Title: Dualities 1

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Field theory dualities in Condensed Matter

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UQM Virtual Summer School
Perimeter Institute, Aug 11, 2020

Plan for today

1. *Basic notions and the particle-vortex duality in (2+1)d*

2. *A fermion-boson duality (a.k.a. 3d bosonization)*

3. *A fermion-fermion duality and the half-filled Landau level*

Historical motivation

- Q: nature of finite T transition for $3d$ superconductor?

$$|(\nabla - i\mathbf{a})\phi|^2 - m|\phi|^2 + \lambda|\phi|^4 + \frac{1}{2e^2} f_{\mu\nu}^2 + \ldots$$

4-ϵ expansion: always 1st order

- However, **on a lattice**, we can map this theory to

$$|\nabla\Phi|^2 + m|\Phi|^2 + \lambda|\Phi|^4 + \ldots$$

Lattice mapping

- Compact boson + $U(1)$ gauge field:

$$Z_s(\beta,e) = \int_{-\pi}^{\pi}(d\theta_i/2\pi)\int_{-\infty}^{\infty}d[A_{i\mu}]\sum_{[n_{i\mu}]=-\infty}^{\infty}\exp[-\tfrac{1}{2}\beta\sum_{i=1}^{N}\sum_{\mu=1}^{3}(\Delta_\mu\theta_i - 2\pi n_{i\mu} - eA_{i\mu})^2 - \tfrac{1}{2}\sum_i|\vec{\Delta}\times\vec{A}_i|^2]$$

- Compact boson:

$$Z_{s'}(\beta',e) = \sum_{[n_{i\mu}]}\int(d\theta_i/2\pi)\exp[-\tfrac{1}{2}\beta'\sum_i|\vec{\Delta}\times\vec{n}_i|^2 - (e^2/8\pi^2)\sum_{i\mu}(\Delta_\mu\theta_i - 2\pi n_{i\mu})^2]$$

- Lattice duality(Peskin; Dasgupta, Halperin):

$$Z_s(\beta,e) \propto (2\pi\beta)^{-3N/2}Z_{s'}(1/\beta,e)$$

Lattice mapping

$\cos(\Delta\theta - A)$

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- Compact boson:

$$Z_{s'}(\beta', e) = \sum_{[n_{i\mu}]} \int (d\theta_i/2\pi) \exp[-\tfrac{1}{2}\beta' \sum_i |\vec{\Delta} \times \vec{n}_i|^2 - (e^2/8\pi^2) \sum_{i\mu} (\Delta_\mu \theta_i - 2\pi n_{i\mu})^2]$$

- Lattice duality(Peskin; Dasgupta, Halperin):

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Particle-vortex duality

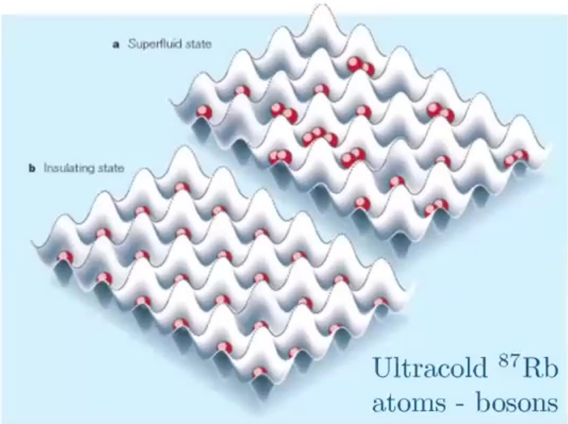
$$|(\nabla - i\mathbf{a})\phi|^2 - m|\phi|^2 + \lambda|\phi|^4 + \frac{1}{2e^2}f_{\mu\nu}^2 + \ldots$$

$$\updownarrow$$

$$|\nabla\Phi|^2 + m|\Phi|^2 + \lambda|\Phi|^4 + \ldots$$

- Conclusion: transition can be continuous and belongs to $O(2)$ Wilson-Fisher universality class

- Also describes $(2+1)d$ quantum superfluid-insulator transition at $T=0$



(Greiner, et. al, 2002; image from Sachdev, 12)

Why does it work?

- Both theories have a global $U(1)$ symmetry, conserved current:

$$j^{\mu}_{\Phi} \quad \Leftrightarrow \quad \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_{\nu}a_{\lambda}$$

Conserved boson charge $\leftrightarrow$ Conserved gauge flux

Φ And ϕ see each other as vortices: particle-vortex duality

- Local operators charged under global $U(1)$

$$\Phi \quad \Leftrightarrow \quad \mathcal{M}_a$$

Flux-insertion operator,
a.k.a. ***monopole***

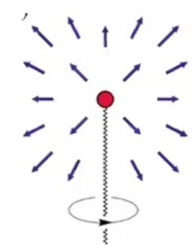


Image from Zyga, phys.org

Mean field phase diagram

$$|(\partial_\mu - iA_\mu)\Phi|^2 + m|\Phi|^2 + \lambda|\Phi|^4 + \ldots$$

$$\updownarrow$$

$$|(\partial - ia_\mu)\phi|^2 - m|\phi|^2 + \lambda|\phi|^4 + \frac{1}{2e^2} f_{\mu\nu}^2 + \frac{i}{2\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda + \ldots$$

- $m<<0$:

  Φ theory: superfluid $<\Phi>\neq 0$, sound (Goldstone) boson

  ϕ theory: ϕ gapped, free photon

- $m>>0$:

  Φ theory: Φ gapped, trivial at long distance (IR)

  ϕ theory: $< \phi > \neq 0$, Higgs phase

A vortex insulator (superfluid) is a particle superfluid (insulator)

Continuum limit: a conjectured duality

$$|D_A \Phi|^2 + \lambda |\Phi|^4 \quad \Leftrightarrow \quad |D_a \phi|^2 + \lambda |\phi|^4 + \frac{1}{2e^2} f_{\mu\nu}^2 + \frac{i}{2\pi} a \wedge dA$$

- Send momentum cut-off $\Lambda \to \infty$, allowed since both theories super-renormalizable ($[e^2]=[\lambda]=[\text{momentum}]$)
- On lattice, this means weak-coupling
- Lattice duality does *not* imply the continuum duality (two lattice theories cannot simultaneously be at weak coupling), but motivates it
- We call the lattice duality ***weak*** and the continuum duality ***strong***

Continuum limit: a conjectured duality

$$|D_A\Phi|^2 + \lambda|\Phi|^4 \quad \Leftrightarrow \quad |D_a\phi|^2 + \lambda|\phi|^4 + \frac{1}{2e^2}f_{\mu\nu}^2 + \frac{i}{2\pi}a \wedge dA$$

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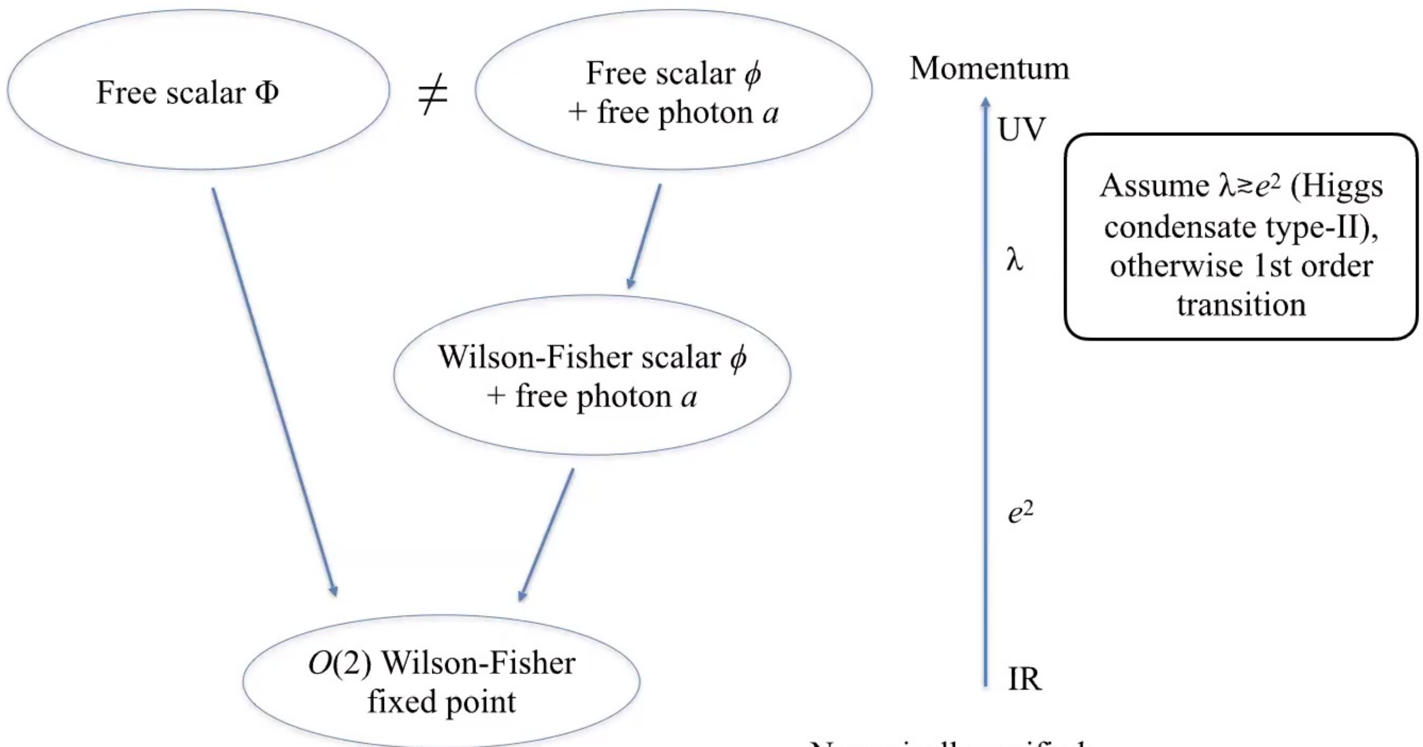
IR duality

$$|D_A\Phi|^2 + \lambda|\Phi|^4 \quad \Leftrightarrow \quad |D_a\phi|^2 + \lambda|\phi|^4 + \frac{1}{2e^2}f_{\mu\nu}^2 + \frac{i}{2\pi}a \wedge dA$$



Numerically verified:
Nguyen and A. Sudbø, 99; Kajantie, et.al, 04

Weak vs. Strong duality

Weak duality:

- Valid for some lattice theories
- Same phase diagram, same set of local operators
- Easier to establish analytically
- Useful for exploring new phases of matter (e.g. topological orders, half-filled Landau level)

Strong duality:

- Valid for continuum field theories (often only in IR)
- Same critical properties (e.g. exponents)
- Often conjectural and needs numerical verification
- Useful for studying critical properties (e.g. deconfined quantum criticality)

(1+1)d analogue: Kramers-Wannier duality

- Quantum transverse field Ising model in $1d$:

$$H = -J\sum_i \sigma_i^z \sigma_{i+1}^z - h\sum_i \sigma_i^x = -h\sum_i \tau^z_{i-\frac{1}{2}}\tau^z_{i+\frac{1}{2}} - J\sum_i \tau^x_{i+\frac{1}{2}}$$

$$\tau^x_{i+\frac{1}{2}} = \sigma_i^z \sigma_{i+1}^z$$

$$\tau^z_{i+\frac{1}{2}} = \prod_{j\leq i} \sigma_i^x$$

- $\tau^z_{i+\frac{1}{2}}$ creates a domain wall (or kink) in the original ferromagnet

- On a ring, τ^z periodic/anti-periodic b.c. depending on total charge — should view τ^z as coupled to a dynamical Z_2 gauge field

Kramers-Wannier in continuum limit

$$(D_B\phi)^2 + r\phi^2 + \lambda\phi^4 \quad \Leftrightarrow \quad (D_b\tilde{\phi})^2 - r\tilde{\phi}^2 + \lambda\tilde{\phi}^4 + i\pi b \cup B$$

- B (b): background (dynamical) $\mathbb{Z}_2$ gauge field, decides whether ϕ ($\tilde{\phi}$) has periodic/anti-periodic b.c.
- $i\pi b \cup B$: assigns a $\mathbb{Z}_2$ charge under B to a nontrivial flux of b (anti-periodic b.c. for $\tilde{\phi}$), very similar to $a \wedge dA$ in particle-vortex
- Original Ising scalar ϕ: flux-insertion operator in b (twist field), again similar to particle-vortex

Kramers-Wannier in continuum limit

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- Not really a "self-duality" — appears to be one only because $\mathbb{Z}_2$ gauge field does not affect IR dynamics
- Logically, this does not simply follow from the lattice duality (but of course we know it's true in this case)

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A Fermion-Boson Duality

$$i\bar{\Psi}\not{D}_A\Psi - \frac{i}{8\pi}AdA \quad \Leftrightarrow \quad |D_b\phi|^2 + \lambda|\phi|^4 + \frac{1}{2e^2}f_{\mu\nu}^2 + \frac{i}{4\pi}bdb + \frac{i}{2\pi}bdA$$

- *Interlude*: LHS really means

$$Z = |\det \not{D}_A| e^{-\frac{i\pi}{2}\eta[A]}$$

η-invariance: gives Hall conductance $\sigma_{xy} = -1/4\pi$,

similar to $-\frac{1}{8\pi}AdA$, but gauge invariant

- Sometimes the above is simply denoted as $[i\bar{\Psi}\not{D}_A\Psi]_{PV,+}$: gauge-invariant but breaks T-reversal (a.k.a. parity-anomaly)

- In our notation, the Dirac term alone has

$$Z = |\det \not{D}_A| e^{-\frac{i\pi}{2}\eta[A] + \frac{i}{8\pi}\int AdA} = |\det \not{D}_A|(-1)^{\mathcal{I}}$$

which preserves T-reversal but not gauge-invariant on its own: surface of topological insulator

(Witten, RMP 16)

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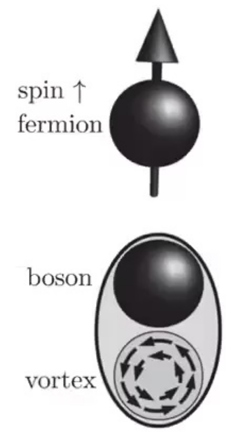
$$U(1)_A : j_\Psi \quad \longleftrightarrow \quad \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial^\nu b^\lambda$$

$$\text{Operator} : \Psi \quad \longleftrightarrow \quad \phi^\dagger \mathcal{M}_b$$

- A relativistic version of flux-attachment



(Figure from Mross, et. al, 17)

Phase diagram

$$i\overline{\Psi}\not{D}_A\Psi - \frac{1}{8\pi}AdA \qquad \longleftrightarrow \qquad |D_b\phi|^2 - |\phi|^4 + \frac{1}{4\pi}bdb + \frac{1}{2\pi}bdA$$

$$m\overline{\Psi}\Psi \qquad \longleftrightarrow \qquad m|\phi|^2$$

$m < 0 : -\frac{1}{4\pi}AdA$ $\qquad$ $m > 0$: trivial

- LHS: integrating out massive $\Psi \rightarrow \frac{m}{|m|}\frac{1}{8\pi}AdA$
- RHS: $\langle\phi\rangle \neq 0 \rightarrow$ trivial Higgs phase
  massive $\phi \rightarrow \frac{1}{4\pi}bdb + \frac{1}{2\pi}bdA \rightarrow -\frac{1}{4\pi}AdA$

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IQHE

- LHS: integrating out massive $\Psi \to \frac{m}{|m|}\frac{1}{8\pi}AdA$

- RHS: $\langle\phi\rangle \neq 0 \to$ trivial Higgs phase
  massive $\phi \to \frac{1}{4\pi}bdb + \frac{1}{2\pi}bdA \to -\frac{1}{4\pi}AdA$

Status of this duality

- As a weak duality: fully established through
    - General arguments
    - Lattice model (Chen et. al,)
    - Coupled-wire construction (Mross et. al,)
    - SUSY mirror dualities + SUSY breaking (Kachru et. al,)
- As a strong duality: remains conjectural
- Competing scenario: bosonic side flows to 1st order transition

Time-reversal Symmetry?

$$i\bar{\Psi}\not{D}_A\Psi \quad \Leftrightarrow \quad |D_b\phi|^2 - |\phi|^4 + \frac{1}{4\pi}bdb + \frac{1}{2\pi}bdA + \frac{1}{8\pi}AdA$$

- LHS: invariant (TI surface)

- RHS: a naive application of *T*:

$$|D_{\tilde{b}}\tilde{\phi}|^2 - |\tilde{\phi}|^4 - \frac{1}{4\pi}\tilde{b}d\tilde{b} - \frac{1}{2\pi}\tilde{b}dA - \frac{1}{8\pi}AdA$$

- Not identical to RHS, but *dual* to it, through particle-vortex duality on $\tilde{\phi}$

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$$\to |D_b\phi|^2 - |\phi|^4 - \frac{1}{2\pi}bd\tilde{b}$$

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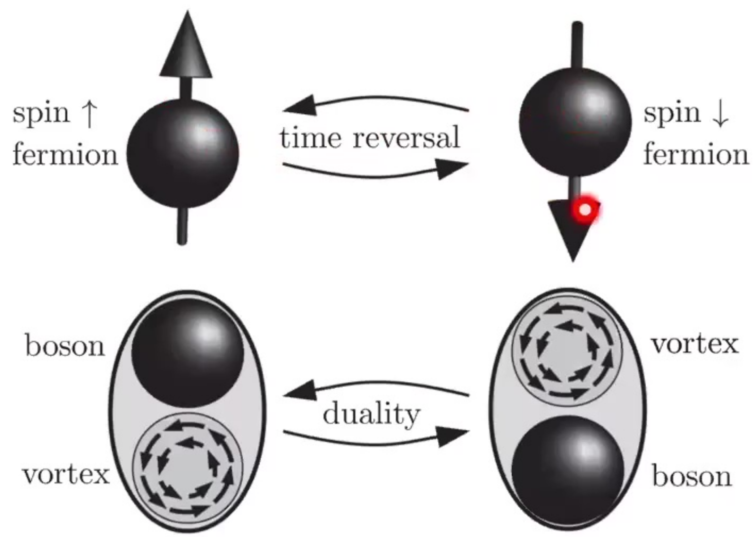
$$\frac{1}{4\pi}(b+A)d(b+A)$$

Time-reversal Symmetry?



(Figure from Mross, et. al, 17)

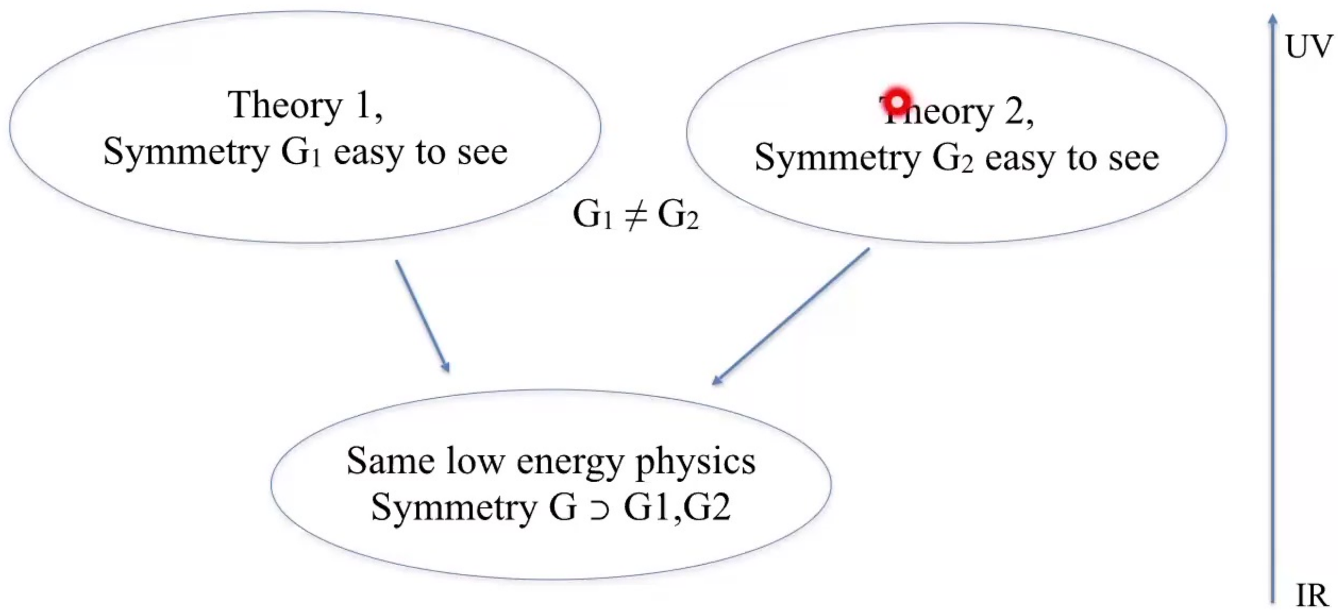
Analogue in (1+1)d: Jordan-Wigner

$$H = -J\sum_i \sigma_i^z \sigma_{i+1}^z - h\sum_i \sigma_i^x = -h\sum_i \tau_{i-\frac{1}{2}}^z \tau_{i+\frac{1}{2}}^z - J\sum_i \tau_{i+\frac{1}{2}}^x$$
$$= iJ\sum_{r=2i} \eta_r \eta_{r+1} + ih\sum_{r=2i+1} \eta_r \eta_{r+1}$$

$$\eta_{2i} = i\sigma_i^z \tau_{i+1/2}^z, \quad \eta_{2i+1} = \tau_{i+1/2}^z \sigma_{i+1}^z$$

- Majorana fermion = Ising spin + domain wall: flux-attachment!
- If impose symmetry $\mathcal{S} : \eta_r \to \eta_{r+1}$, then $J = h$ and system enforced to be gapless
- What does $\mathcal{S}$ look like in other pictures?
  $\mathcal{S} : \sigma_i \to \tau_{i+1/2}, \tau_{i+1/2} \to \sigma_{i+1}$, exchanges spin and domain wall.
  Very similar to time-reversal in $3d$ bosonization

Emergent symmetry from duality



Powerful in understanding emergent symmetries in exotic quantum phase transitions (e.g. deconfined criticality)

Things I didn't get to talk about

- Relation between these (2+1)d dualities to S-dualities (electric-magnetic dualities) of (3+1)d U(1) gauge theory (U(1) quantum spin liquids)

- Other applications in condensed matter

  - Deconfined quantum criticality

  - Topological phases

  - Quantum Hall plateau transitions

  - Superconductor-insulator transitions

  - Quantum phase transitions of Kitaev spin liquids

“Deriving” other dualities

- Starting from

$$\mathcal{L}_1[A] \quad \Leftrightarrow \quad \mathcal{L}_2[A]$$

- Operation T:

$$\mathcal{L}_1[A] + \frac{1}{4\pi} AdA \quad \Leftrightarrow \quad \mathcal{L}_2[A] + \frac{1}{4\pi} AdA$$

- Operation S:

$$\mathcal{L}_1[a] + \frac{1}{2\pi} adA \quad \Leftrightarrow \quad \mathcal{L}_2[A] + \frac{1}{2\pi} adA$$

- Can generate infinitely many dualities

A fermion-fermion duality

$$\bar{\psi} i \not{D}_A \psi \quad \Leftrightarrow \quad \bar{\chi} i \not{D}_a \chi - \frac{1}{4\pi} a dA$$

- A fermionic version of particle-vortex duality: χ behaves as a 4π vortex of ψ
- Related to $3d$ bosonization through some S, T operations
- Passes all "kinematic" checks: same symmetries, same set of local operators, same phase diagram upon mass deformation…
- Explicit coupled-wire realization exists

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Summary

Theory 1 $\approx$ Theory 2

Hard questions in (1) may be simpler in (2) if

1. Theory (2) is solvable
2. Certain properties (like symmetries) are manifest in Theory (2)
3. It's easier to make a guess in Theory (2)