Title: Quantum Raychaudhuri Equation: Implications for spacetime singularities and the quantum origin of Lambda

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Series: Quantum Gravity

Date: September 24, 2020 - 2:30 PM

URL: http://pirsa.org/20090002

Abstract: The Raychaudhuri equation predicts the convergence of geodesics and gives rise to the singularity theorems. The quantum Raychaudhuri equation (QRE), on the other hand, shows that quantal trajectories, the quantum equivalent of the geodesics, do not converge and are not associated with any singularity theorems. Furthermore, the QRE gives rise to the quantum corrected Friedmann equation. The quantum correction is dependent on the wavefunction of the perfect fluid whose pressure and density enter the Friedmann equation. We show that for a suitable choice of the wavefunction this term can be interpreted as a small positive cosmological constant.

DM_and_DE_from_BEC_Talk__2020_ (11).pdf - Adobe Acrobat Reader DC

Quantum Raychaudhuri Equation: implications for spacetime singularities and the quantum origin of Λ

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24 September 2020

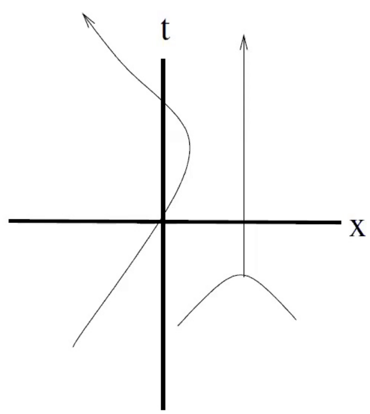
Overview

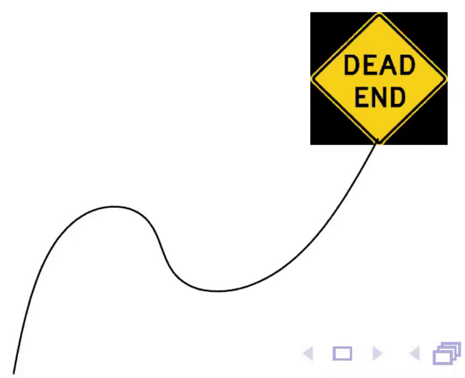
1. Raychaudhuri Equation and Singularity Theorems
2. Quantum Raychaudhuri Equation and no Singularity Theorems
3. Quantum Friedmann Equation
4. Dark Matter and Dark Energy from a Bose-Einstein Condensate
5. Summary and Conclusions

Raychaudhuri Equation and Singularity Theorems-1

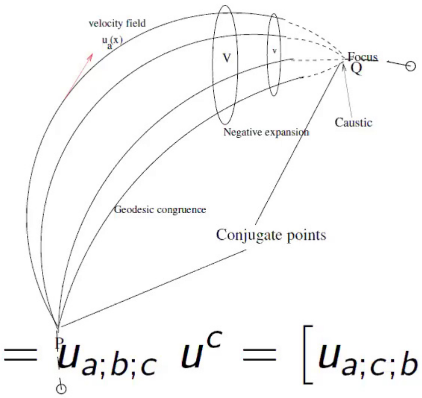
Singularities in Spacetime

- Spacetime pathologies signal breakdown of all physics, start/end of time (big-bang/big-crunch), black holes, ...
- Curvature $\rightarrow \infty$ is indicative, but neither a necessary, nor a sufficient condition for singular spacetimes
- E.g., infinite curvature takes infinite τ to reach (*not singular*), or hole cut off from finite curvature spacetime (*singular*)
- But, **Geodesic incompleteness** $=$ necessary and sufficient condition for singularities





Raychaudhuri Equation and Singularity Theorems-2



Velocity field $u_a = u_a(x) \Rightarrow \frac{du_{a;b}}{d\tau} = u_{a;b;c}\, u^c = \left[u_{a;c;b} + R_{cba}{}^d u_a\right] u^c$

$$= \Big(\underbrace{u_{a;c}u^c}_{=0\ (\text{geod.eqn.})}\Big)_{;b} - u^c{}_{;b}u_{a;c} + R_{cba}{}^d u^c u_d = -u^c{}_{;b}u_{a;c} + R_{cbad}\; u^c u^d$$

Symmetric part: $\sigma_{ab} = u_{(a;b)} - \frac{1}{3}h_{ab}\theta$

Anti-symmetric part and trace: $\omega_{ab} = u_{[a;b]};\ \ \theta = h^{ab}u_{a;b};\ ,\ h_{ab} = g_{ab} - u_a\, u_b$

Decomposition: $u_{a;b} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab}$

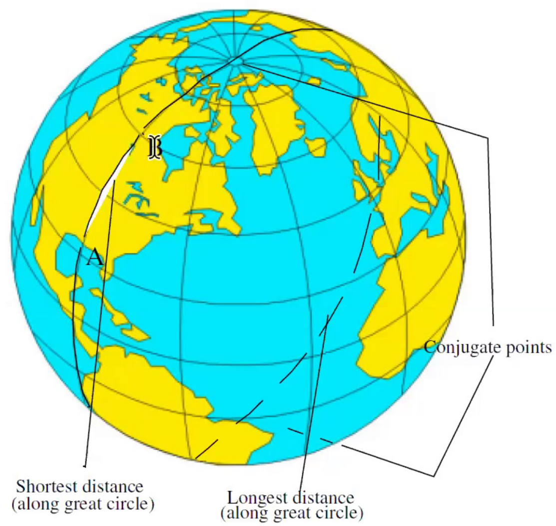
$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd}u^c u^d \ < 0$$ *[Raychaudhuri Equation]*

If $\theta_0 = \theta(0) < 0$ (initially converging)

Focus/caustic for $\tau \leq \frac{3}{|\theta_0|}$ *(geodesics end in finite time!)*

Raychaudhuri Equation and Singularity Theorems-3

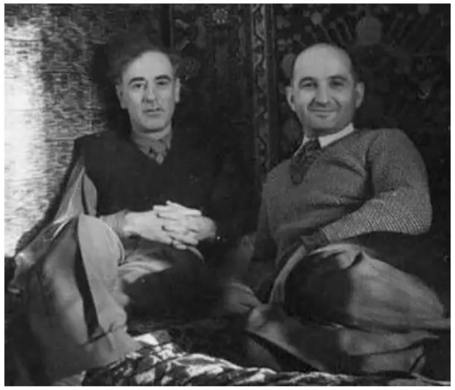
- Conjugate points due to Raychaudhuri equation.
- Geodesics are no longer maximal length curves.



- Maximal geodesics predicted by global arguments, on the other hand.
- Sufficiently long geodesics cannot exist. Geodesics are incomplete.
- Singularities! (Singularity theorems) - for most spacetimes
- A. K. Raychaudhuri (1955), L. D. Landau, E. M. Lifshitz (c.1959), R. Penrose (1965), S. W. Hawking and R. Penrose (1970)

Raychaudhuri Equation and Singularity Theorems-4

A. K. Raychaudhuri (1924–2005)
Kolkata, India



L. D. Landau (1908–1968) E. M. Lifshitz (1915–1985)
Russia

Raychaudhuri Equation and Singularity Theorems-5

- *Generality:* All reasonable spacetimes are singular. Physics breaks down! (gravity universal & attractive)
- Fluid picture: velocity field $\vec{u}(x)$
- But, classical $\rightarrow$ quantum? (Expectation values, Ehrenfest type theorem?)
- First, find a 'quantum velocity field'

Quantum Raychaudhuri Equation - 1

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd}u^c u^d + \underbrace{Tr[(u_{a;c}\,u^c)_{;b}]}_{=0}$$

First, for simplicity:

'Non-relativistic' Raychaudhuri Equation $(R_{cd}u^c u^d \to \nabla^2 V_{class})$:

$$\frac{d\theta}{dt} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - \nabla^2 V + \underbrace{\left(\frac{d\vec{v}}{dt} + \vec{\nabla}V_{class}\right)}_{=0}$$

How does this change on quantization?

Quantum Raychaudhuri Equation - 2

Classical $\rightarrow$ Quantum:

$$m\frac{d\vec{v}}{dt} = -m\vec{\nabla}V_{class} \;\rightarrow\; -\frac{\hbar^2}{2m}\nabla^2\psi + V_{class}\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\psi(\vec{x},t) = \mathcal{R}e^{iS}$$

(Normalizable, single-valued, $\mathcal{R}$, S= Real functions)

$$\vec{v}\,(\vec{x},t) = \frac{d\vec{x}}{dt} \equiv \frac{\hbar}{m}\,\mathcal{I}m\left(\frac{\vec{\nabla}\psi}{\psi}\right) = \frac{\hbar}{m}\,\vec{\nabla}S(\vec{x},t) \leftarrow \text{quantum velocity field!}$$

Real and imaginary parts of the Schrödinger equation $\rightarrow$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \quad \text{(Probability conservation)}$$

$$m\frac{d\vec{v}}{dt} = -m\vec{\nabla}V_{class} + \underbrace{\frac{\hbar^2}{2m}\vec{\nabla}\left(\frac{1}{\mathcal{R}}\nabla^2\mathcal{R}\right)}_{-m\vec{\nabla}V_Q} \quad \text{(Newton's law + quantum potential } V_Q\text{!)}$$

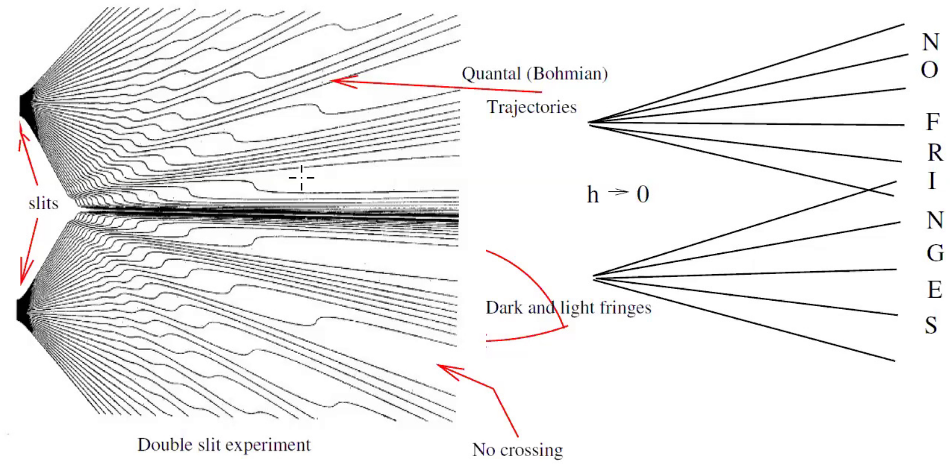
$$V_Q = -\frac{\hbar^2}{2m}\left(\frac{1}{\mathcal{R}}\nabla^2\mathcal{R}\right)$$

Quantum Raychaudhuri Equation-3

- Initially, particles distributed as $\rho(0) = |\psi(0)|^2$ ('quantum equilibrium')
- Prob. conservation $\Rightarrow$ they remain distributed as $\rho(t) = |\psi(t)|^2$
- Each particle follows individual trajectories, subjected to $V + V_Q$

  (quantal/Bohmian trajectories)
- Make measurement in the end

Quantum Raychaudhuri Equation - 4



(C. Phillipides, C. Dewdney, B. J. Hiley, *Quantum Interference and the Quantum Potential*, il nuovo cim. **B52**, no.1 (1979) 15-28)

- All correct quantum predictions
- Only dynamical input → Schrödinger equation
- Classical limit: trajectories → with $\hbar \to 0$ (interference disappears)
- L. de Broglie (1927), D. Bohm (1952), superconductivity, superfluidity, Bose-Einstein condensates (Gross-Pitaevskii equation)
- Note $V_Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\mathcal{R}}{\mathcal{R}} = -V_{class} + constant$ if $\psi =$ stationary state.

Valid picture!

Quantum Raychaudhuri Equation - 5

L. de Broglie (1892–1987)

D. Bohm (1917–1992)

Quantum Raychaudhuri Equation - 6

$$\frac{d\theta}{dt} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - \nabla^2 V + \underbrace{\left(\frac{d\vec{v}}{dt} + \vec{\nabla} V_{class}\right)}_{\frac{\hbar^2}{2m}\vec{\nabla}\left(\frac{1}{\mathcal{R}}\nabla^2\mathcal{R}\right)}$$

$V \to V_{class} + V_Q/m$ in Raychaudhuri equation

$$\frac{d\theta}{dt} = -\frac{1}{3}\,\theta^2 - \sigma_{ab}\sigma^{ab} - \nabla^2 V + \frac{\hbar^2}{2m^2}\nabla^2\left(\frac{1}{\mathcal{R}}\nabla^2\mathcal{R}\right)$$

Quantum correction, repulsive @ short-distances

Quantum Raychaudhuri Equation - 7

Now: Relativistic: Classical fluid (geodesics) $\to$ Quantum fluid (Ψ)

Klein-Gordon equation: $\left[\Box + \frac{m^2c^2}{\hbar^2}\right]\Psi = 0$

$$\Psi(x) = \mathcal{R}(x)\, e^{iS(x)} \ , \ \mathcal{R}, S \in \mathbb{R} \ ,$$

$$k_a = \partial_a S \ , \quad u_a = c\frac{dx_a}{d\tau} = \frac{\hbar k_a}{m} \ , \quad \vec{v} = \frac{d\vec{x}}{dt} = -c^2\frac{\vec{\nabla}S}{\partial^0 S}$$

Imaginary part of the KG equation: $\partial^a\left(\mathcal{R}^2\partial_a S\right) = 0$

Real part of the KG equation: $k^2 = \frac{(mc)^2}{\hbar^2} + \frac{\Box\mathcal{R}}{\mathcal{R}}$

$$u^b_{;a}u^a = \frac{\hbar^2}{m^2}\left(\frac{\Box\mathcal{R}}{\mathcal{R}}\right)^{;b} \neq 0 \quad \left(\text{i.e. geodesic equation } + V_Q = \frac{\hbar^2}{m^2}\frac{\Box\mathcal{R}}{\mathcal{R}}\right)$$

Quantum Raychaudhuri equation

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\,\theta^2 - \sigma_{ab}\sigma^{ab} - R_{cd}u^c u^d + \frac{\hbar^2}{m^2}h^{ab}\left(\frac{\Box\mathcal{R}}{\mathcal{R}}\right)_{;a;b} \leftarrow \textit{Quantum Correction}$$

S. Das, Phys. Rev. **D89** (2014) 084068 [arXiv/1311.6539]

Quantum Raychaudhuri Equation - 8

Quantum Raychaudhuri equation

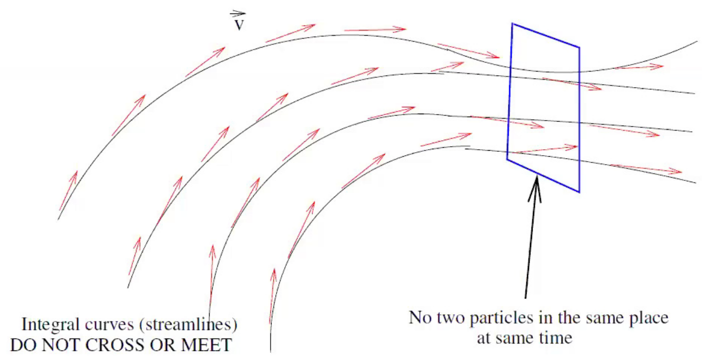
$$\frac{d\theta}{d\tau} = -\frac{1}{3}\,\theta^2 - \sigma_{ab}\sigma^{ab} - R_{cd}u^c u^d + \frac{\hbar^2}{m^2} h^{ab}\left(\frac{\Box\mathcal{R}}{\mathcal{R}}\right)_{;a;b} \leftarrow \text{Quantum}$$

- $+$ term
- Wavefunction dependent
- The quantum term prevents focusing and formation of conjugate points

Quantum Raychaudhuri Equation and no Singularity Theorems

No-crossing of quantal trajectories

$$\vec{v} = \frac{d\vec{x}}{dt} = -c^2 \frac{\vec{\nabla} S}{\partial^0 S}$$



- No focusing, no conjugate points, geodesics go on forever
- No singularities! (all because of $\hbar$)

Application to Cosmology: Raychaudhuri $\rightarrow$ Friedmann equation

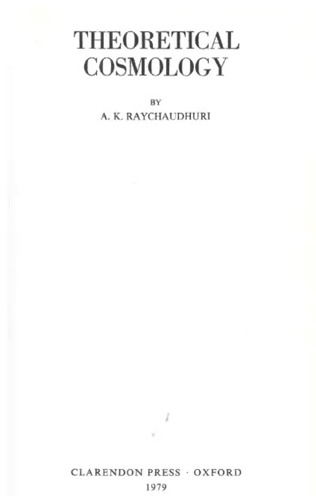
FLRW Universe

$$ds^2 = -dt^2 + a(t)^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad [a = scale\ factor = 1\ Now]$$

Raychaudhuri $\rightarrow$ Friedmann equation

$$\theta = 3\,\frac{\dot{a}}{a}\ , \quad R_{cd}u^c u^d \rightarrow \frac{4\pi G}{3}(\rho + 3p)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,(\rho + 3p)$$



THEORETICAL
COSMOLOGY

BY
A. K. RAYCHAUDHURI

CLARENDON PRESS · OXFORD
1979

Quantum Raychaudhuri → Friedmann Equation

FLRW Universe: $ds^2 = -dt^2 + a(t)^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$ [a = *scale factor*]

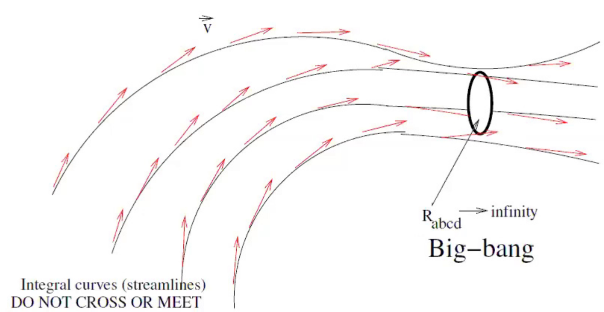
$$\theta = 3\,\frac{\dot a}{a}\ , \quad R_{cd}u^c u^d \rightarrow \frac{4\pi G}{3}(\rho + 3p)$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \underbrace{\frac{\hbar^2}{3m^2} h^{ab}\left(\frac{\Box\mathcal{R}}{\mathcal{R}}\right)_{;a;b}}_{\frac{\Lambda_Q}{3}}$$

$$\Lambda_Q = \frac{\hbar^2}{m^2} h^{ab}\left(\frac{\Box\mathcal{R}}{\mathcal{R}}\right)_{;a;b} = \text{Wavefunction-dependent}$$

Consequences

- No crossing (e.g. at the Big bang)

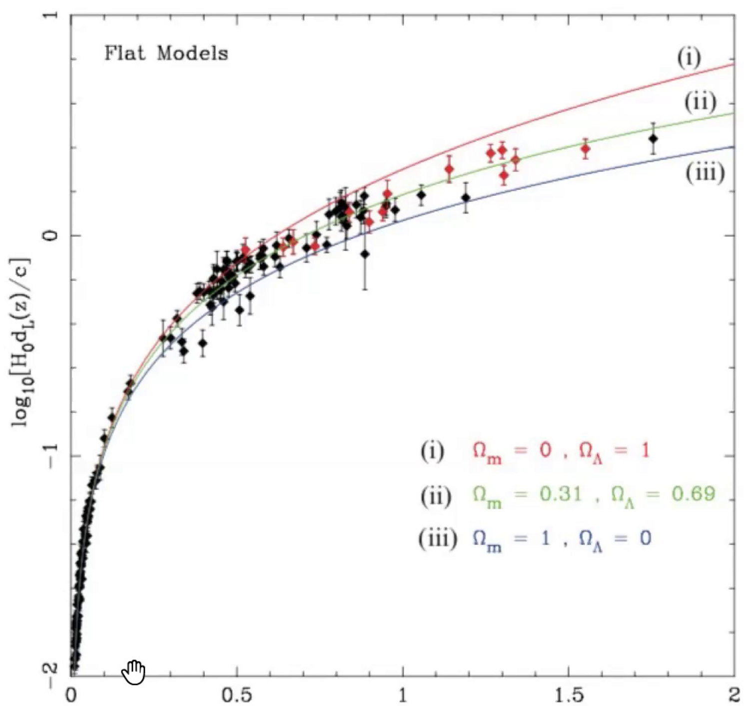


S. Das, IJMPD **23**, No. 12, 1442017 (2014) [arXiv/1405.4011]

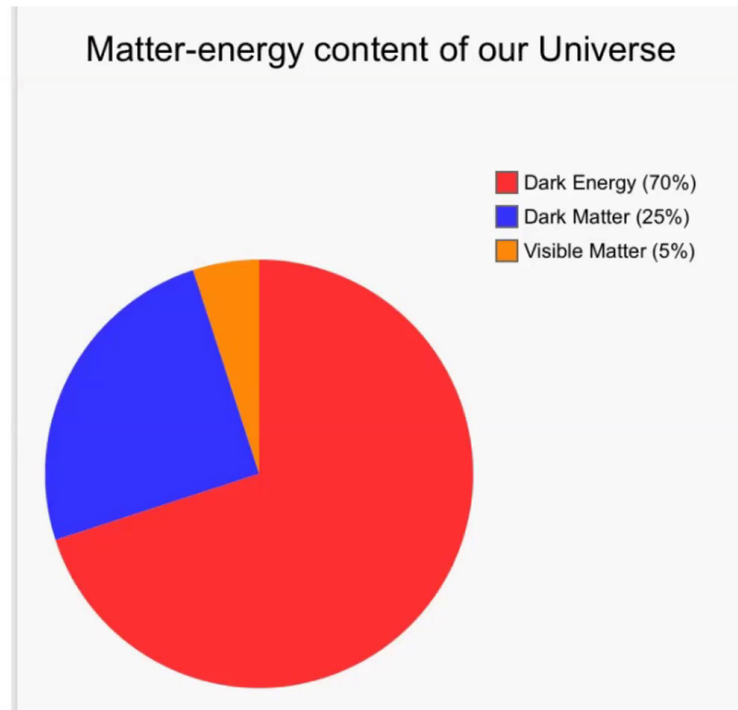
Dark Matter, Dark Energy

Luminosity distance $d_L(\Omega_\Lambda, \Omega_M, z) = \frac{(1+z)}{H_0} \int_0^z \frac{dz}{\sqrt{\underbrace{\Omega_\Lambda}_{0.7} + \underbrace{\Omega_M}_{0.3}(1+z)^3}}$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{crit}} \;,\quad \Omega_M = \frac{\rho_M}{\rho_{crit}}$$



(Pic: Copeland et al, hep-th/0603057)



Questions

- What constitutes Dark Matter?
- What constitutes Dark Energy/Λ?
- Why is Λ positive?
- Why is Λ tiny, about $10^{-123}\ell_{Pl}^{-2}$ where ℓ_{Pl} is the Planck length?

  $(\rho = \int_0^{k_{max}} dk\, k^2 \sqrt{k^2 + m^2} \approx k_{max}^4 > 10^{50}\, \rho_\Lambda)$
- Currently $\rho_{DM} \approx \underbrace{\rho_\Lambda}_{\frac{\Lambda c^2}{4\pi G}} \approx \underbrace{\rho_{crit}}_{\frac{3H_0^2}{8\pi G}} \approx 10^{-26}\, kg\ m^{-3}$

  Why? The *'coincidence problem'*

Bose-Einstein Condensate (BEC) as Dark Matter - 1

- Cold
- Dark
- Light bosons as DM $\Rightarrow$ no small scale structure
- Macroscopic *Quantum state*
- BEC $\Rightarrow$ DE ($\approx$ DM/Λ) via its (repulsive) Quantum Potential
- Few assumptions and free parameters

Bose-Einstein Condensate (BEC) as Dark Matter - 2

Is the critical temperature (below which a BEC forms) high enough?

Critical temperature $= T_c$
Universe temperature $= T(a)$
Boson mass $= m\ eV/c^2$
$\rho_{DM} = 0.25\,\rho_{crit}/a^3$
No. density $= \frac{N}{V} = 0.25\,\frac{\rho_{crit}}{m\,a^3}$

$$T_c(a) = \frac{\hbar c}{k_B}\left(\frac{(N/V)\,\pi^2}{\eta\zeta(3)}\right)^{1/3} = \frac{\hbar c}{k_B}\left(\frac{(0.25\,\rho_{crit}/ma^3)\,\pi^2}{\eta\zeta(3)}\right)^{1/3} = \frac{4.9}{m^{1/3}\ a}\ K$$

$$T(a) = \frac{3.7}{a}\ K\ ,\quad a = \text{scale factor}$$

$$T(a) < T_c(a) \ \rightarrow\quad m < 6\ eV/c^2 \ \Rightarrow \text{BEC forms in the early universe}$$

.

BEC density = DM density

Quantum Potential of BEC

BEC $\Rightarrow \Psi \Rightarrow$ *Quantum potential!*

Quantize

Macroscopic BEC wavefunction $\quad \Psi = \frac{R_0}{a^{3/2}}\, e^{-r^2/\sigma^2} = \mathcal{R}(x)$

$$\rho_{DM} = |\Psi|^2 \propto \frac{1}{a^3}\ , \quad \int dV |\Psi|^2 = N$$

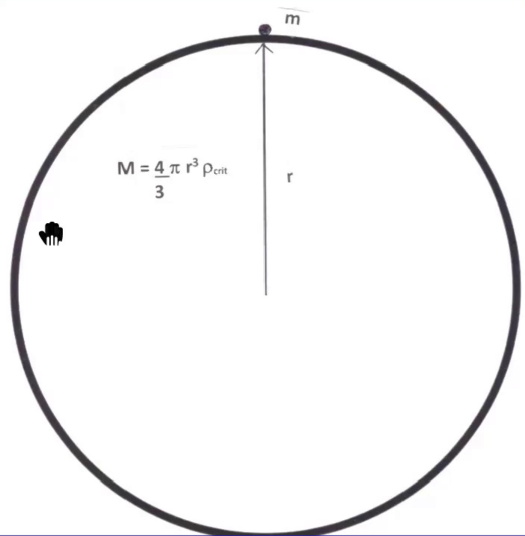
$$\frac{\ddot{a}}{a} = -\frac{4\pi G \rho_{crit}}{3} + \frac{\Lambda_Q}{3}$$

$$\Lambda_Q = \frac{\hbar^2}{m^2c^2}\, h^{ab} \left(\frac{\Box \mathcal{R}}{\mathcal{R}}\right)_{;a;b} = 24 \left(\frac{\hbar}{mc}\right)^2 \frac{1}{\sigma^4} = \text{constant!}$$

S. Das, R. K. Bhaduri, Class. Quant. Grav. **32** 105003 (2015) [arXiv:1411.0753]

S. Das, R. K. Bhaduri, Phys. News (special S. N. Bose anniversary issue) arXiv:1808.10505

BEC Wavefunction Ψ - 1



Newtonian limit, $R_{space} = 0$, $R_{spacetime} = 10^{-123}\,\ell_{Pl}^{-2}$

$$m\ddot{r} = -\frac{GMm}{r^2} = -\frac{G(\frac{4}{3}\pi r^3\,\epsilon\,\rho_{crit})m}{r^2}\ , \quad r = r_0\,a(t)\ ,\ \epsilon \approx 0.25$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,\epsilon\,\rho_{crit} = -\omega^2$$ Raychaudhuri Equation

BEC in a harmonic trap for $t \ll H_0^{-1}$ (14 Gyr)

$$\text{Quantize} \rightarrow \ \Psi = R(a)\,e^{-\frac{m\omega r^2}{2\hbar}} = R(a)\,e^{-\frac{m(4\pi G\,\epsilon\,\rho_{crit}/3)^{1/2}r^2}{2\hbar}} = \frac{R_0}{a^{3/2}}\,e^{-\frac{r^2}{\sigma^2}}$$

$$\sigma^2 = \frac{2\hbar}{m(4\pi G\epsilon\rho_{crit}/3)^{1/2}}$$

BEC Wavefunction Ψ - 2

$$a(t) = a_0 + a_1(t) = \text{constant} + \text{slowly varying}$$

$$\Psi = \Psi_0 + \Psi_1 = \text{time-indep.} + \text{slowly varying}$$

$$\mathcal{R} = \frac{R_0}{a^{3/2}}\, e^{-(r^2/\sigma)^2} = \frac{R_0}{a_0^{3/2}} - \left(\frac{3R_0}{2\, a_0^{5/2}}\right) a_1\, e^{-(r/\sigma)^2}$$
$$= \text{time-indep.} + \text{slowly varying}$$

$$\Lambda_Q = \Lambda_Q^{(0)} + \Lambda_Q^{(1)} = \text{constant} + \text{slowly varying}$$

How slow is slow? $(\frac{a_1}{a_0}|_{t_1} \ll 1)$

$a(t) \propto (t - t_0)^{\frac{2}{3(1+w)}}$ *(Matter/radiation. $p = w\rho,\ w = 0, \frac{1}{3}$)*

$a(t) = a_0 e^{H_0\, t}$ *(de Sitter. $p = -\rho,\ w = -1$)*

$\Delta t \equiv t - t_1 \ll t_1 - t_0$ *(Matter/radiation)*

current time - ref.time ≪ ref.time - BB

$\Delta t \equiv t - t_1 \ll H_0^{-1} \simeq 16\ Gyr$ *(de Sitter)*

Don't go too far in the past!

Λ_Q from quantum potential

$$\Psi = R(a)\, e^{-\frac{m(4\pi G\, \epsilon\, \rho_{crit}/3)^{1/2} r^2}{2\hbar}} = \frac{R_0}{a^{3/2}}\, e^{-\frac{r^2}{\sigma^2}} = \mathcal{R}$$

$$\Lambda_Q = \frac{\hbar^2}{m^2} h^{ab} \left(\frac{\Box \mathcal{R}}{\mathcal{R}}\right)_{;a;b} = 8\,\pi\, G \epsilon\, \rho_{crit}$$ *(independent of m!)*

$$\rho_\Lambda = \frac{\Lambda}{4\pi G} = 2\,\epsilon\,\rho_{crit}$$

$$\rho_{DM} = \epsilon \rho_{crit}$$

$$\frac{\rho_\Lambda}{\rho_{DM}} = 2$$

$$\rho_\Lambda \approx \rho_{DM}$$

Summary

- What constitutes DM? *BEC*
- What constitutes DE? *Quantum potential of the BEC*
- Why is Λ positive?
  Because negative gravitational potential $\Rightarrow$ positive Quantum Potential
- Why is $\rho_{DM} \approx \rho_\Lambda \approx \rho_{crit}$?
  Because $|$Quantum potential$|$ = $|$classical potential$|$ for stationary states

Remarks

- We get $\rho_\Lambda =$ ~~3~~ ρ_{DM}, because (i) $\rho_{DM} \propto 1/a^3, \rho_\Lambda \propto$ constant
  (ii) all bosons not in the ground state
- Prediction: ultralight bosons of $m < 6eV/c^2$
- What are these bosons? gravitons, axions, ...
- Interacting DM and DE model and data fitting *(M. Sharma, S. Sur, SD [work in progress])*
- Prediction: Λ has changed in the far past and will change in the future