Title: Diophantine approximation as Cosmic Censor for AdS black holes

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Series: Strong Gravity

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Abstract: The statement that general relativity is deterministic finds its mathematical formulation in the celebrated â€˜Strong Cosmic Censorship Conjectureâ€™ due to Roger Penrose. I will present my recent results on this conjecture in the case of negative cosmological constant and in the context of black holes. It turns out that this is intimately tied to Diophantine properties of a suitable ratio of mass and angular momentum of the black hole.

Diophantine Approximation as Cosmic Censor for AdS black holes

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Strong Gravity seminar
Perimeter Institute

24 September 2020

Dynamics in General Relativity

- General relativity is a *dynamic* theory.
- Einstein equations are hyperbolic and admit well-posed initial value formulation.

In vacuum:

$$\mathrm{Ric}(g) = \Lambda g. \qquad \text{(EE)}$$

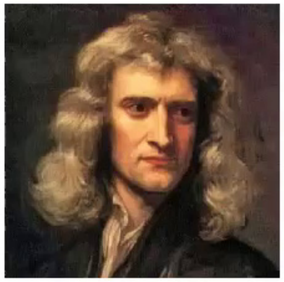
Depending on context: Cosmological constant $\Lambda = 0, \Lambda > 0, \Lambda < 0$.

Theorem (Choquet-Bruhat).

For appropriate initial data $(\Sigma^3, \bar{g}, k)$*, there exists a unique maximal future development* $(\mathcal{M}^4, g)$ *solving the Einstein equations* (EE).

DETERMINISM IN GENERAL RELATIVITY

Look at Newtonian Gravity first.



Newtonian gravity for N point masses x_i:

$$m_i \ddot{x}_i = F_i$$

$$F_i = -\sum_{j \neq i} \frac{m_i m_j (x_i - x_j)}{|x_i - x_j|^3}$$

Newtonian gravity is **deterministic** because either

1. **a unique solution exists globally**
2. **or we exit the validity of the theory,** $|F_i| \to +\infty$**.**

DETERMINISM IN GENERAL RELATIVITY

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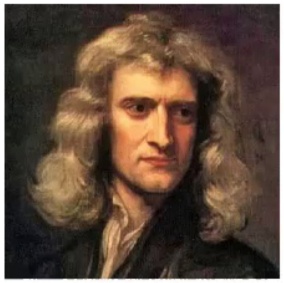

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Back to Einstein... Is it deterministic like Newtonian Gravity?

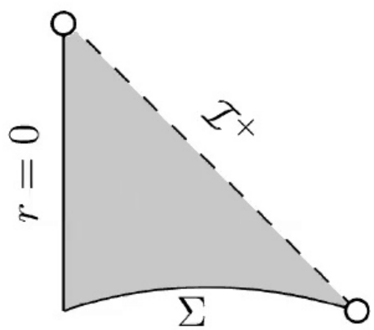
First, look at explicit solutions to Einstein equations for $\Lambda = 0$ (same for $\Lambda > 0$ and $\Lambda < 0$):

I) Minkowski spacetime
II) non-rotating Schwarzschild black hole
III) rotating Kerr black hole

I) MINKOWSKI SPACETIME

Initial data $\Sigma = \mathbb{R}^3$, $\bar{g} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ give

$$\mathcal{M} = [0, \infty) \times \mathbb{R}^3, \quad g = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$



Minkowski space $(\mathcal{M}, g)$ determined from data on Σ.

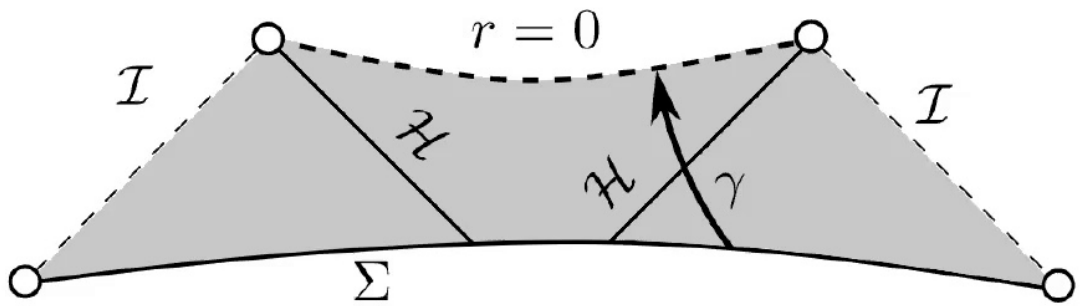
- Minkowski space is **geodesically complete**: global existence and uniqueness
- Global existence for small perturbations of initial data: Stability of Minkowski space (Christodoulou–Klainerman, '93)

Determinism <u>holds true</u>: Global existence and uniqueness (geodesically complete).

II) THE SCHWARZSCHILD BLACK HOLE

Static, nonrotating black hole: Parametrized by mass $M > 0$

$$g = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$



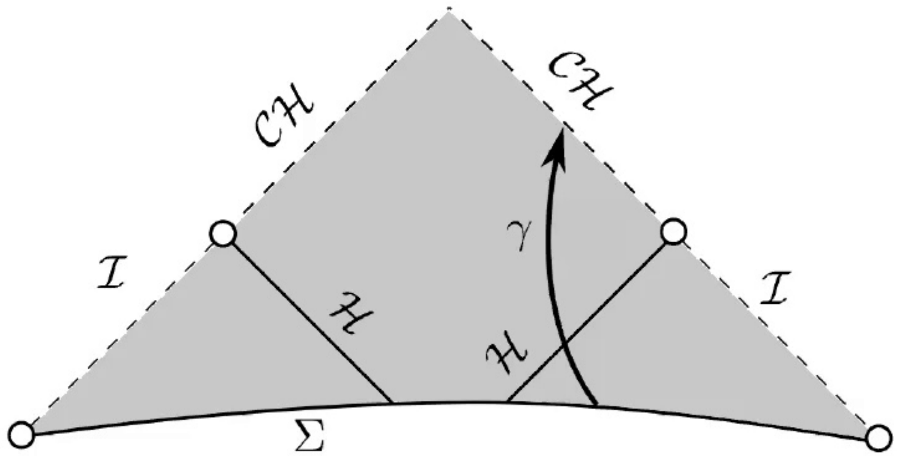
Schwarzschild solution determined from data posed on Σ.

- Geodesically **incomplete**: observer γ reaches singularity $\{r = 0\}$ in finite time.
- **Strong** singularity at $\{r = 0\}$: γ torn apart by **infinite** tidal deformations!
  (Schwarzschild **inextendible** as a **continuous** spacetime)

Determinism holds true: Geodesically incomplete but γ exits the validity of the theory.

III) THE KERR BLACK HOLE

Rotating black hole: Parametrized by mass M and angular momentum $|a| < M$

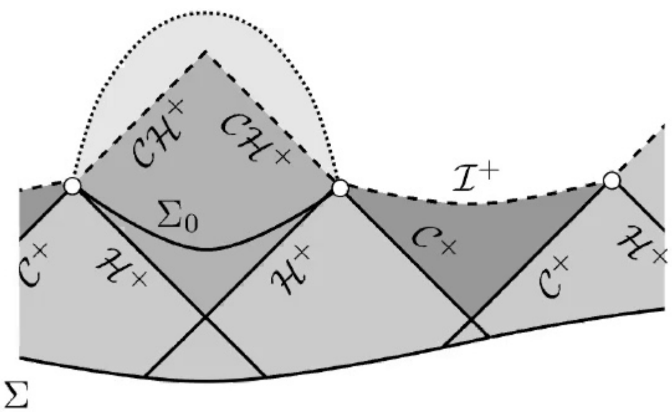


Kerr solution (darker shaded region) determined by initial data posed on Σ.

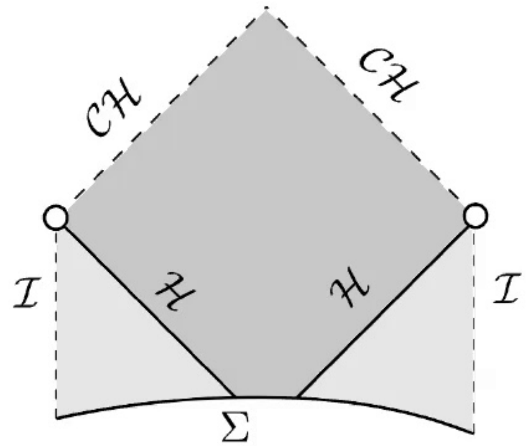
- *Like* Schwarzschild, Kerr is **incomplete**: γ reaches Cauchy horizon $\mathcal{CH}$ in finite time.
- *Unlike* Schwarzschild, tidal deformations for γ remain **finite**.
- Kerr is **smoothly extendible** and general relativity is valid for γ.
- Fate of γ is **undetermined**.

Determinism fails: Geodesically incomplete, yet general relativity is still valid for γ.

KERR–DE SITTER ($\Lambda > 0$) AND KERR–ANTI-DE SITTER ($\Lambda < 0$)



Penrose diagram of Kerr(RN)–de Sitter ($\Lambda > 0$)



Penrose diagram of
Kerr(RN)–Anti-de Sitter ($\Lambda < 0$)

- **Kerr–de Sitter** ($\Lambda > 0$) and **Kerr–Anti-de Sitter** ($\Lambda > 0$) are **smoothly extendible** beyond the Cauchy horizon $\mathcal{CH}$.

STRONG COSMIC CENSORSHIP CONJECTURE

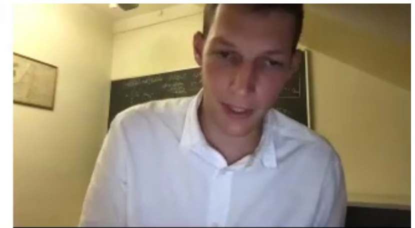
Minkowski, Schwarzschild and Kerr: highly symmetric solutions.

What is the **generic** behavior?

Penrose (1974): "Generically, general relativity is deterministic."

Conjecture 1 (C^0-formulation of Strong Cosmic Censorship).
For ***generic*** *initial data for the Einstein equations, the arising solution is inextendible as a spacetime* ***with continuous metric****.*

Strong Cosmic Censorship: Generically, if the evolution is **incomplete**, then observers are torn apart by **infinite tidal deformations** (as in Schwarzschild).



STRONG COSMIC CENSORSHIP CONJECTURE FOR $\Lambda \geq 0$

Bad news: Conjecture 1 (C^0-formulation of SCC) is false for $\Lambda = 0$ and $\Lambda > 0$!

- All solutions settling down to Kerr are **continuously extendible**. (Dafermos–Luk,'17) Thus, SCC is false!
- Key ingredient: sufficiently fast decay of gravitational waves in the exterior (polynomial $\Lambda = 0$ and exponential $\Lambda > 0$)
- $\Lambda = 0$: Weaker formulation put forward by Christodoulou expected to be true. (Chandrasekhar–Hartle, Poisson–Israel, Ori, . . . , Dafermos–Shlapentokh-Rothman, Luk–Oh, Van de Moortel)
- $\Lambda > 0$: Even Christodoulou formulation may fail, but maybe not—story complicated! (Brady–Chambers, Brady–Moss–Myers, Brady–Poisson . . . , Dafermos, Cardoso–Costa–Destounis–Hintz–Jansen, Dafermos–Shlapentokh-Rothman, Dias–Eperon–Reall–Santos)

STRONG COSMIC CENSORSHIP FOR $\Lambda < 0$

Conjecture 1 (C^0-formulation of SCC) open for $\Lambda < 0$ up until now!

Turns out: Story for $\Lambda < 0$ is **completely different**.

- Methods of proof for $\Lambda \geq 0$ manifestly fail.
- Attractive possibility: Conjecture 1 may be true for $\Lambda < 0$. (Determinism in better shape)
- Connections to small divisors and to Diophantine approximation arise.
- Resolution depends crucially on the notion of genericity.

I will present an unexpected—mixed—resolution
of the linear scalar analogue of Conjecture 1.

CONJECTURE 2: LINEAR ANALOG OF C^0-FORMULATION OF SCC

Reissner–Nordström–AdS:

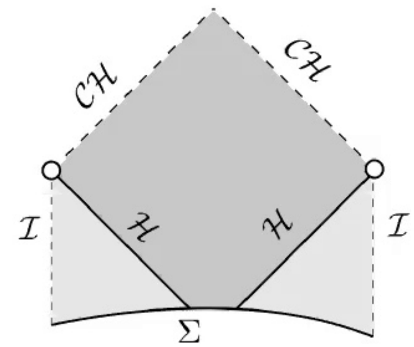
$$g_{\mathrm{RN-AdS}} = -f\mathrm{d}t^2 + f^{-1}\mathrm{d}r^2 + r^2\gamma_{\mathbb{S}^2}$$

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}; \;\; l^2 = -\frac{3}{\Lambda}$$

Kerr–AdS: $g_{\mathrm{Kerr-AdS}}$: more complicated.

Linear scalar perturbations ψ of a rotating black hole:

$$\begin{cases} \Box_g \psi - \frac{2}{3}\Lambda\psi = 0. \\ (\psi, n_\Sigma \psi) := (\psi_0, \psi_1) \text{ smooth data} \\ r\psi = 0 \text{ at } \mathcal{I} \end{cases} \qquad \text{(WE)}$$



RN–AdS and Kerr–AdS black hole

Conjecture 2 (Linear scalar analog of Conjecture 1).

Generic linear scalar perturbations ψ solving (WE) *blow up in amplitude $|\psi| \to +\infty$ at the Cauchy horizon $\mathcal{CH}$ for RN–AdS or Kerr–AdS black holes.*

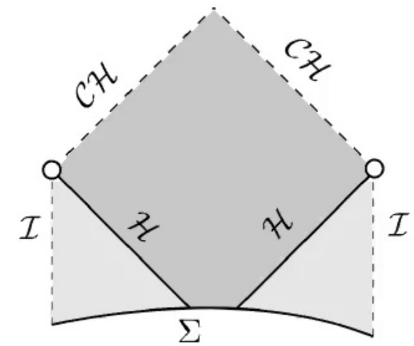
THEOREM: **REISSNER–NORDSTRÖM–ADS**

Theorem (K. '18, *).
Conjecture 2 is false for Reissner–Nordström–AdS.

More precisely, all linear scalar perturbations ψ of Reissner–Nordström–AdS remain uniformly bounded in the black hole interior

$$|\psi| \leq C.$$

Moreover, ψ extends continuously across the Cauchy horizon $\mathcal{CH}$.



* Kehle, C. *Uniform boundedness and continuity at the Cauchy horizon for linear waves on Reissner–Nordström–AdS black holes*, Commun. Math. Phys. (2020); arxiv:1812.06142

Kerr–AdS

Resolution depends on **Diophantine properties** of dimensionless mass and angular momentum $\mathfrak{m} := M\sqrt{-\Lambda}$, $\mathfrak{a} := a\sqrt{-\Lambda}$!

THEOREM: **KERR–ADS**

$\mathcal{P} := \{(\mathfrak{m}, \mathfrak{a}) :$ subextremal Kerr–AdS parameter below Hawking–Reall bound$\}$.

Theorem (K., ’20,*).

Conjecture 2 is <u>true</u> for <u>Baire-generic</u> Kerr–AdS black holes.

More precisely, generic scalar linear perturbations ψ of Kerr–AdS ***blow up***

$$|\psi| \to +\infty$$

at the Cauchy horizon for a set $\mathcal{P}_{\text{Blow-up}} \subset \mathcal{P}$ of dimensionless black hole parameters $(\mathfrak{m}, \mathfrak{a})$ with the following properties:

- $\mathcal{P}_{\text{Blow-up}}$ *is* ***dense****,*
- $\mathcal{P}_{\text{Blow-up}}$ *is* ***Baire-generic****,*
- $\mathcal{P}_{\text{Blow-up}}$ *is* ***Lebesgue-exceptional*** *(Lebesgue null set).*

(Baire-generic: Countable intersection of open and dense sets)

“Linear analog of Strong Cosmic Censorship is respected for Baire-generic black holes”

* Kehle, C. *Diophantine approximation as Cosmic Censor for Kerr–AdS black holes*, arXiv:2007.12614

Conjecture: Kerr–AdS

Conjecture (K., 2020).

Conjecture 2 is false for Lebesgue-generic Kerr–AdS black holes.

More precisely, all linear scalar perturbations ψ remain ***bounded***

$$|\psi| \leq C$$

at the Cauchy horizon for a set $\mathcal{P}_{\text{Bounded}} \subset \mathcal{P}$ of dimensionless black hole parameters $(\mathfrak{m}, \mathfrak{a})$ with the following properties:

- $\mathcal{P}_{\text{Bounded}}$ *is* ***dense****,*
- $\mathcal{P}_{\text{Bounded}}$ *is* ***Lebesgue-generic*** *(full Lebesgue measure),*
- $\mathcal{P}_{\text{Bounded}}$ *is* ***Baire-exceptional****.*

"Linear analog of Strong Cosmic Censorship is false for Lebesgue-generic black holes"

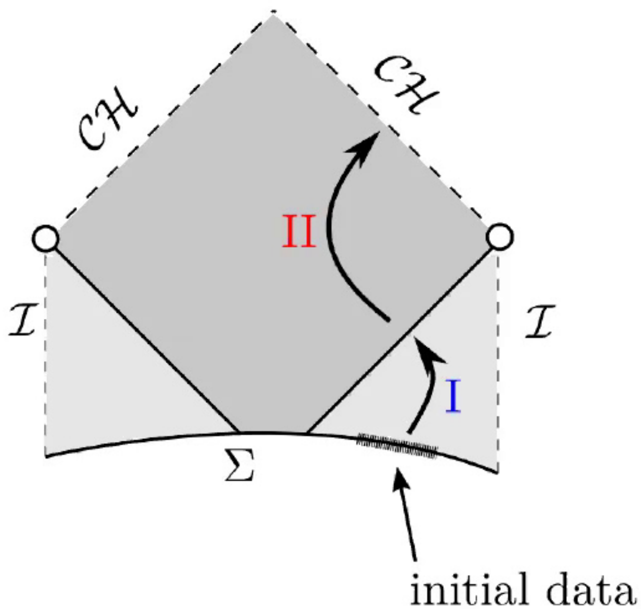
Recall: **Conjecture 2 is holds true for Baire-generic Kerr–AdS black holes and false for Reissner–Nordström–AdS black holes!**

HEURISTICS

Actual proof is intricate and technical. I will outline heuristics arguments which led me to discover the connection to Diophantine approximation.

I. Exterior Propagation; $\Sigma \rightarrow$ event horizon: Take a sum of quasinormal modes

II Interior Propagation; Event horizon $\rightarrow$ Cauchy horizon: Based on the interior scattering results (K.–Shlapentokh-Rothman, '18)



Exterior Propagation

Theorem (Holzegel–Smulevici, '13,'14).
We have

$$|\psi| \leq \frac{C}{\log(t^*)}$$

*on the exterior and the decay rate is **sharp**.*

- Compare: polynomial ($\Lambda = 0$) and exponential ($\Lambda > 0$) decay
- Slow decay is tied to stable trapping
- Quasinormal modes (QNMs) converging to the axis exponentially fast
- Related: possible instability of AdS (black holes) (Dafermos–Holzegel, Bizon–Rostworoski, Green–Maillard–Lehner–Liebling, Moschidis)

Exterior Propagation

QNMs: Finite energy solutions of the form

$$\psi(r, t^*, \theta, \phi) = R(r) Y_{m\ell}(\theta, \phi) e^{-i\omega t^*}.$$

QNMs on Reissner–Nordström–AdS (similar Kerr–AdS): $(\omega_{n,\ell})_{\ell \in \mathbb{N}, n=1,\dots,C(\ell)}$

$$\operatorname{Re} \omega_{n,\ell} \sim c\ell$$
$$\operatorname{Im} \omega_{n,\ell} \sim -e^{-\tilde{c}\ell}$$

Slowly decaying QNM $\leftrightarrow$ high frequency: ℓ and $|\omega|$ large.

Consider sum of QNMs along event horizon:

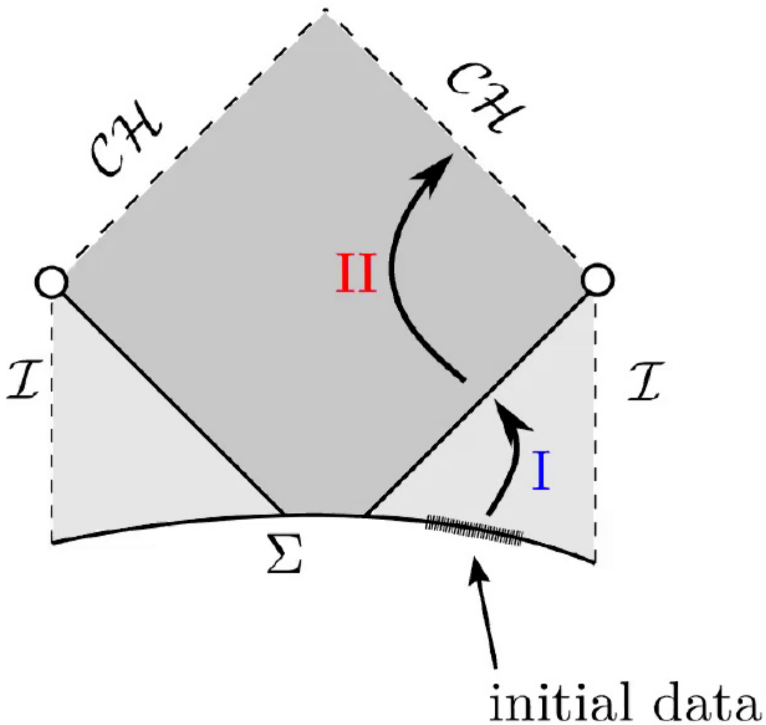
$$\psi_{\mathcal{H}}(v, \theta, \phi) = \sum_{m\ell} a_{m\ell} \, R(r_+) Y_{m\ell}(\theta, \phi) \exp(ic\ell v) \exp(-e^{-\tilde{c}\ell} v)$$

Proposition: $a_{m\ell} \to 0$ decay exponentially

From Exterior to Interior Propagation

I. Exterior Propagation; $\Sigma \to$ event horizon: Sum of quasinormal modes

II. Interior Propagation; Event horizon $\to$ Cauchy horizon: Based on the interior scattering results (K.–Shlapentokh-Rothman, '18)



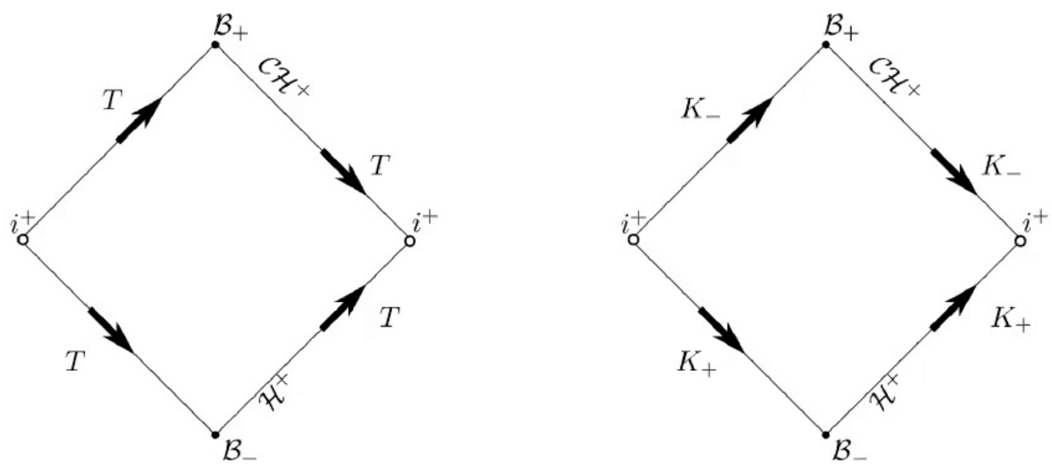
INTERIOR PROPAGATION

Scattering operator from event horizon to Cauchy horizon

$\xrightarrow{\text{in Fourier space}}$

Multiplication by reflection coefficient $\mathfrak{R}(\omega, m, \ell)$

$\mathfrak{R}(\omega, m, \ell)$ has pole at zero frequency w.r.t. generator of Cauchy horizon
(K.-Shlapentokh-Rothman, '18 [1])



Generators of the Cauchy horizon:

Reissner–Nordström–AdS: $T = \partial_t$; Kerr–AdS: $K_- = T + \omega_- \Phi$

[1] Kehle, C. and Shlapentokh-Rothman, Y. *A scattering theory for linear waves on the interior of Reissner–Nordström black holes*, Ann. Henri Poincaré 20 (2019), no. 5, 1583–1650, arxiv:1804.05438

INTERIOR PROPAGATION: **REISSNER–NORDSTRÖM–ADS**

$$\mathfrak{R}(\omega, m, \ell) \sim \frac{1}{\omega} + \text{analytic \& bounded}$$

Recall: $\psi_{\mathcal{H}} = \text{sum of QNMs}$

$$\mathcal{F}(\psi_{\mathcal{H}}) \sim \sum_{m\ell} a_{m\ell}\,\delta(\omega - c\ell - ie^{-\tilde{c}\ell})Y_{m\ell}(\theta,\phi), \qquad a_{m\ell} \to 0 \text{ exponentially}$$

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Hence,

$$\psi_{\mathcal{CH}}(u = 0, \theta, \phi) \sim \mathcal{F}^{-1} \circ \mathfrak{R} \circ \mathcal{F}(\psi_{\mathcal{H}}) \sim \sum_{m\ell} a_{m\ell} \, \text{pv} \int_{\mathbb{R}} \frac{1}{\omega} \delta(\omega - c\ell - ie^{-\tilde{\ell}}) \mathrm{d}\omega Y_{m\ell}(\theta, \phi)$$

$$\sim \sum_{m\ell} a_{m\ell} \frac{1}{c\ell + ie^{-\tilde{c}\ell}} Y_{m\ell}(\theta, \phi) < C$$

Quasinormal modes in exterior decouple from scattering poles in interior!

Boundedness $|\psi| \leq C$ at Cauchy horizon on Reissner–Nordström–AdS

INTERIOR PROPAGATION: **KERR–ADS**

$$\mathfrak{R}(\omega, m, \ell) \sim \frac{1}{\omega - \omega_- m} + \text{analytic \& bounded}$$

$$\psi_{\mathcal{CH}}(u=0,\theta,\phi) \sim \sum_{m\ell} a_{m\ell} \mathcal{F}^{-1} \circ \mathfrak{R} \circ \mathcal{F}(\psi_{\mathcal{H}})$$

$$\sim \sum_{m\ell} a_{m\ell} \, \text{pv} \int_{\mathbb{R}} \frac{1}{\omega - \omega_- m} \delta(\omega - c\ell - ie^{-\tilde{c}\ell}) \mathrm{d}\omega S_{m\ell}(\theta,\phi)$$

$$\sim \sum_{m\ell} a_{m\ell} \frac{1}{\omega_- m - c\ell - ie^{-\tilde{c}\ell}} S_{m\ell}(\theta,\phi)$$

Quasinormal modes in exterior couple to zero-frequency poles in interior!

DIOPHANTINE APPROXIMATION

Recall: $a_{m\ell} \to 0$ exponentially.

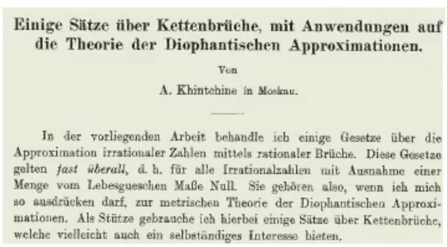
Blow-up: $|\omega_- m - c\ell| < \frac{1}{e^{\ell} e^{|m|}}$ for ∞-many (m, ℓ).

$$\mathcal{R} := \left\{ \frac{\omega_-}{c} \in \mathbb{R} : \left| \frac{\omega_-}{c} - \frac{\ell}{m} \right| < \frac{1}{e^{\ell} e^{|m|}} \text{ for infinitely many } m, \ell \right\}. \tag{1}$$

The set $\mathcal{R}$ are **exponential Liouville numbers**: **Baire-generic, Lebesgue exceptional**

The $\mathcal{P}_{\text{Blow-up}}$ in Theorem 2 has a similar structure but is not exactly $\mathcal{R}$.

$\Rightarrow$ **Blow-up or boundedness for ψ on Kerr–AdS depends on Diophantine properties of $\mathfrak{m}$, $\mathfrak{a}$.**



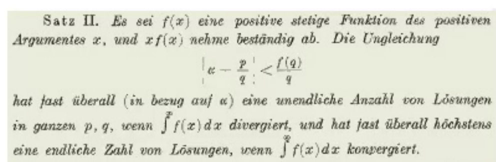
Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen.

Von

A. Khintchine in Moskau.

In der vorliegenden Arbeit behandle ich einige Gesetze über die Approximation irrationaler Zahlen mittels rationaler Brüche. Diese Gesetze gelten *fast überall*, d. h. für alle Irrationalzahlen mit Ausnahme einer Menge vom Lebesgueschen Maße Null. Sie gehören also, wenn ich mich so ausdrücken darf, zur metrischen Theorie der Diophantischen Approximationen. Als Stütze gebrauche ich hierbei einige Sätze über Kettenbrüche, welche vielleicht auch ein selbständiges Interesse bieten.

Mathematische Annalen (1924)



Satz II. *Es sei $f(x)$ eine positive stetige Funktion des positiven Argumentes x, und $x f(x)$ nehme beständig ab. Die Ungleichung*

$$\left| \alpha - \frac{p}{q} \right| < \frac{f(q)}{q}$$

hat fast überall (in bezug auf α) eine unendliche Anzahl von Lösungen in ganzen p, q, wenn $\int^{\infty} f(x)\,dx$ divergiert, und hat fast überall höchstens eine endliche Zahl von Lösungen, wenn $\int^{\infty} f(x)\,dx$ konvergiert.

Khintchine's Theorem

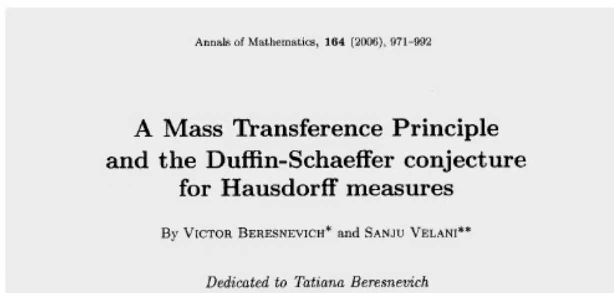
FRACTAL DIMENSIONS OF $\mathcal{P}_{\text{Blow-up}}$

The set $\mathcal{R}$ from heuristic discussion satisfies:

- $\mathcal{R}$ has full packing dimension: $\dim_P(\mathcal{R}) = 1$
- $\mathcal{R}$ has logarithmic Hausdorff dimension: $\dim_H(\mathcal{R}) = \log$

Recall: $\mathcal{P}_{\text{Blow-up}}$ subset of dimensionless parameters $\mathfrak{m}, \mathfrak{a}$.

- We prove: $\mathcal{P}_{\text{Blow-up}}$ has full packing dimension $\dim_P(\mathcal{P}_{\text{Blow-up}}) = 2$
- Open problem: $\mathcal{P}_{\text{Blow-up}}$ has Hausdorff dimension $\dim_H(\mathcal{P}_{\text{Blow-up}}) = 1 + \log$.



Annals of Mathematics, **164** (2006), 971–992

A Mass Transference Principle and the Duffin-Schaeffer conjecture for Hausdorff measures

By VICTOR BERESNEVICH* and SANJU VELANI**

Dedicated to Tatiana Beresnevich

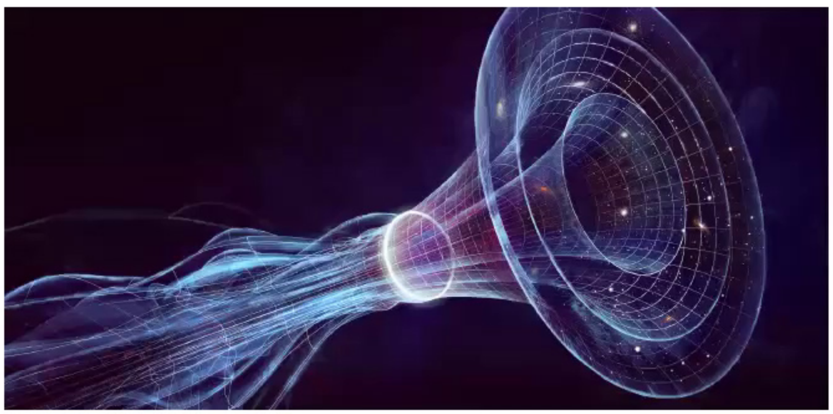
Annals of Mathematics (2006)

SUMMARY

Conjecture 2: Linear scalar analog of C^0-formulation of Strong Cosmic Censorship

Reissner–Nordström–AdS: Conjecture 2 is **false**.

Kerr–AdS: Conjecture 2 is **true** or **false** depending on **Diophantine properties** and the notion of **genericity** imposed.
(cf. KAM theory)



Quanta magazine

Thank you!