Title: Entanglement in hybrid quantum circuits

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Collection: Online School on Ultra Quantum Matter

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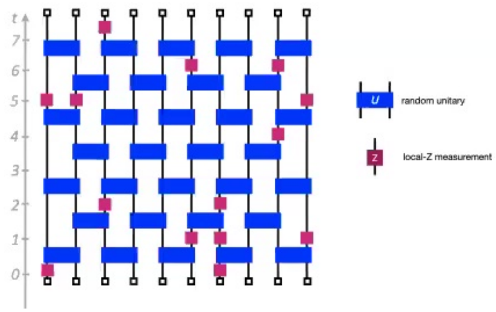
Entanglement in Hybrid Quantum Circuits

Ultra Quantum Matter Summerschool
8/14/20

MPA Fisher

Open, non-equilibrium, dynamical, quantum phases/transitions



Measurement driven entanglement transition

UQM: Mostly equilibrium…

Focus of UQM Summerschool:

- Quantum Systems in Equilibrium
- Ground states
- Exotic order (topological/fracton)
- Quantum criticality

CMT, HET
(Thermodynamic limit)

Some other topics:

- Non-equilibrium quantum systems
- Open quantum systems
- Role of measurements (observer)

Quantum Info
(Often few body)

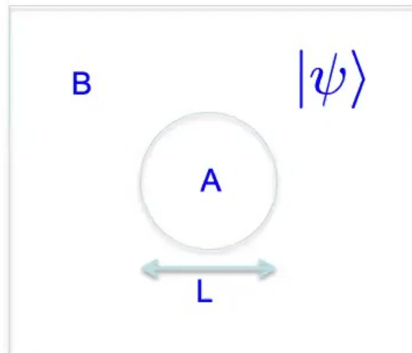
This talk:
Quantum phases/transitions driven by measurements
(in open, non-equilibrium systems, in thermodynamic limit)

Common thread: Entanglement Entropy

Entanglement entropy

Single eigenstate $|\psi\rangle$

Pure-state density matrix: $\hat{\rho} = |\psi\rangle\langle\psi|$



Spatial Bi-partition: Regions A and B

Reduced density matrix in A

$$\hat{\rho}_A = Tr_B(\hat{\rho})$$

Entanglement entropy:

$$S_A(L) = -Tr_A(\hat{\rho}_A \ln \hat{\rho}_A)$$

Spatial Scaling of Entanglement entropy

Ground states:

Gapped - area law:

$$S_A(L) \sim L^{d-1}$$

Gapless - area law, or log enhancement; 1d CFT/Fermi liquids

$$S_A(L) \sim L^{d-1} \ln(L)$$

Ground states manifest spatial locality

Excited eigenstates with finite energy-density

Volume-law entanglement entropy

$$S_A(L) = sL^d$$

Finite energy-density eigenstates are non-local

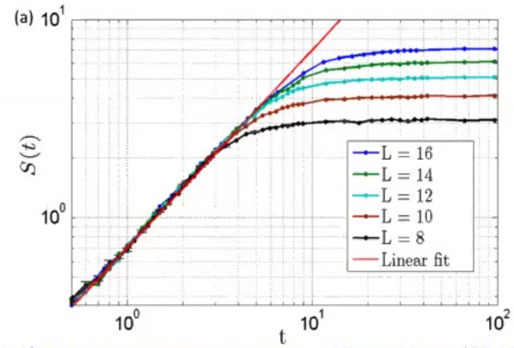
Entanglement **dynamics** out of equilibrium (closed system)

1) Quantum Quench

Evolve unentangled initial state w/ Hamiltonian

$$H = \sum_{i=1}^{L} \left(g\sigma_i^x + h\sigma_i^z + J\sigma_i^z\sigma_{i+1}^z \right)$$

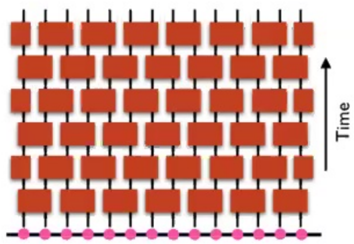
Entanglement spreads ballistically,
even though energy diffuses

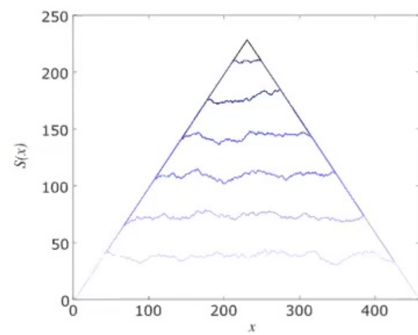


Half-cut entanglement entropy Kim + Huse (2013)

2) Unitary Dynamics with no energy conservation

Quantum circuit: evolve Qubits w/ (random) unitary gates
Initial state: unentangled product state
Entanglement spreads ballistically, into maximal entropy state

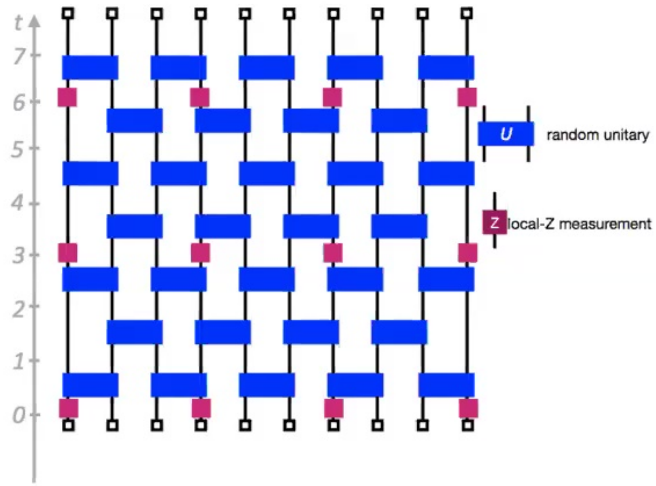




Nahum, Ruhman, Vijay, Haah (2017)

How to control (entanglement) entropy growth?

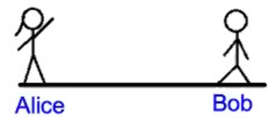
Via Measurements



Measurement driven entanglement transition

Entanglement and measurements

Alice and Bob share a singlet state



$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle_{AB} - |\downarrow\uparrow\rangle_{AB}]$$

Maximum entanglement entropy:

$$S_A = S_B = \ln(2)$$

Alice measures her spin:

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle_{AB} - |\downarrow\uparrow\rangle_{AB}] \Rightarrow \begin{cases} |\psi'\rangle = |\uparrow\rangle_A \otimes |\downarrow\rangle_B \\ |\psi'\rangle = |\downarrow\rangle_A \otimes |\uparrow\rangle_B \end{cases}$$

After measurment, direct product state

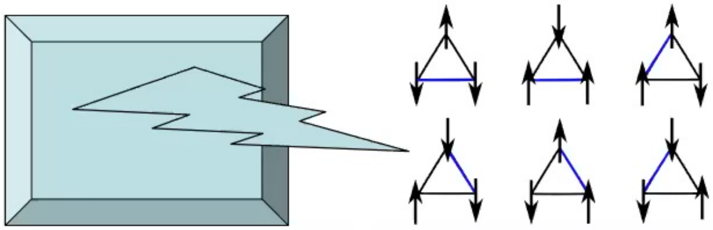
$$S_A = S_B = \ln(2) \Rightarrow S'_A = S'_B = 0$$

(Local) Measurement induces disentanglement

Open Quantum Systems

Two classes:

System coupled to a bath (environment)



System is monitored by an “observer”

- Initial pure density matrix becomes mixed
- Environment “measures” system, but results lost
- Decoherence
- Dynamics of density matrix evolves w/ (e.g.) Lindblad equation

- Initial pure state is measured and stays pure
- ”Observer” keeps track of measurements
- Wavefunction evolves as a pure state
- Dynamics described in terms of (wavefunction) quantum trajectories

Measurement-driven entanglement transition

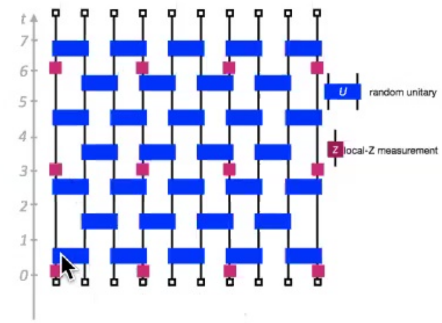
Extended systems w/ measurements (“monitored”)

“Hybrid Quantum Circuit”
w/ both unitary and measurement gates

- Unitary evolution induces entanglement growth
- Measurements induce disentanglement

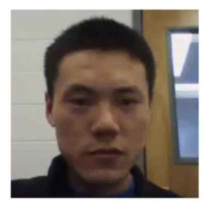
Explore competition between unitary evolution and measurements

(by following wavefcn quantum trajectories)



- Li, Chen, MPAF (2018/2019)
- Skinner, Ruhman, Nahum (2018)
- Chan, Nandkishore, Pretko, Smith (2018)
- Choi, Bao, Qi, Altman (2019)
- Gullans, Huse (2019)
- Many more...

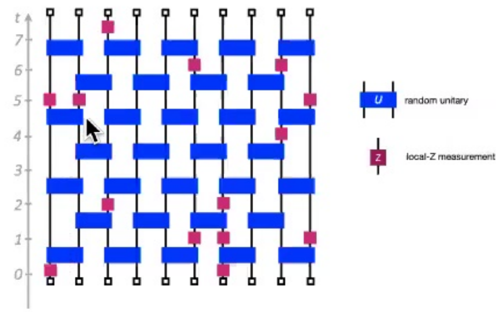
Yaodong Li



Xiao Chen

"Hybrid" Quantum Circuit

Initial density matrix $\hat{\rho}_0$



2-Qubit Unitaries:

$$\hat{\rho} \rightarrow \hat{U}\hat{\rho}\hat{U}^{\dagger}$$

1-Qubit Measurements

$$\hat{\rho} \rightarrow \frac{\hat{P}_{\pm}\hat{\rho}\hat{P}_{\pm}}{Tr[\hat{P}_{\pm}\hat{\rho}\hat{P}_{\pm}]}$$

Projection operators $\hat{P}_{\pm} = (1 \pm \hat{Z})/2$

Measurement outcome probabilities $Tr[\hat{P}_{\pm}\hat{\rho}\hat{P}_{\pm}]$

Canonical Model:

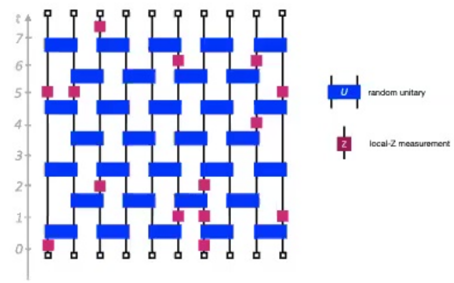
- **Randomly chosen 2-Qubit unitaries**
- **Single qubit measurements made with probability, p**

Single parameter: $p \in [0, 1]$

Phase diagram for hybrid circuit?

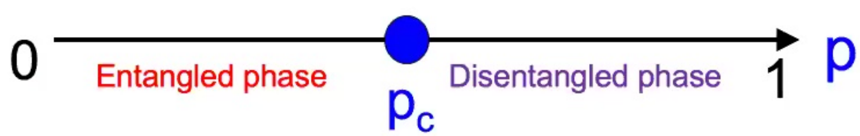
Initial state (say unentangled) $\hat{\rho}_0 = |\psi_0\rangle\langle\psi_0|$

- Run circuit dynamics to long times $\hat{\rho}_t = |\psi_t\rangle\langle\psi_t|$
- Density matrix stays **pure**
- Compute bipartite entanglement entropy $S_A(t) = -Tr_A(\hat{\rho}_A \log \hat{\rho}_A)$



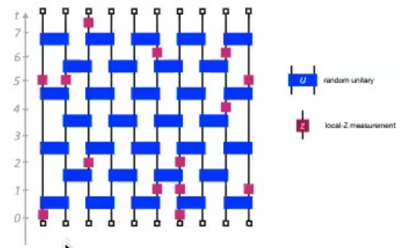
Phase Diagram?

- p=0; No measurement, Volume law entanglement
- p=1; Measure every Qubit, no entanglement (area law)
- Transition at p=p_c ?



Entanglement transition

Numerics on Hybrid Circuits



Direct simulation very challenging for large L
(since the Hilbert space grows as 2^L)

Employ Quantum information "technology":

- **"Stabilizers"** to encode special **"codeword"** quantum states
- Evolve stabilizers with **Clifford unitaries**
- Measurements of Z-component of spin

Gottesman-Knill Theorem: Such quantum circuits can be efficiently simulated on a classical computer (accessing >500 Qubits, say)

Pauli Strings, Stabilizers and Codewords

(Nielsen + Chuang)

Pauli operators for a single Qubit $\{1, \sigma_x, \sigma_y, \sigma_z\} \rightarrow \{1, X, Y, Z\}$

Pauli String Operators for L Qubits: $g = 1_1 Y_2 X_3 I_4 X_5 ... Z_L$

g

Stabilizers and "codewords":

$|\psi\rangle$ is a "codeword" state if "stabilized" by L independent, commuting Pauli string operators $g_j|\psi\rangle = |\psi\rangle$

Example 1: $|\psi\rangle = |00, ...0\rangle$ is stabilized by $g_j = Z_j$

Example 2: $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is stabilized by $g_1 = Z_1 Z_2$

$g_2 = X_1 X_2$

Clifford Unitaries/Dynamics

Clifford unitaries take Pauli string operators into other Pauli string operators $\hat{U}\hat{g}U^{\dagger} = \hat{g}'$

Unitary evolution of a "codeword" state: follow the dynamics of the L stabilizers:

If $|\psi\rangle$ stabilized by g_j then $|\psi'\rangle = U|\psi\rangle$ stabilized by $g'_j = U g_j U^{\dagger}$

Measurements and Stabilizers

Consider a projective measurement of a codeword $g_j|\psi\rangle = |\psi\rangle$

$$|\psi\rangle \rightarrow P_{\pm}|\psi\rangle \qquad P_{\pm} = (1 \pm Z_j)/2$$

Measuring Z-component of j^{th} qubit

If Z_j anticommutes with g_1 and commutes with $g_2, \ldots, g_L$ (say) the stabilizers are modified under the measurement as:

$$\{g_1, g_2, ..., g_L\} \rightarrow \{\pm Z_j, g_2, ..., g_L\}$$ when the "result" of the measurement is ± 1

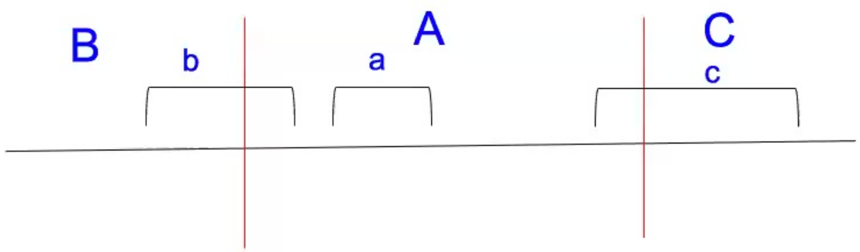
Entanglement and Stabilizers

Stabilizer length

length=6

$$g = 1_1 1_2 X_3 1_4 Z_5 Y_6 1_7 Z_8 1_9 1_{10}$$

Entanglement entropy S_A



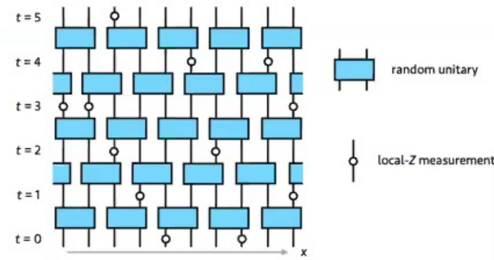
Denote number of stabilizers starting in A and ending in A,B,C as n_a, n_b, n_c

Entanglement: $$S_A = \frac{(n_b + n_c)}{2} \log(2)$$

Hybrid Clifford Circuit

All 2-Qubit unitaries taken from the Clifford group:

$$|\psi_t\rangle \rightarrow |\psi_{t+1}\rangle = U|\psi_t\rangle$$



All single Qubit measurements taken from Pauli group

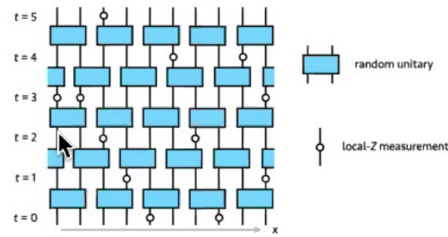
$$|\psi\rangle \rightarrow \frac{P_{\pm}|\psi\rangle}{\sqrt{p_{\pm}}} \qquad P_{\pm} = \tfrac{1}{2}(1 \pm Z)$$

Make measurements with probability p

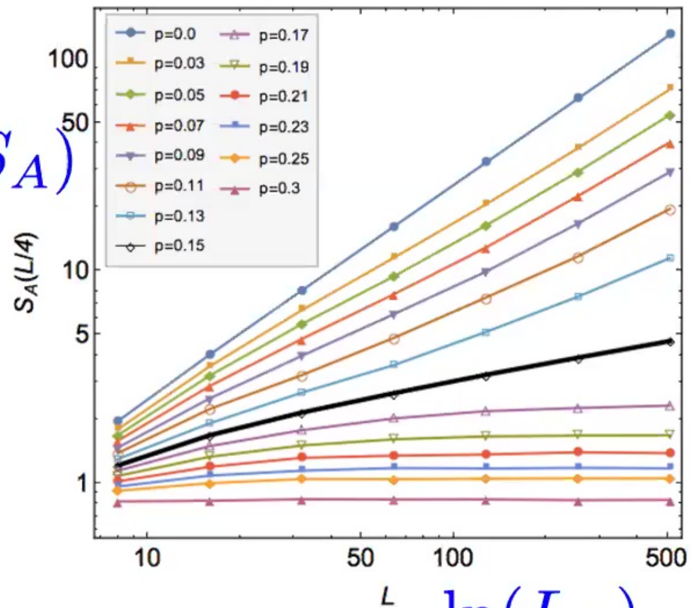
Simulate Clifford quantum circuits on classical computer
(accessing >500 Qubits)

Entanglement Entropy

w/ pure initial state,
long-time steady-state of Clifford circuit



$\ln(S_A)$



$\ln(L_A)$

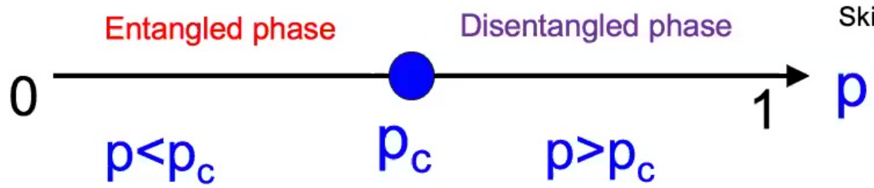
Volume law
entanglement

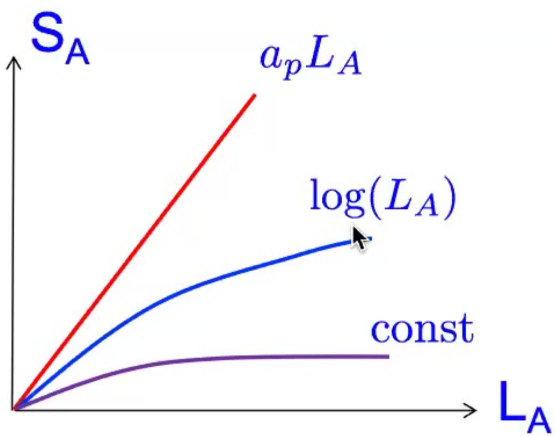
Increasing
measurement rate

Area law
entanglement

Entanglement Transition

Li, Chen, MPAF (2018)
Skinner, Ruhman, Nahum (2018)

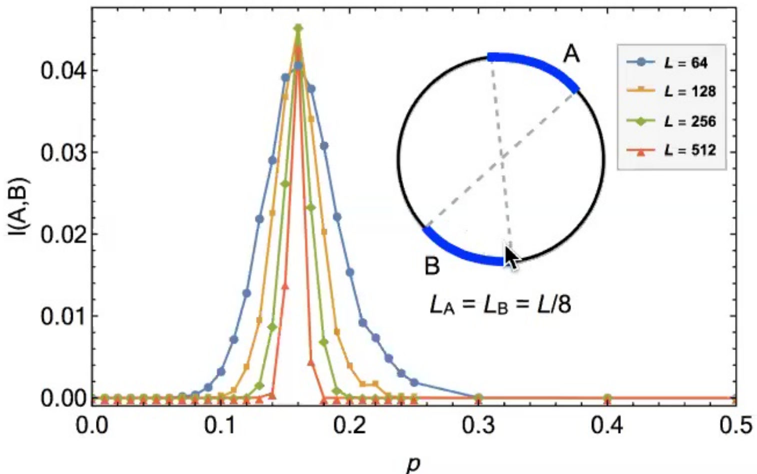




$$S_A(L_A) \sim \begin{cases} a_p L_A; & p < p_c \\ \log(L_A); & p = p_c \\ \text{const}; & p > p_c \end{cases}$$

Mutual Information: Locates transition

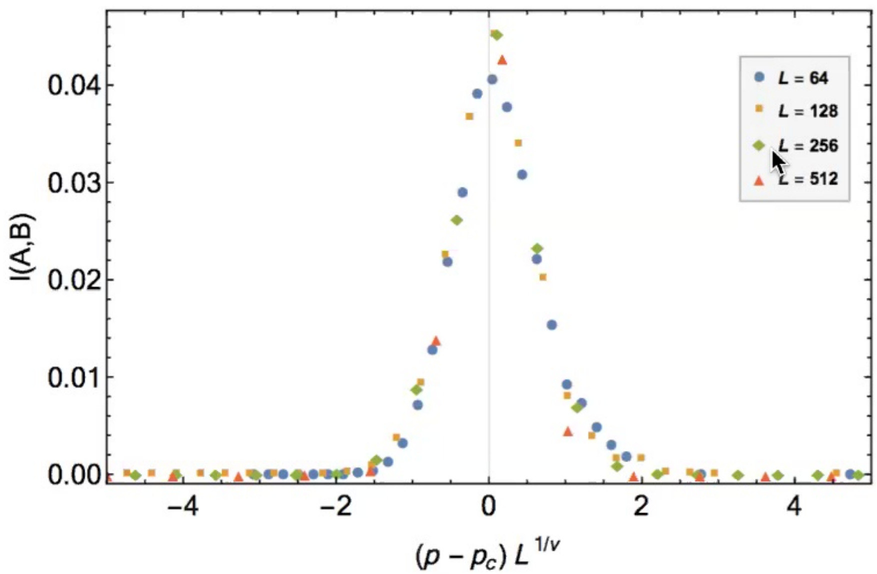
$$I_{AB} = S_A + S_B - S_{AB}$$



$$I_{AB}(L \to \infty) = \begin{cases} 0; & p \neq p_c \\ \text{const}; & p = p_c \end{cases}$$

Critical Properties of Transition:

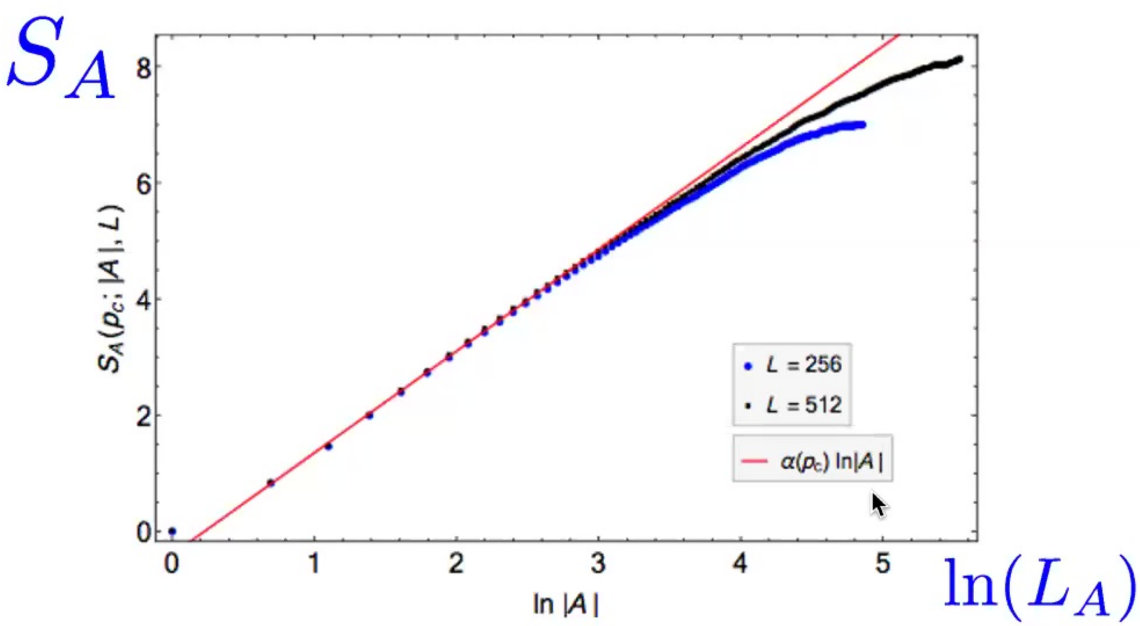
Data Collapse for Mutual Information



$$\nu \approx 1.4$$

Log Scaling at Criticality ($p=p_c$)

$$S_A(L_A) = \alpha_c \log(L_A) \qquad \alpha_c \approx 1.6$$



Resembles 1+1 CFT ground state

Conformal Symmetry at criticality ($p=p_c$)

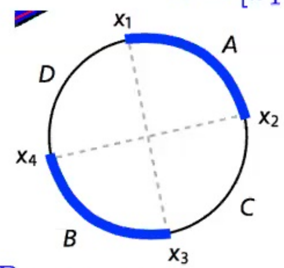
If have underlying conformal field theory, then mutual information depends only on the cross ratio

$$I_{AB} = f(\eta)$$

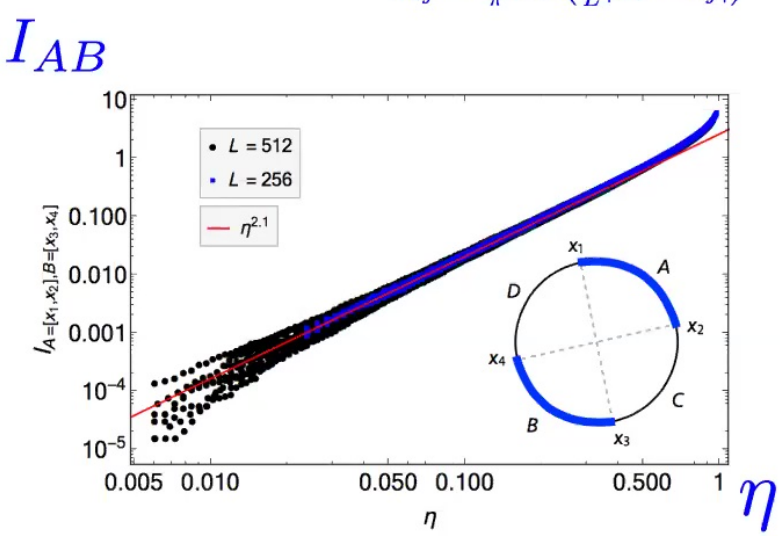
$$\eta \equiv \frac{x_{12}x_{34}}{x_{13}x_{24}}$$

$$x_{ij} = \frac{L}{\pi}\sin\left(\frac{\pi}{L}|x_i - x_j|\right)$$

$A = [x_1, x_2]$

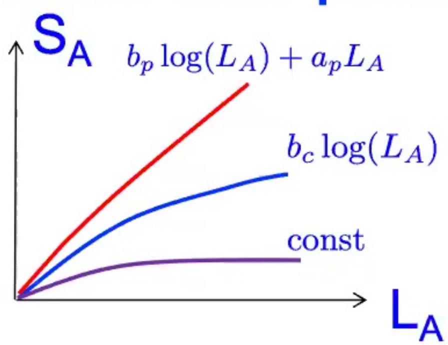


$B = [x_3, x_4]$

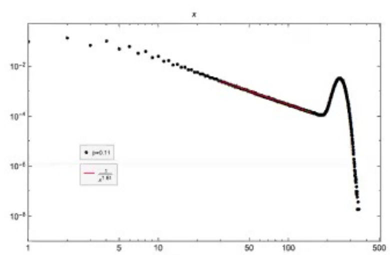


Nature of the volume law phase?

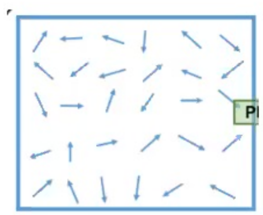
"Background" logarithm in volume law phase?



- Clifford numerics using stabilizer distribution function



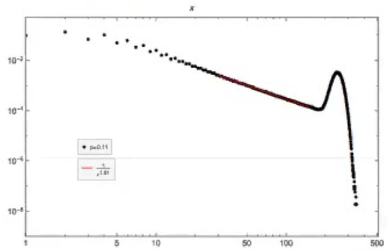
- Mapping to a stat mech model
  - Entanglement domain walls
  - Capillary wave theory

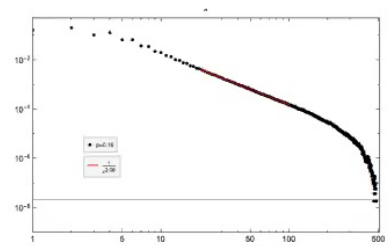


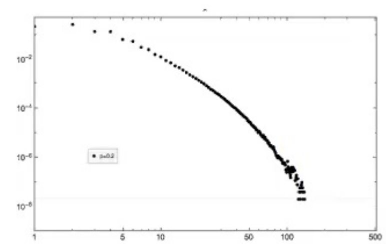
Stabilizer length distribution function

Increasing measurement rate

$\ln D(x)$







$\ln x$

$p<p_c$ | $p=p_c$ | $p>p_c$

$$D(x) \approx \frac{1}{x^2} + \delta(x - L/2)$$

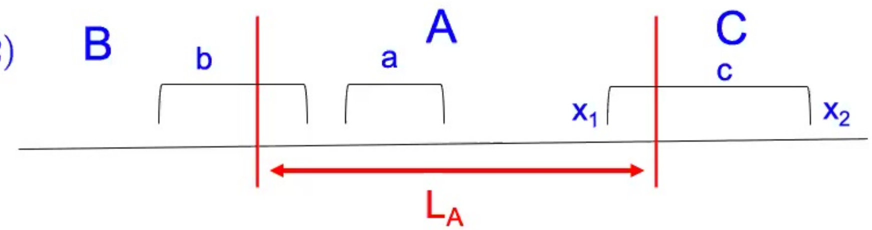
$$D(x) \approx \frac{1}{x^2}$$

$$D(x) \approx \frac{1}{x^2} e^{-x/\xi}$$

log from Stabilizer length distribution

Entanglement entropy follows: $S_A(L_A) \approx \int_0^{L_A} dx_1 \int_{L_A}^{L} dx_2 D(x_1 - x_2, L)$

$S_A = \frac{(n_b + n_c)}{2} \log(2)$



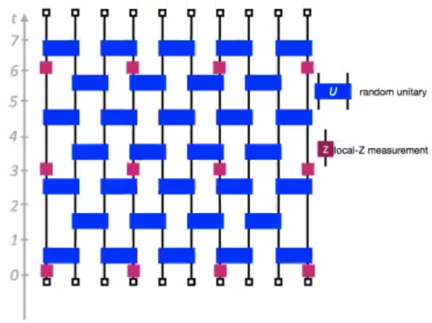
Background log

$$S_A(L_A) \approx \begin{cases} \log(L_A) + L_A; & p < p_c \\ a_c \log(L_A); & p = p_c \\ \log(\xi); & p > p_c \end{cases}$$

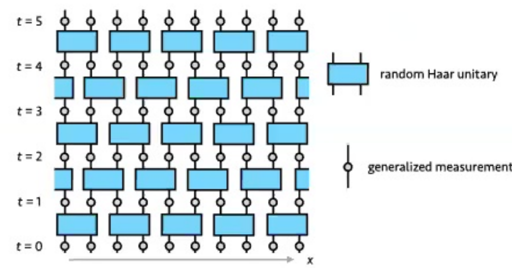
Mapping to Stat Mech (spin) model

Jian, You, Vasseur, Ludwig (2019)
Bao, Choi, Altman (2019)

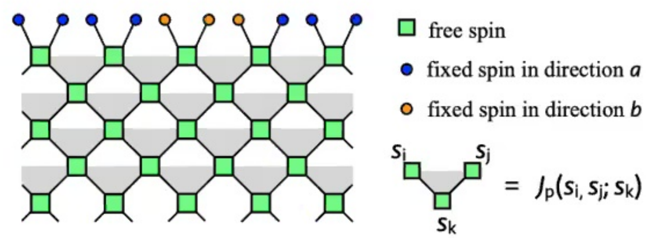
Random Clifford circuit
w/ projective measurements



Random Haar circuit w/
generalized measurements

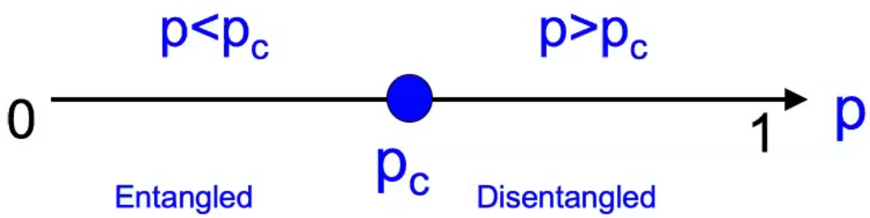


Haar circuit mapped to
“Generalized Potts” model
(spins live on space-time
manifold, 2d)

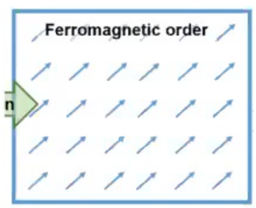


Phases in Stat mech model

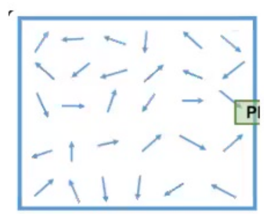
Volume law phase = “ordered” phase of stat mech model
Area law phase = “disordered” (paramagnetic) phase of stat mech model
Critical point = phase transition in stat mech model



“Ordered” phase of spin-model

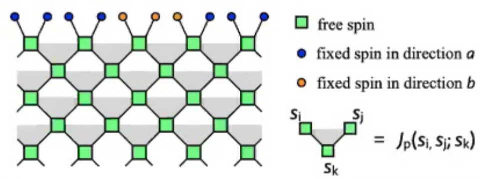


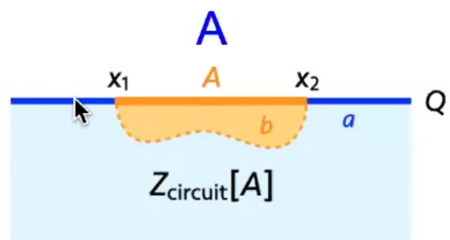
“Paramagnetic” phase of spin model



Entanglement entropy from Stat Mech model

$S_A = F_A =$ free energy cost for changing boundary conditions in region A





Volume law (ordered) phase:
Free energy of domain wall

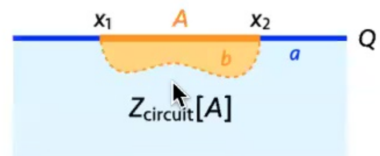
$$S_A = F_A \sim L_A$$

Area law (paramagnetic) phase:
Free energy from end-points

$$S_A = F_A = O(1)$$

Entanglement domain walls in volume law phase

S_A = Domain wall free energy



Capillary wave theory (MFT): $F_{CW} = -\ln(Z_{CW})$

$$Z_{CW}(A) = e^{-\beta\sigma|A|} \int Dy(x) e^{-\beta\sigma \int_{x_1}^{x_2} (\partial_x y)^2}$$

$y(x)$ parameterizes domain wall

Background log

$$S_A \approx F_{CW}(A) = \beta\sigma L_A + \frac{3}{2} \ln L_A$$

Fan, Vijay, Vishwanath, You (2020)
Li, MPAF (2020)

Volume law term (surface energy)

Transverse fluctuations of entanglement domain wall

Purification in Clifford circuit w/ mixed initial state

Gullans, Huse (2019)

Start dynamics in maximally mixed state, maximum entropy

$$\hat{\rho}_0 = \frac{1}{2^L}\hat{1}$$

$$S(t=0) = L \ln 2$$

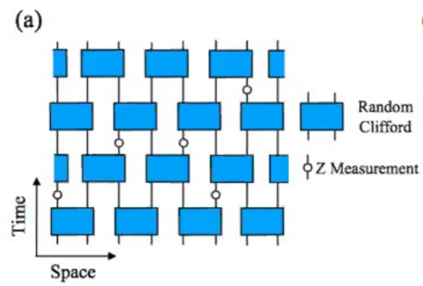
Run Clifford circuit:

Mixed state has fewer stabilizers than qubits

$$\{g_1, ..., g_m\} \quad m < L$$

$$S = (L-m)\ln 2$$

Initially m=0, but measurements can increase number of stabilizers (and decrease entropy)



- If Z_j commutes w/ but is independent of $\{g_j\}$ then m -> m+1

$$\{g_1, ..., g_{m+1}\} = \{g_1, ..., g_m\} \cup Z_j$$

- If Z_j anticommutes with g_1 and commutes with $g_2,...,g_m$ (say)

$$\{g_1, g_2, ..., g_m\} \to \{Z_j, g_2, ..., g_m\}$$

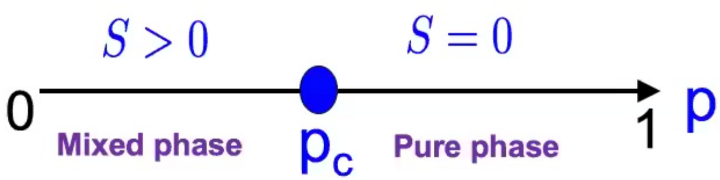
Purification Transition in hybrid Clifford circuit

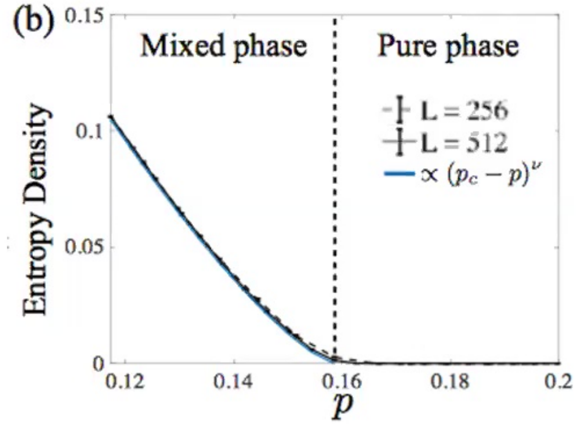
Gullans, Huse (2019)

Run Clifford circuit from mixed state at t=0, to t=cL

Compute thermal entropy at time t

$$S_t = -Tr(\rho_t \ln \rho_t)$$





Purification transition = entanglement transition

mixed ρ_0 pure $\rho_0 = |\psi_0\rangle\langle\psi_0|$

Quantum Error Correcting Code in mixed phase

Ippoliti, Gullans, Gopalakrishnan, Huse Khemani (2020)
Li, MPAF (2020)

Shor's 9-Qubit (stabilizer) QECC

[L,k,d] = [9,1,3]

L = 9 physical qubits
k = 1 logical qubit
L- k = 8 stabilizers

$$|\bar{0}\rangle = \frac{(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|\bar{1}\rangle = \frac{(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle)}{2\sqrt{2}}$$

Logical operators $\bar{Z}, \bar{X}$

$$\overline{Z}|\overline{x}\rangle = (-1)^x|\overline{x}\rangle$$
$$\overline{X}|\overline{x}\rangle = |\overline{x} \oplus 1\rangle$$

$$\begin{aligned}
g_1 &= Z_1\ Z_2\ I_3\ I_4\ I_5\ I_6\ I_7\ I_8\ I_9,\\
g_2 &= I_1\ Z_2\ Z_3\ I_4\ I_5\ I_6\ I_7\ I_8\ I_9,\\
g_3 &= I_1\ I_2\ I_3\ Z_4\ Z_5\ I_6\ I_7\ I_8\ I_9,\\
g_4 &= I_1\ I_2\ I_3\ I_4\ Z_5\ Z_6\ I_7\ I_8\ I_9,\\
g_5 &= I_1\ I_2\ I_3\ I_4\ I_5\ I_6\ Z_7\ Z_8\ I_9,\\
g_6 &= I_1\ I_2\ I_3\ I_4\ I_5\ I_6\ I_7\ Z_8\ Z_9,\\
g_7 &= X_1\ X_2\ X_3\ X_4\ X_5\ X_6\ I_7\ I_8\ I_9,\\
g_8 &= I_1\ I_2\ I_3\ X_4\ X_5\ X_6\ X_7\ X_8\ X_9,\\
\overline{Z} &= X_1\ X_2\ X_3\ X_4\ X_5\ X_6\ X_7\ X_8\ X_9,\\
\overline{X} &= Z_1\ Z_2\ Z_3\ Z_4\ Z_5\ Z_6\ Z_7\ Z_8\ Z_9.
\end{aligned}$$

d = code distance: "Shortest logical operator"

A code w/ distance d can correct any error acting on (d-1)/2 arbitrary qubits

With d=3, Shor code can correct any arbitrary single qubit error

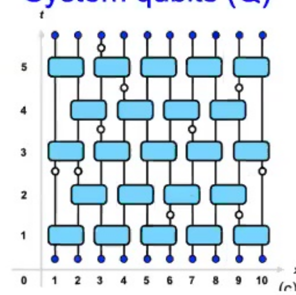
Dynamically generated QECC in mixed phase

At t=0, purify mixed state w/ ancilla qubits (R), $|\Psi_{QR}\rangle$

$$\hat{\rho}_0 = Tr_R|\Psi_{QR}\rangle\langle\Psi_{RQ}| = \frac{1}{2^L}\hat{1} \qquad S_0 = L\ln 2$$

Evolve system Qubits (Q) with Hybrid circuit

System qubits (Q)



R

After time t have an (L,k,d) QEC stabilizer code

L = physical qubits
k = logical qubits = entropy of code space
m = L-k = stabilizers

$$k = S_t = -Tr(\hat{\rho}_t \log \hat{\rho}_t)$$

Code rate=k/L = const.

Code distance??

Code distance for Clifford circuit

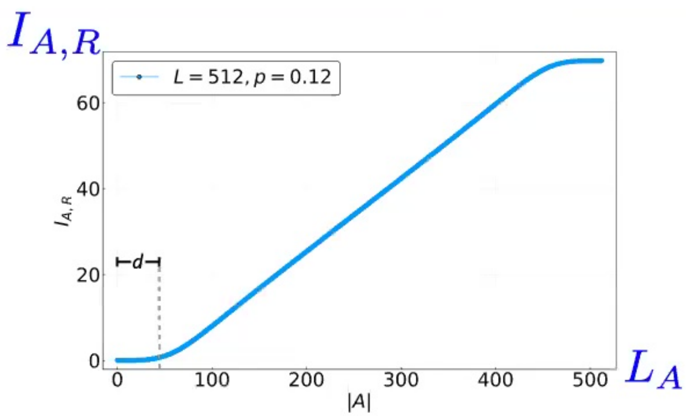
Mutual information numerically
for Clifford circuit

Extract code distance from condition;

$$I_{A,R} \approx 0; \quad |A| < d$$

Code distance scaling w/ L

$$d \sim L^{0.38}$$



Clifford numerics

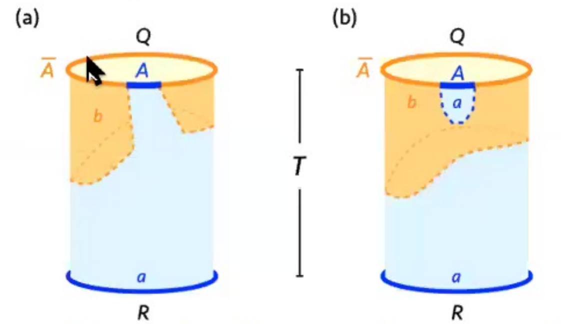
Code distance from entanglement domain walls

2 competing domain wall configs

(a) dominates for $|A| > d$

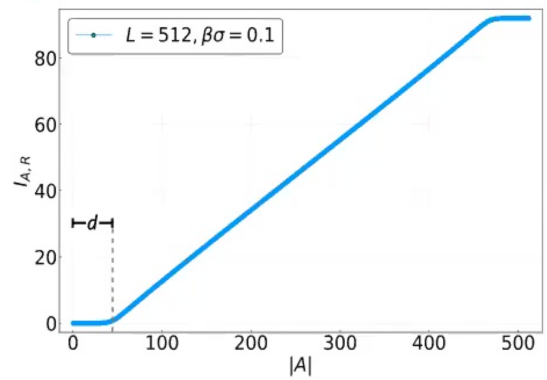
(b) dominates for $|A| < d$

Equating two gives d



$I_{A,R} > 0$ $\quad$ $I_{A,R} = 0$

$I_{A,R}$



L_A

Capillary wave theory

Capillary wave theory

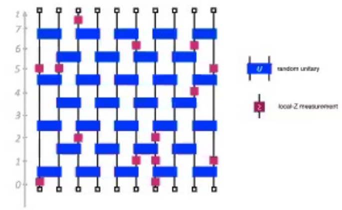
$d_{CW} \sim \frac{1}{2\beta\sigma}(\ln L + \ln T)$

Disagrees w/ Clifford numerics

$d \sim L^{0.38}$

Experimental Access??

Quantum trajectories required to access the transition



Challenge:

- To extract S_A (or other observables) requires multiple copies of same wavefunction $|\psi\rangle$
- To reproduce $|\psi\rangle$ requires post selection on O(Lt) measurement outcomes, choosing among 2^{Lt} possible (random) outcomes, to get copy of $|\psi\rangle$

Overcoming challenge??

(1) Two proposals by Choi, Bao, Altman (2019); Gullans, Huse (2019)

(2) Employ Clifford circuits

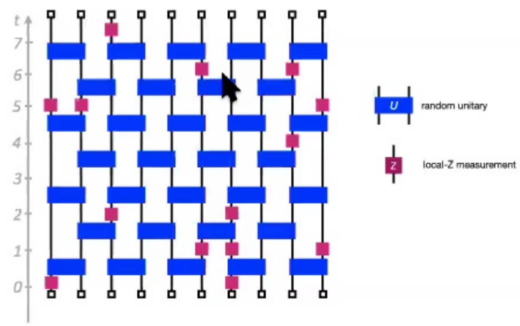
- Measurement outcomes *can* be forced, using error correction to “undo” a wrong measurement outcome, then try again
- Clifford gates can be implemented on a quantum computer

Summary: Entanglement Transitions

Quantum Entanglement Transition:

Competition between unitary induced entanglement and measurement induced disentanglement

(Entanglement transition = purification transition)



Open/future:

- Universality class of transition (CFT)?
- Role of randomness?
- Higher dimensions?
- Systems w/ feedback?
- Phases of QECC's? Phases of Quantum computers?
- Other dynamical phases in monitored open systems?
- Experimental observation of entanglement transition?

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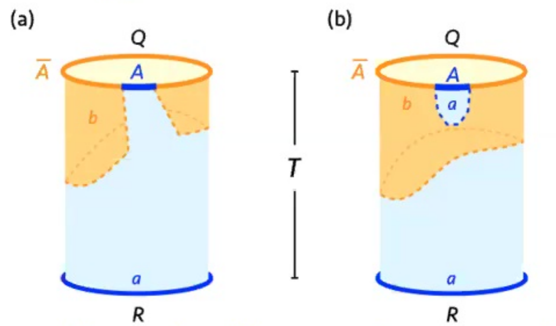
Equating two gives d



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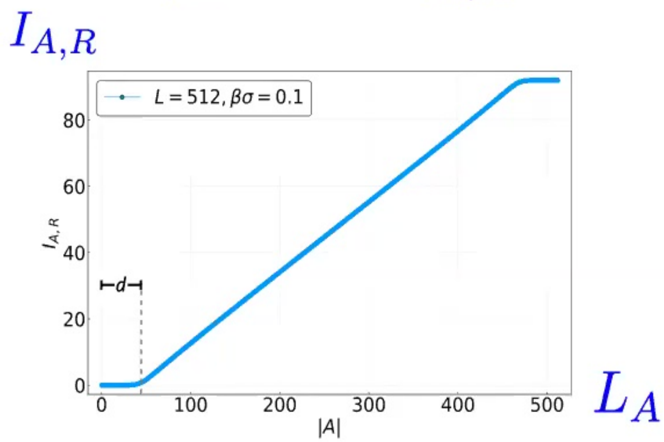

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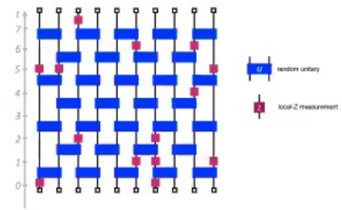

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