooguri-Vafa],[Gukov-Schwarz-Vafa],[Dimofte-Gaiotto-Gukov],[Chun-Gukov-Roggenkamp],[Gukov-Putrov-Vafa],[Gukov-Pei-Putrov-Vafa], ...

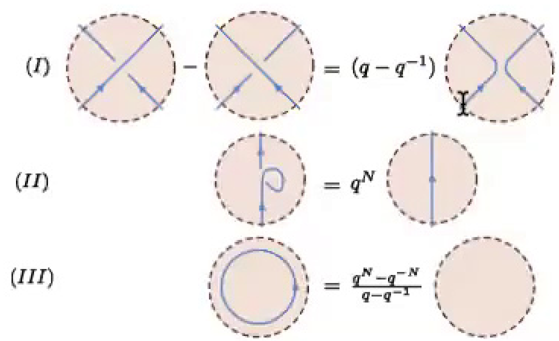
In the IR the finite extent of L in the $\mathbb{R}_h$-direction is suppressed, it looks like a line defect in the 4d class-S theory, which is 1/4-BPS and only preserves $U(1)_P$ rotation symmetry and $U(1)_R$ R-symmetry. We could still define PSC to count its BPS ground states.

If L is isotopic to a simple closed curve on C at a fixed $\mathbb{R}_h$-coordinate, then the line defect is 1/2-BPS and preserves the full rotation symmetry.

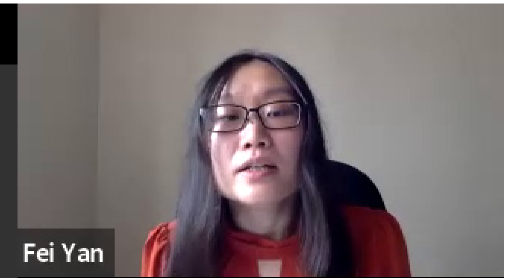


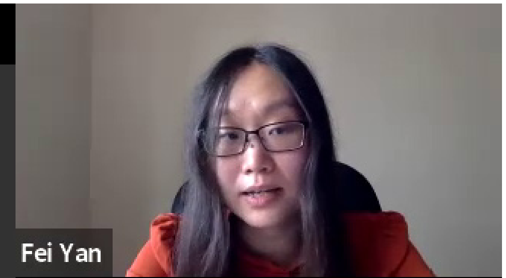
The UV skein algebra

The space of such line defects equipped with OPE is described by the $\mathfrak{gl}(N)$ HOMFLY skein algebra of $M = C \times \mathbb{R}_h$, defined as the space of formal $\mathbb{Z}[q^{\pm 1}]$-linear combinations of framed oriented links (up to isotopy) in M, modulo the following relations:



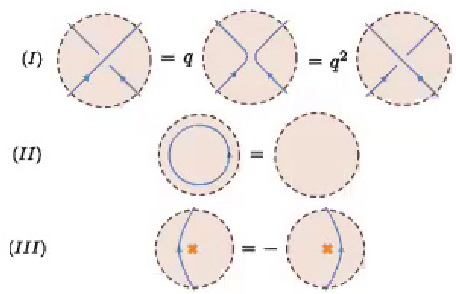
The multiplication is defined by "stacking" links along the $\mathbb{R}_h$-direction.





The IR skein algebra

The IR line defects correspond to framed oriented links $\widetilde{L} \subset \widetilde{M} = \widetilde{C} \times \mathbb{R}$.
The OPE algebra is (twisted) $\mathfrak{gl}(1)$ skein algebra of $\widetilde{M}$, defined as the space of formal $\mathbb{Z}[q^{\pm 1}]$-linear combinations of framed oriented links (up to isotopy) in $\widetilde{M}$, modulo the following relations:



This skein algebra is indeed isomorphic to the quantum torus.
Given a charge $\gamma \in H_1(\widetilde{C}, \mathbb{Z})$, X_γ corresponds to the class of certain loop on $\widetilde{C}$ in class γ. The DSZ pairing corresponds to the intersection pairing in $H_1(\widetilde{C}, \mathbb{Z})$.

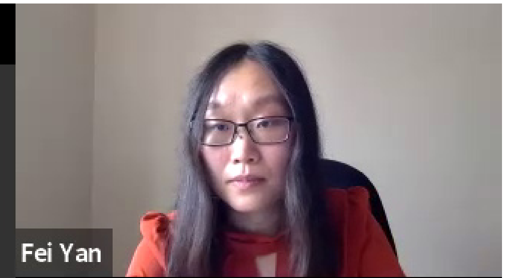
The UV-IR map

To compute framed protected spin characters, or equivalently to construct the UV-IR map for line defects, we need a map F from the UV skein algebra to the IR skein algebra, sending links $L \subset M$ to links $\widetilde{L} \subset \widetilde{M}$:

$$F(L) = \sum_{\widetilde{L}} \alpha(\widetilde{L})\widetilde{L}, \quad \alpha(\widetilde{L}) \in \mathbb{Z}[q^{\pm 1}].$$

The map F has to respect line defects OPE, i.e. it has to be a homomorphism from the $\mathfrak{gl}(N)$ HOMFLY skein algebra of M to the (twisted) $\mathfrak{gl}(1)$ skein algebra of $\widetilde{M}$.

F could be viewed as quantization of the classical non-abelianization map. [Gaiotto-Moore-Neitzke],[Hollands-Neitzke], ...

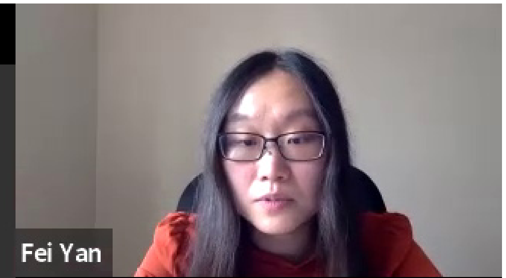


Construction of the UV-IR map

With Andrew Neitzke, we constructed the UV-IR map F completely for $N = 2$, almost for $N = 3$, many pieces working for $N > 3$.

What's our basic strategy?

- Compactifying the time direction and introducing a twist by $(-q)^{2J_3} q^{2I_3}$, the reduced theory is effectively a twisted and Ω-deformed 5d $\mathcal{N} = 2$ $U(N)$ SYM on $M \times \mathbb{R}^2_\epsilon$, with insertion of fundamental Wilson line along the link $L \subset M$.
- The theory is put in a background which generically breaks $U(N) \to U(1)^N$, determined by the covering $\widetilde{M} \to M$.
- Theoretically $F(L)$ could be obtained by computing the partition function of the 5d $\mathcal{N} = 2$ $U(N)$ SYM in the above background, and then expanding it in terms of partition function of the IR effective theory, with IR Wilson lines insertions.



Construction of the UV-IR map

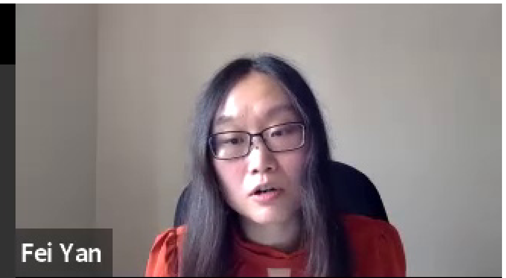
We did not carry out our construction in such localization paradigm. Rather, we combine the above physical picture with certain bootstrap methods.

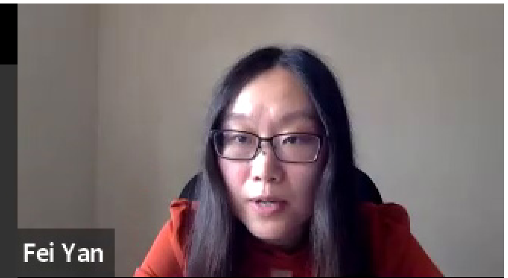
Concretely, our construction of the UV-IR map:

$$F(L) = \sum_{\widetilde{L}} \alpha(\widetilde{L})\widetilde{L}, \quad \alpha(\widetilde{L}) \in \mathbb{Z}[q^{\pm 1}].$$

consists of two steps:

- Step 1: Enumerate all possible $\widetilde{L} \subset \widetilde{M}$ consisting of local pieces.
- Step 2: To each $\widetilde{L}$ we assign a factor $\alpha(\widetilde{L}) \in \mathbb{Z}[q^{\pm 1}]$.

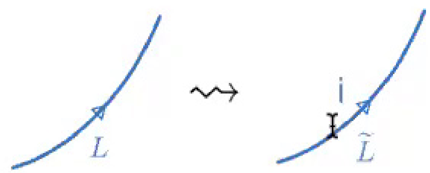




Construction of the UV-IR map: Step 1

To enumerate $\widetilde{L} \subset \widetilde{M}$, we assemble local pieces of the following two kinds:

- Locally away from branch points of $\widetilde{M} \to M$, $U(N)$ is broken into $U(1)^N$. The fundamental representation of $U(N)$ is decomposed into its N weight spaces, corresponding to the N sheets of $\widetilde{M}$.
  The fundamental Wilson line in the $U(N)$ theory decomposes locally into N Wilson lines of the $U(1)^N$ theory:

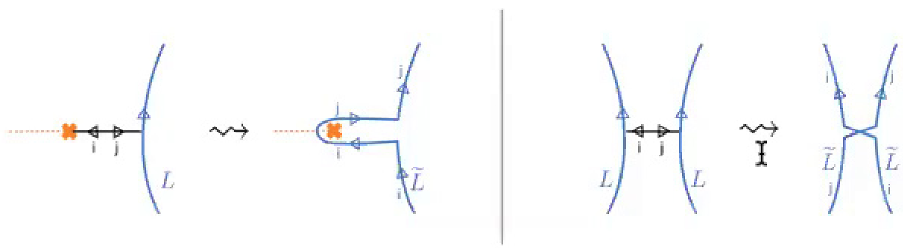


  The local strand of $L \subset M$ is mapped to its N preimages under the covering $\widetilde{M} \to M$.

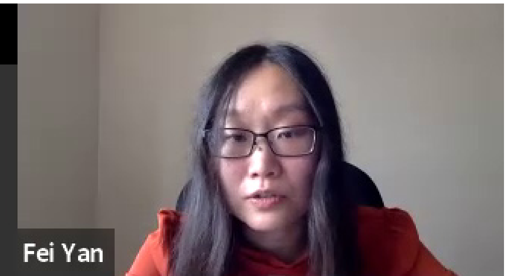
Construction of the UV-IR map: Step 1

- The second type of local piece represents trajectories from massive W-bosons. W-bosons are labeled by two sheets i, j of $\widetilde{M}$. The BPS condition determines that, W-bosons must travel along the so-called BPS ij-trajectories [Gaiotto-Moore-Neitzke].

  For $N = 2$, W-bosons could travel to/from branch locus of $\widetilde{M} \to M$. They could also be exchanged between different strands of L.



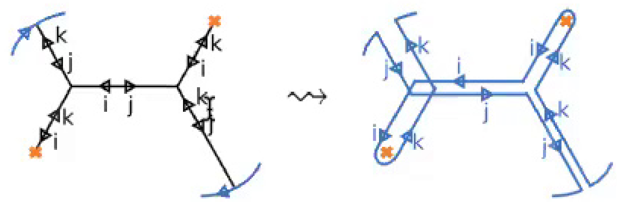
  For $N > 2$, W-bosons could travel along complicated web trajectories.



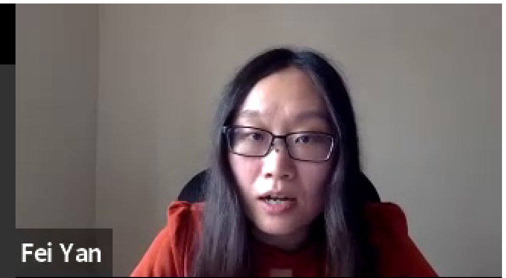
Construction of the UV-IR map: Step 2

Each $\widetilde{L}$ carries a weight factor $\alpha(\widetilde{L}) \in \mathbb{Z}[q^{\pm 1}]$, which also consists of various local contributions. We determined these weight factors by bootstrap, with the constraint that UV-IR map F is a homomorphism from the UV skein algebra to the IR skein algebra.

For example, web trajectories of W-bosons contribute a weight factor of $\pm(q - q^{-1})^m$, which vanishes as $q \to 1$.



In practice, this could be determined in a reductive way using isotopy invariance constraints.



Intermediate Summary

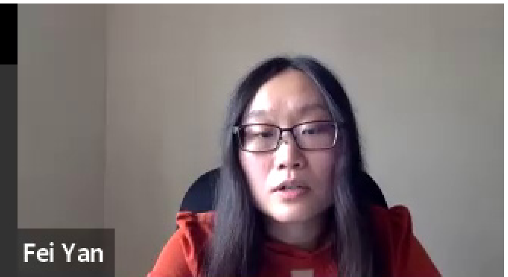
We constructed UV-IR map F computing protected spin characters for a new kind of line defects in class-S theories. F is a map from the UV skein algebra to the IR skein algebra. Concretely it maps links $L \subset M = C \times \mathbb{R}$ to formal combinations of links $\widetilde{L} \subset \widetilde{M} = \widetilde{C} \times \mathbb{R}$:

$$F(L) = \sum_{\widetilde{L}} \alpha(\widetilde{L})\widetilde{L}, \quad \alpha(\widetilde{L}) \in \mathbb{Z}[q^{\pm 1}].$$

- When $M = \mathbb{R}^3$, $F(L)$ computes a familiar link invariant:

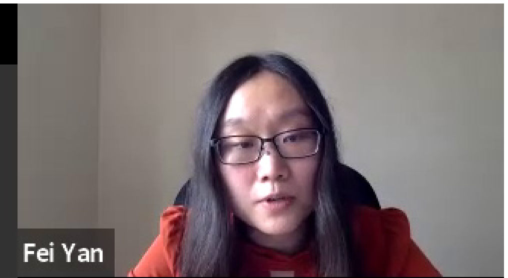
$$F(L) = q^{Nw(L)} P_{\mathrm{HOMFLY}}(L, a = q^N, z = q - q^{-1}),$$

  where $w(L)$ is the self-linking number of L.
  For $N = 2$, our construction reproduces the vertex model for Jones polynomials, interpreted in physics language by Witten and Gaiotto-Witten.



Intermediate Summary

- In certain cases, the UV-IR map has been obtained using a different method.
  For $M = C \times \mathbb{R}$ and $N = 2$, Bonahon-Wong constructed quantum trace map.
  For $M = C \times \mathbb{R}$ and at special loci in the Coulomb branch where the BPS ij-trajectories are nearly collinear, Gabella computed PSC using spectral network with additional R-matrix factors.



Wall-Crossing behavior

The UV-IR map F depends on the covering $\widetilde{M} \to M$. As we move on the Coulomb branch, F could change discontinuously. This is called (framed) wall-crossing. The wall-crossing behavior of F is controlled by the BPS spectrum of the bulk 4d theory.

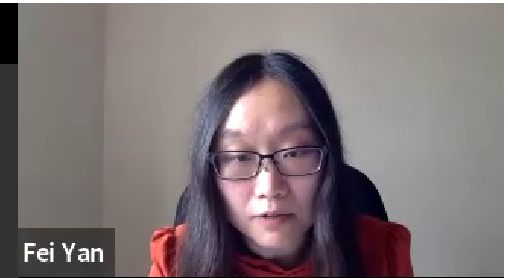
[Kontsevich-Soibelman], [Gaiotto-Moore-Neitzke], [Dimofte-Gukov],...

For example, across a wall corres. to a BPS hypermultiplet with charge γ:

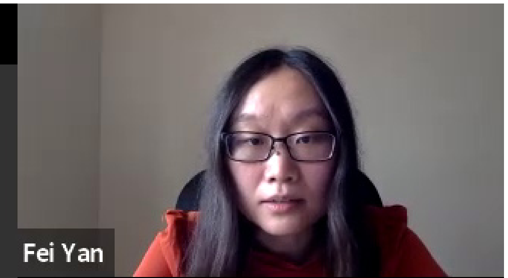
$$F(L) \to E_q(X_\gamma) F(L) E_q(X_\gamma)^{-1},$$

where $E_q(x)$ is the quantum dilogarithm.

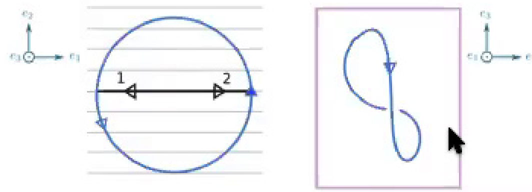
For $N = 2$, we have checked our map F has the expected wall-crossing behavior.



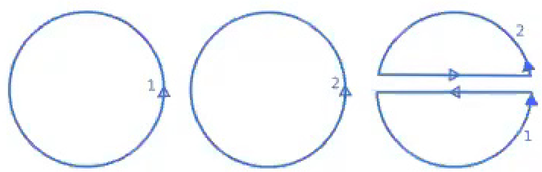
A simple example in $M = \mathbb{R}^3$ $(N = 2)$



We take $M = \mathbb{R}^3$ and $\tilde{M} = \mathbb{R}^3 \cup \mathbb{R}^3$. The BPS 12-trajectories are straight lines in (say) x^1-direction. We take L to be an unknot.



There are three lifts of L:

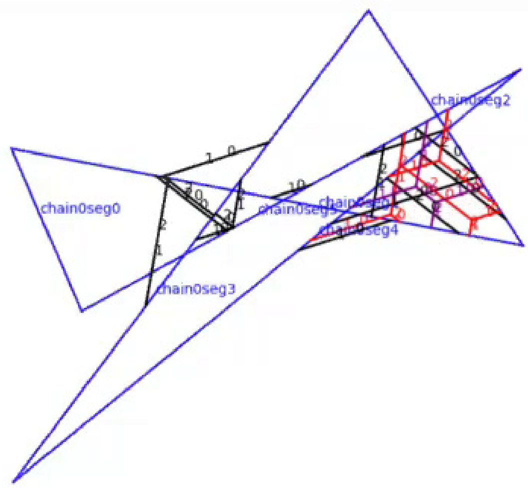


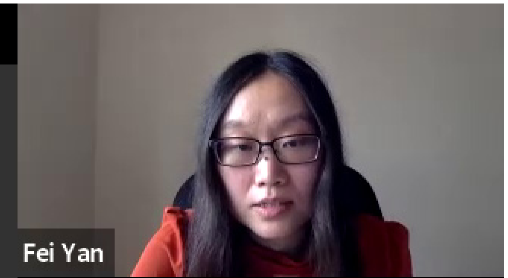
Their contributions sum up to

$$F(L) = q^{-1} + q^{-1} + (q - q^{-1}) = q + q^{-1}$$

Another example in $M = \mathbb{R}^3$ ($N = 3$)

We take $M = \mathbb{R}^3$, and $\widetilde{M} = \mathbb{R}^3 \cup \mathbb{R}^3 \cup \mathbb{R}^3$. There are now three different kinds of BPS trajectories, as straight lines along three different directions. We take L to be a trefoil. There are in total 51 lifts of L.

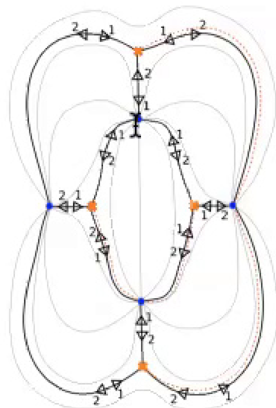


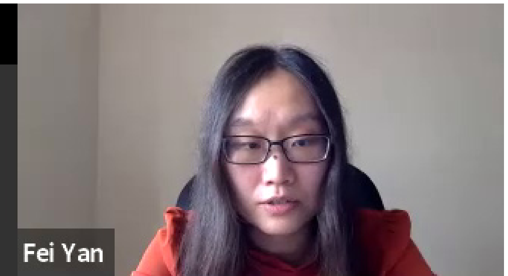


PSC computation in $SU(2)$ with $N_f = 4$

We take $N = 2$ and $M = C \times \mathbb{R}$, where C is a four-punctured sphere. This corresponds to the $SU(2)$ gauge theory with four fundamental hypers. We go to a point in the moduli space where the Seiberg-Witten curve is given by

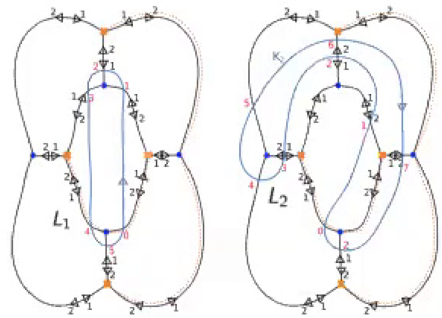
$$\tilde{C} = \{\lambda : \lambda^2 + \phi_2 = 0\} \subset T^*C, \quad \phi_2 = -\frac{z^4 + 2z^2 - 1}{2(z^4 - 1)^2} dz^2.$$





PSC computation in $SU(2)$ with $N_f = 4$

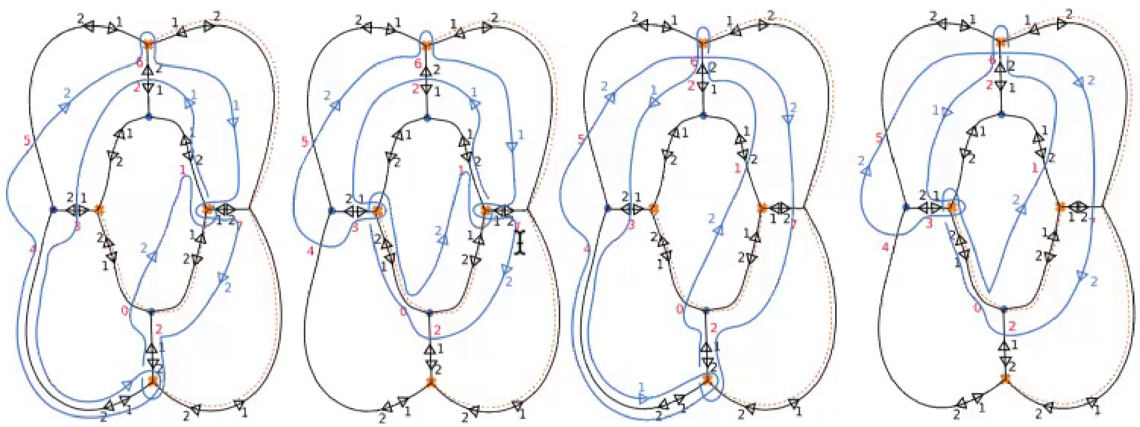
We consider two 1/2-BPS line defects:



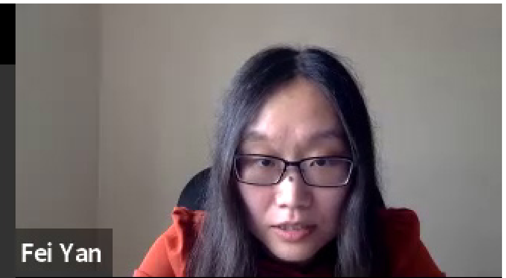
- L_1 corresponds to a fundamental Wilson line in certain duality frame, $F(L_1)$ is an expansion in 7 X_γ's with coefficients 1.
  [Gaiotto-Moore-Neitzke]
- L_2 has BPS states with non-trivial spin:

$$F(L_2) \supset \left(1 + (1 + q^2 + q^{-2})\right) X_\gamma$$

PSC computation in $SU(2)$ with $N_f = 4$

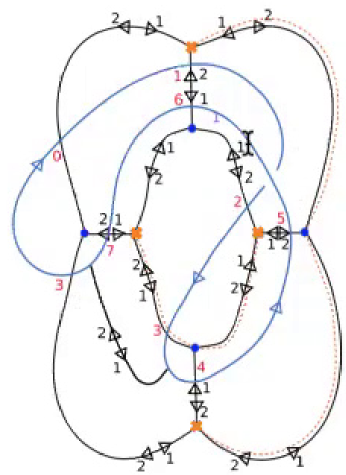


$X_{2\gamma_1-\gamma_2+\mu_1+\mu_3-2\mu_4}$ $\quad q^{-2}X_{2\gamma_1-\gamma_2+\mu_1+\mu_3-2\mu_4}$ $\quad q^{2}X_{2\gamma_1-\gamma_2+\mu_1+\mu_3-2\mu_4}$ $\quad X_{2\gamma_1-\gamma_2+\mu_1+\mu_3-2\mu_4}$

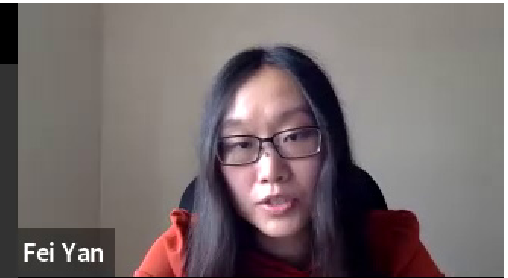


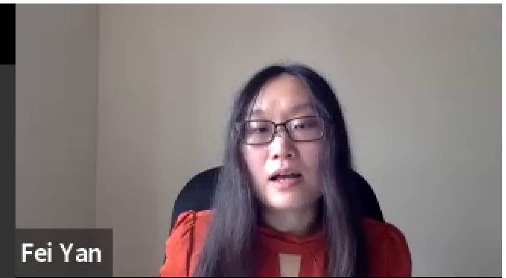
PSC computation in $SU(2)$ with $N_f = 4$

Here is an example of a 1/4-BPS line defect:



$F(L)$ is an expansion in terms of X_γ with 31 different charges.





General three-manifold M

We have taken $M = C \times \mathbb{R}$. What about more general three-manifold M?

Reducing the 6d theory (with surface defect) on M gives a 3d $N = 2$ theory with line defect insertion. One could also make a perturbation, labeled by $\widetilde{M} \to M$, under which the bulk theory flows to a Lagrangian theory in the IR. [Dimofte-Gaiotto-Gukov],[Cecotti-Córdova-Vafa],[Dimofte-Gaiotto-van der Veen],...

Question: how does a UV line defect decompose in terms of IR line defects?

Here the UV-IR map should go from the $\mathfrak{gl}(N)$ skein module of M to the (twisted) $\mathfrak{gl}(1)$ skein module of $\widetilde{M}$.

General three-manifold M

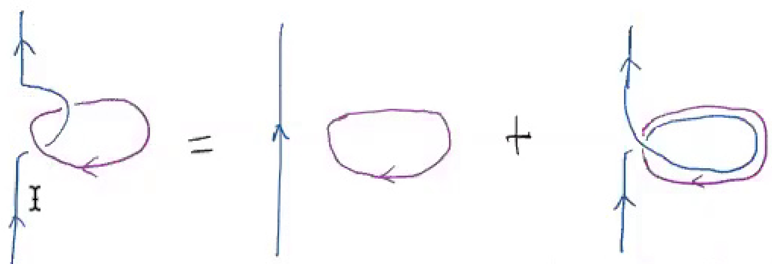
For $N = 2$, our construction could been formulated in a covariant way. (For $N > 2$ it also seems possible.)

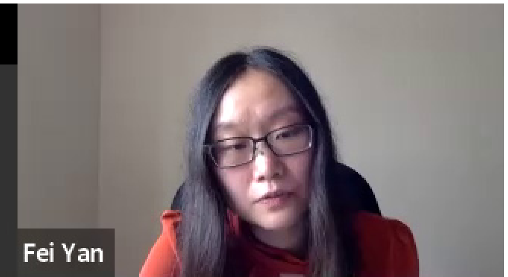
We expect most of our construction to apply to more general 3-manifolds M that admit ideal triangulations, with certain new ingredients.

In particular, $\widetilde{M}$ has an new kind of singularity, one in each tetrahedron of the triangulation.

[Freed-Neitzke],[Cecotti-Córdova-Vafa],[Dimofte-Gaiotto-van der Veen]

We need extra skein relations in the (twisted) $\mathfrak{gl}(1)$ skein module of $\widetilde{M}$:



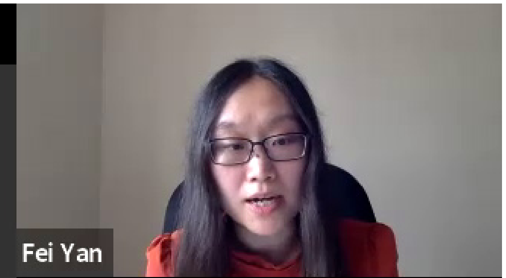


Interlude

- In our construction of the UV-IR map to compute framed protected spin characters, we have effectively turned on half Ω-background along a spacial $\mathbb{R}^2_\epsilon$-plane.

Interlude

- In our construction of the UV-IR map to compute framed protected spin characters, we have effectively turned on half Ω-background along a spacial $\mathbb{R}^2_\epsilon$-plane.
- On the other hand, there has been recent developments in exact WKB analysis and non-perturbative approaches to the quantization of Seiberg-Witten curves in 4d $N = 2$ theories. In that context, one also turns on half Ω-background.
- It is tempting to ask whether one could combine the above two stories by turning on the full Ω-background.



Exact WKB and 4d $N = 2$ gauge theories

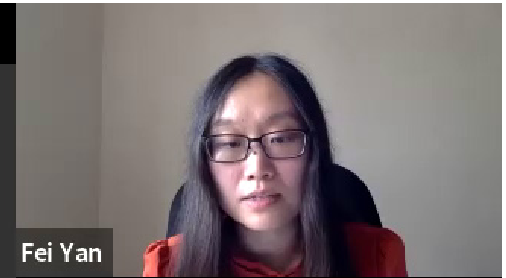
Recently there are many interesting discoveries on relations between quantum mechanical systems and supersymmetric gauge theories.

[Dorey-Tateo],[Nekrasov-Shatashvili], [Nekrasov-Rosly-Shatashvili],[Gaiotto-Moore-Neitzke],[Alday-Gaiotto-Tachikawa], [Gaiotto],[Grassi-Hatsuda-Marino],[Neitzke-Hollands],[Jeong-Nekrasov],[Grassi-Marino],[Grassi-Gu-Marino],[Gaiotto-Lee-Wu]...

The center piece is the Schrödinger equation (and its higher analogue):

$$\left[\hbar^2 \partial_z^2 + P(z)\right] \psi(z) = 0.$$

- Exact WKB methods to study the monodromy data [Voros],[Kawai-Takei]...
  Geometric formulation using abelianization [Gaiotto-Moore-Neitzke],[Hollands-Neitzke]...
- Spectral problems in quantum mechanics
- Quantization of Seiberg-Witten curve in 4d $N = 2$ gauge theories
  Deformation of chiral ring for surface defects in half-Ω background



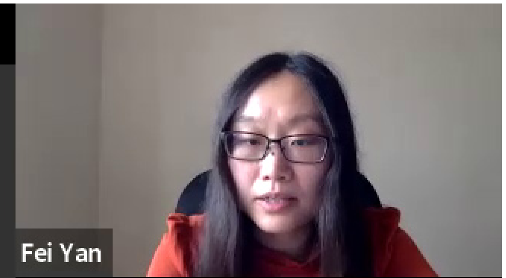
The Voros symbol $\mathcal{X}_\gamma(\hbar)$

In abelianization language, the Voros symbol is:

$$\mathcal{X}_\gamma(\hbar) = \mathrm{Hol}_\gamma \nabla^{\mathrm{ab}}(\hbar), \quad \gamma \in H_1(\Sigma, \mathbb{Z}).$$

It could be obtained via different methods:

- Borel resummation of the exponentiated WKB periods.
- Ratios of Wronskians of certain distinguished local solutions to the Schrödinger equation: asymptotically decaying solutions as z goes into a singularity; or eigenvectors of the monodromy around a loop.
- Computable by instanton calculous in 4d $N = 2$ gauge theories.
- Computable by certain TBA-like integral equations.

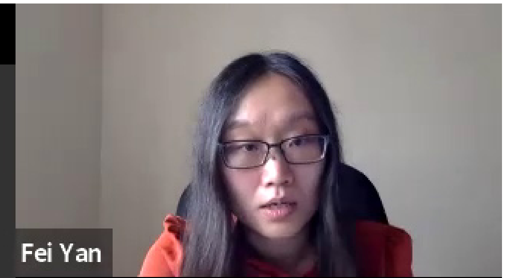


Integral equation for $\mathcal{X}_\gamma(\hbar)$

The Voros symbol $\mathcal{X}_\gamma(\hbar)$ obey the following TBA-like integral equation: [Gaiotto],[Gaiotto-Moore-Neitzke]

$$\mathcal{X}_\gamma(\hbar) = \exp\left[\frac{Z_\gamma}{\hbar} + \frac{1}{4\pi i}\sum_{\mu}\Omega(\mu)\langle\gamma,\mu\rangle\int_{\hbar'\in\mathbb{R}_{-}Z_\mu}\frac{d\hbar'}{\hbar'}\frac{\hbar'+\hbar}{\hbar'-\hbar}\log(1+\mathcal{X}_\mu(\hbar'))\right]$$

- Here Z_γ is the central charge (classical period).
- $\Omega(\mu)\in\mathbb{Z}$ counts BPS states with charge μ, $\mu\in H_1(\Sigma,\mathbb{Z})$.
- Solutions to the integral equation solve the Riemann-Hilbert problem formulated by Gaiotto-Moore-Neitzke. The integrand encodes the jumping behavior across a BPS $\mathcal{K}$-wall.



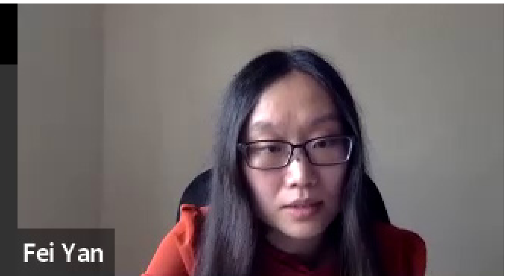
Integral equation for $\mathcal{X}_\gamma(\zeta)$

Question: Is there a q-deformed version of the integral equation? Perhaps by turning on the other half Ω-deformation?

So far, unclear how to approach this from the point of view of Ω-deformation...

The previous integral equation is the conformal limit [Gaiotto] of another integral equation [Gaiotto-Moore-Neitzke] obeyed by $\mathcal{X}_\gamma(\zeta)$. One could try to introduce q-deformation there. (ongoing with Moore and Neitzke.)

Reducing class-S theory on S^1, $\mathcal{X}_\gamma(\zeta)$ corresponds to the VEV of infrared line defects $X_\gamma(\zeta)$ wrapped around S^1. The integrals in TBA equation represents corrections to the semi-classical VEV.

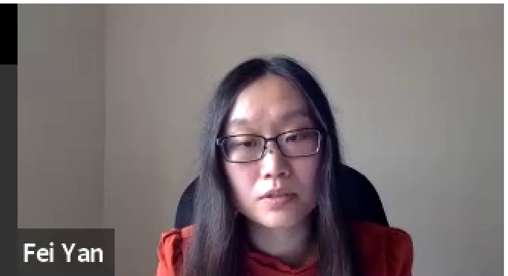


Integral equation for $X_\gamma(\zeta)$

We introduce q-deformation by turning on half Ω-background along spacial $\mathbb{R}^2_\epsilon$. The IR line defects OPE become non-commutative, described by the quantum torus:

$$X_{\gamma_1}(\zeta) * X_{\gamma_2}(\zeta) = (-q)^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1+\gamma_2}(\zeta).$$

- The integral equation we want should be an operator equation respecting the quantum torus.
- The integrand should be able to reproduce the wall-crossing behavior by conjugation of quantum dilogarithm.



Integral equation for $X_\gamma(\zeta)$

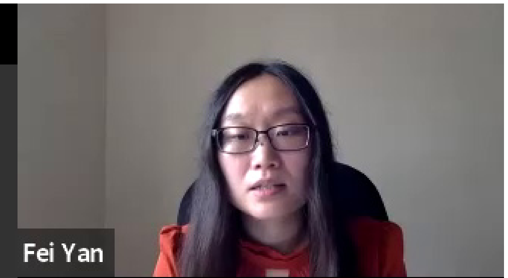
For generic q, this seems rather difficult. Simplifications happen when q is a root of unity.

Consider the space of ground states of 4d IR theory on Melvin space $\mathbb{R}^2 \times_q S^1$. This space form a representation of the quantum torus. Via pullback of the UV-IR map F, it also forms a representation of the UV skein algebra.

When q is a root of unity, this is a finite-dimensional representation, labeled by flat complex connections on C. [Bonahon-Wong]

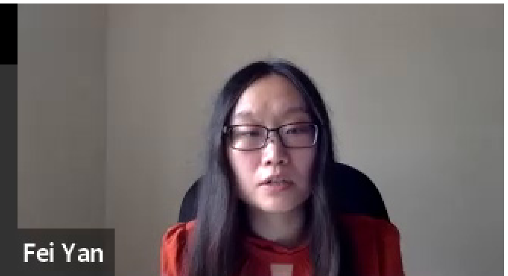
In this case, we could explicitly write down $X_\gamma(\zeta)$ in certain matrix representation, and try to formulate the integral equation for individual matrix entries.

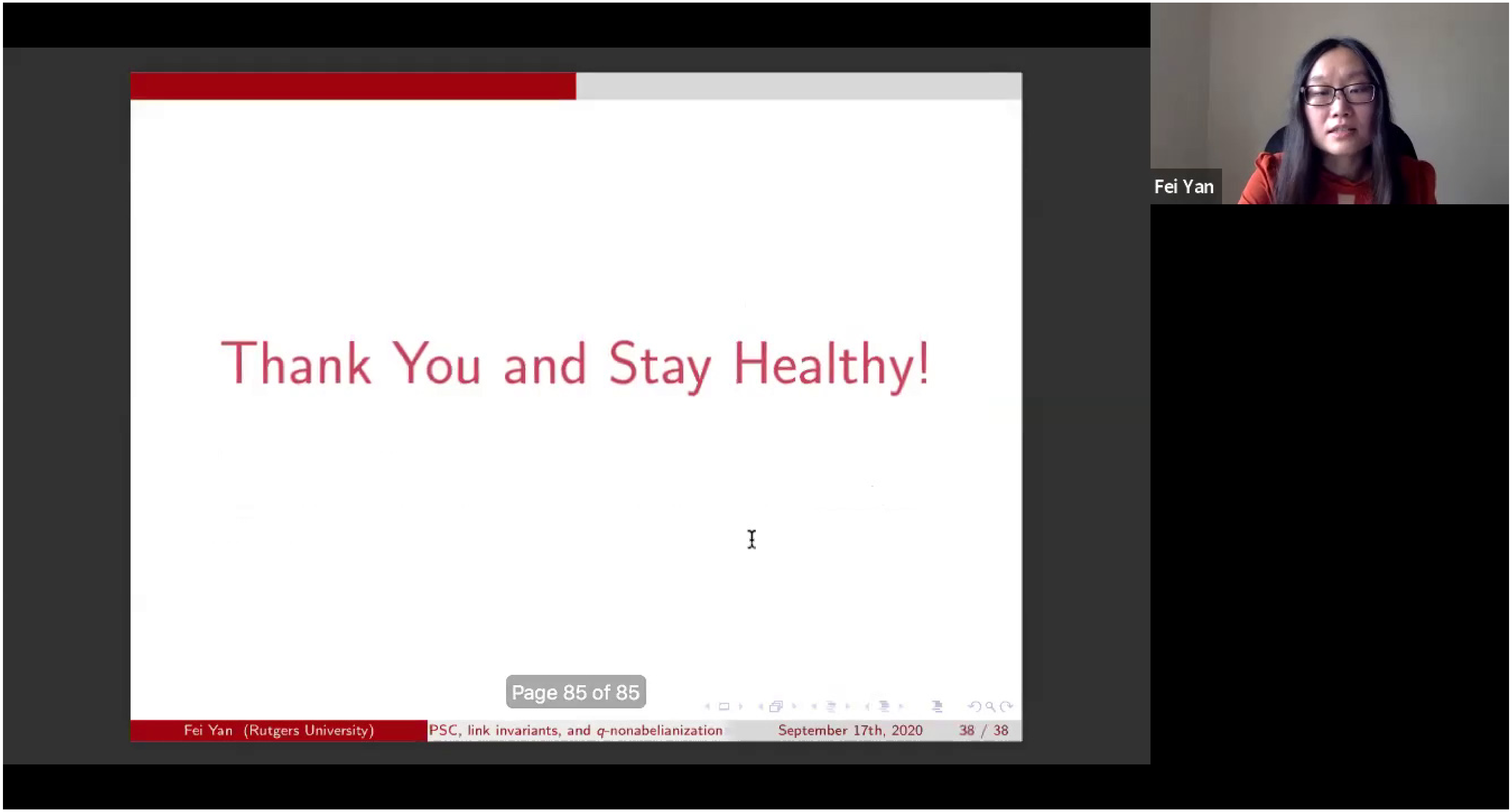
Work in progress...



Summary

- We have constructed a link "invariant" via UV-IR map. It reduces to well-known link invariants when $M = \mathbb{R}^3$.
- The UV-IR map counts BPS ground states with spin for line defects in class-S theories.
- It also has possible connections to exact WKB analysis in 4d $N = 2$ theories.




Fei Yan
Thank You and Stay Healthy!
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