Title: SYK criticality and correlated metals

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# SYK criticality and correlated metals

Subir Sachdev

Online School on Ultra Quantum Matter
Perimeter Institute for Theoretical Physics
August 10, 2020

PHYSICS
HARVARD

# Ordinary metals and quasiparticles

## What are quasiparticles ?

- **Quasiparticles are additive excitations:** The low-lying excitations of the many-body system can be identified as a set $\{n_\alpha\}$ of quasiparticles with energy $\varepsilon_\alpha$

$$E = \sum_\alpha n_\alpha \varepsilon_\alpha + \sum_{\alpha,\beta} F_{\alpha\beta} n_\alpha n_\beta + \ldots$$

In a lattice system of $N$ sites, this parameterizes the energy of $\sim e^{\alpha N}$ states in terms of poly$(N)$ numbers.

# Ordinary metals and quasiparticles

- Quasiparticles eventually collide with each other. Such collisions eventually leads to thermal equilibration in a chaotic quantum state, but the equilibration takes a long time. In a Fermi liquid, this time diverges as

$$\tau_{\rm eq} \sim \frac{\hbar E_F^3}{U^2 (k_B T)^2} \quad , \quad \text{as } T \rightarrow 0,$$

  where $U$ is the strength of interactions, and $E_F$ is the Fermi energy.

- Similarly, a quasiparticle model implies a resistivity

$$\rho = \frac{m^*}{ne^2} \frac{1}{\tau} \sim U^2 T^2 \quad \text{with} \quad \tau \sim \tau_{\rm eq}$$

- These times are much longer than the 'Planckian time' $\hbar/(k_B T)$, which we will find in systems without quasiparticle excitations.

$$\tau \sim \tau_{\rm eq} \gg \frac{\hbar}{k_B T} \quad , \quad \text{as } T \rightarrow 0.$$

Remarkable recent observation of 'Planckian' strange metal transport in cuprates, pnictides, magic-angle graphene, and ultracold atoms: the resistivity, $\rho$, is

$$\rho = \frac{m^*}{ne^2}\,\frac{1}{\tau}$$

with a universal scattering rate

$$\frac{1}{\tau} \approx \frac{k_B T}{\hbar}\,,$$

independent of the strength of interactions!

| Material | | $n$ ($10^{27}$ m$^{-3}$) | $m^*$ ($m_0$) | $A_1 / d$ ($\Omega$ / K) | $h / (2e^2 T_F)$ ($\Omega$ / K) | $\alpha$ |
|---|---|---|---|---|---|---|
| Bi2212 | $p = 0.23$ | 6.8 | 8.4 ± 1.6 | 8.0 ± 0.9 | 7.4 ± 1.4 | 1.1 ± 0.3 |
| Bi2201 | $p \sim 0.4$ | 3.5 | 7 ± 1.5 | 8 ± 2 | 8 ± 2 | 1.0 ± 0.4 |
| LSCO | $p = 0.26$ | 7.8 | 9.8 ± 1.7 | 8.2 ± 1.0 | 8.9 ± 1.8 | 0.9 ± 0.3 |
| Nd-LSCO | $p = 0.24$ | 7.9 | 12 ± 4 | 7.4 ± 0.8 | 10.6 ± 3.7 | 0.7 ± 0.4 |
| PCCO | $x = 0.17$ | 8.8 | 2.4 ± 0.1 | 1.7 ± 0.3 | 2.1 ± 0.1 | 0.8 ± 0.2 |
| LCCO | $x = 0.15$ | 9.0 | 3.0 ± 0.3 | 3.0 ± 0.45 | 2.6 ± 0.3 | 1.2 ± 0.3 |
| TMTSF | $P$ = 11 kbar | 1.4 | 1.15 ± 0.2 | 2.8 ± 0.3 | 2.8 ± 0.4 | 1.0 ± 0.3 |

**Slope of *T*-linear resistivity vs Planckian limit in seven materials.**

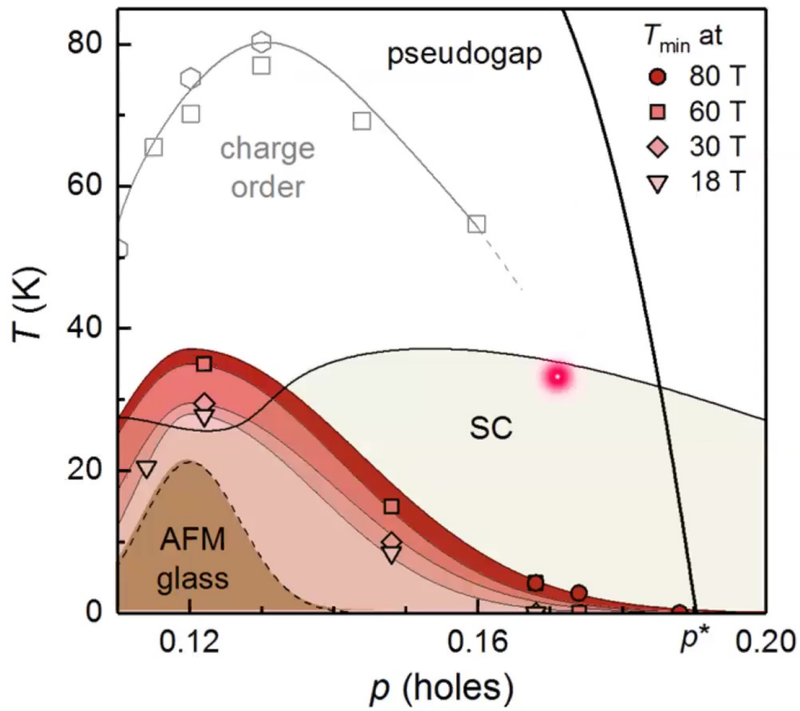
$$\frac{1}{\tau} = \alpha \, \frac{k_B T}{\hbar}$$

A. Legros, S. Benhabib, W. Tabis, F. Laliberté, M. Dion, M. Lizaire, B. Vignolle, D. Vignolles, H. Raffy, Z. Z. Li, P. Auban-Senzier, N. Doiron-Leyraud, P. Fournier, D. Colson, L. Taillefer, and C. Proust, Nature Physics **15**, 142 (2019)

# Hidden magnetism at the pseudogap critical point of a high temperature superconductor

Nature Physics doi: 10.1038/s41567-020-0950-5

Mehdi Frachet[1]†, Igor Vinograd[1]†, Rui Zhou[1,2], Siham Benhabib[1], Shangfei Wu[1], Hadrien Mayaffre[1], Steffen Krämer[1], Sanath K. Ramakrishna[3], Arneil P. Reyes[3], Jérôme Debray[4], Tohru Kurosawa[5], Naoki Momono[6], Migaku Oda[5], Seiki Komiya[7], Shimpei Ono[7], Masafumi Horio[8], Johan Chang[8], Cyril Proust[1], David LeBoeuf[1*], Marc-Henri Julien[1*]




**Quasi-static magnetism in the pseudogap state of $La_{2-x}Sr_xCuO_4$.** Temperature – doping phase diagram representing $T_{min}$, the temperature of the minimum in the sound velocity, at different fields. Since superconductivity precludes the observation of $T_{min}$ in zero-field, the dashed line (brown area) represents the extrapolated $T_{min}(B=0)$. While not exactly equal to the freezing temperature $T_f$ (see Fig. 2), $T_{min}$ is closely tied to $T_f$ and so is expected to have the same doping dependence, including a peak around $p = 0.12$ in zero/low fields (ref. 2). Onset temperatures of charge order are from ref. 33 (squares) and 35 (hexagons).

# Will describe a series of increasingly realistic (partly) solvable random models of correlated metals

1. Quantum matter with quasiparticles: random matrix model

2. Quantum matter without quasiparticles: the complex SYK model

No "Mottness"

3. Random *J* model (insulator)

   *RG analysis and exact exponent*

4. Random Hubbard and *t-J* models

   *Numerical results*

5. Random t-*J* model (metals): *exact exponents*

# A simple model of a metal with quasiparticles

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$\frac{1}{N} \sum_i c_i^\dagger c_i = \mathcal{Q}$$

$t_{ij}$ are independent random variables with $\overline{t_{ij}} = 0$ and $\overline{|t_{ij}|^2} = t^2$

Fermions occupying the eigenstates of a *N* x *N* random matrix

# A simple model of a metal with quasiparticles

Feynman graph expansion in $t_{ij..}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

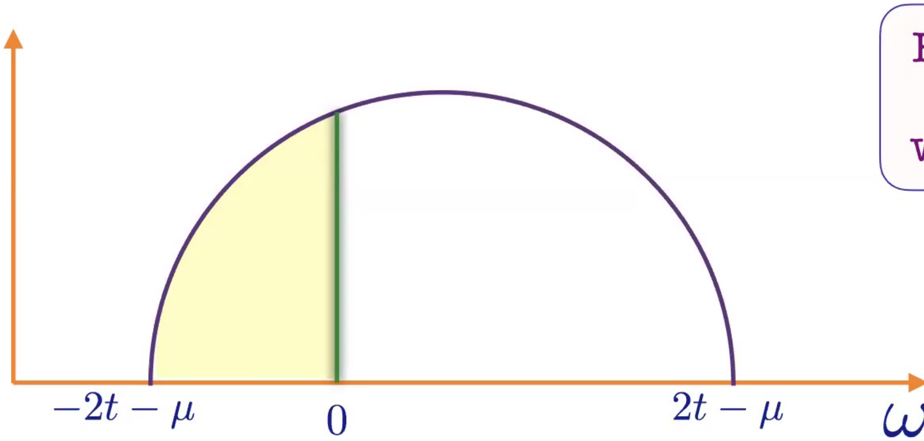
$$G(\tau) \equiv -T_\tau \left\langle c_i(\tau) c_i^\dagger(0) \right\rangle$$

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = t^2 G(\tau)$$

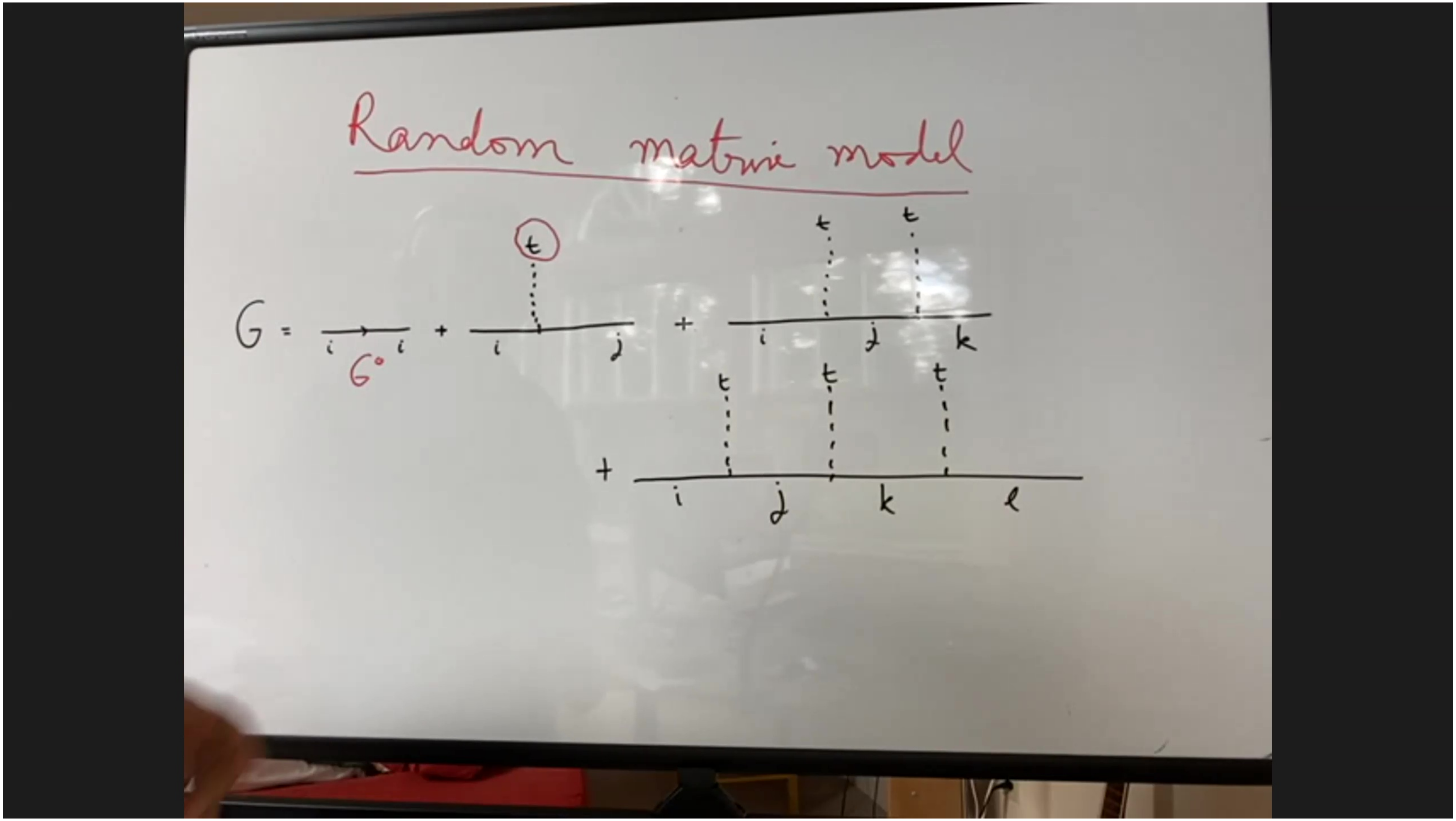
$$G(\tau = 0^-) = \mathcal{Q}.$$

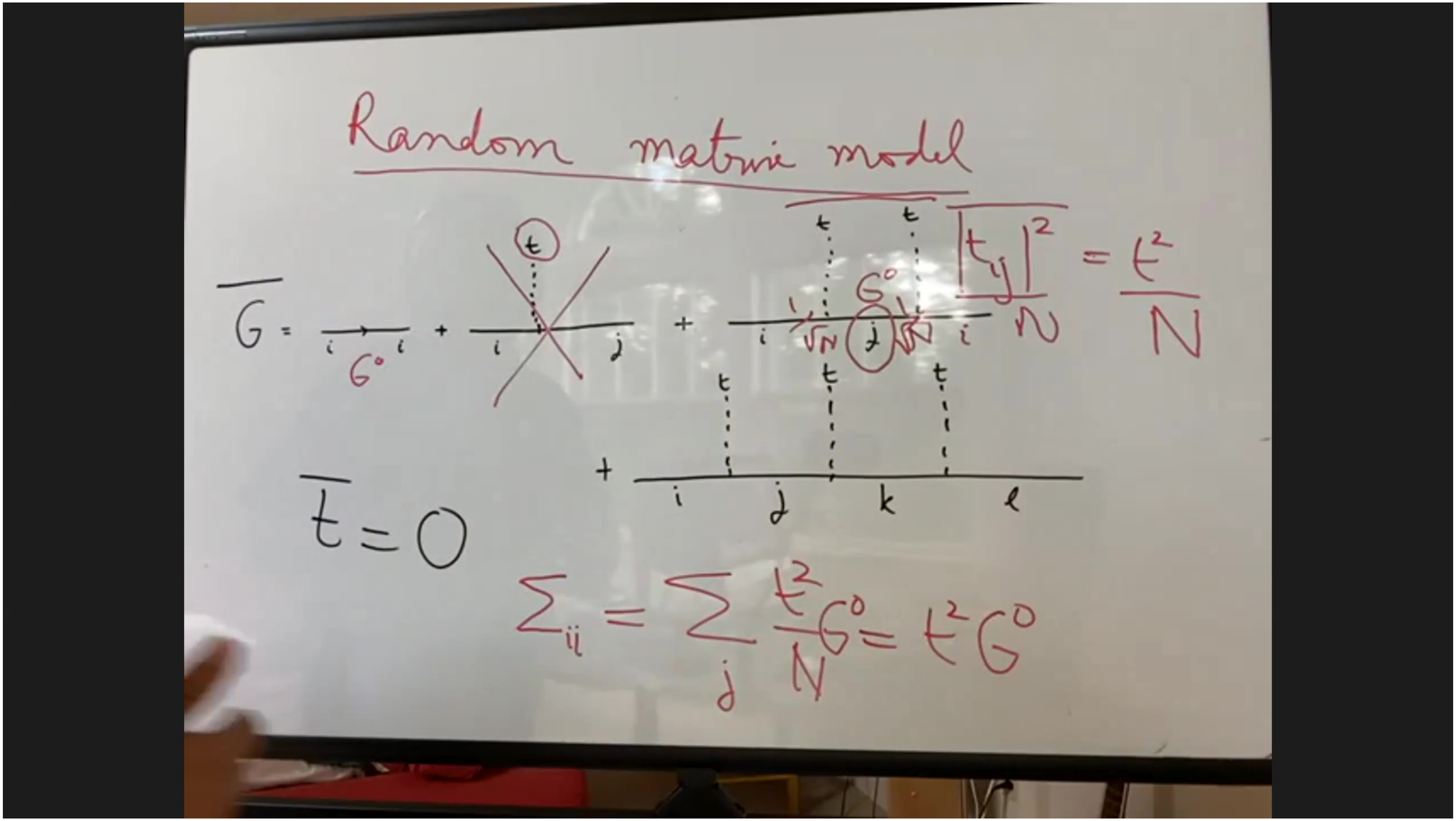
$G(\omega)$ can be determined by solving a quadratic equation: yields $G(\tau) \sim 1/\tau$.

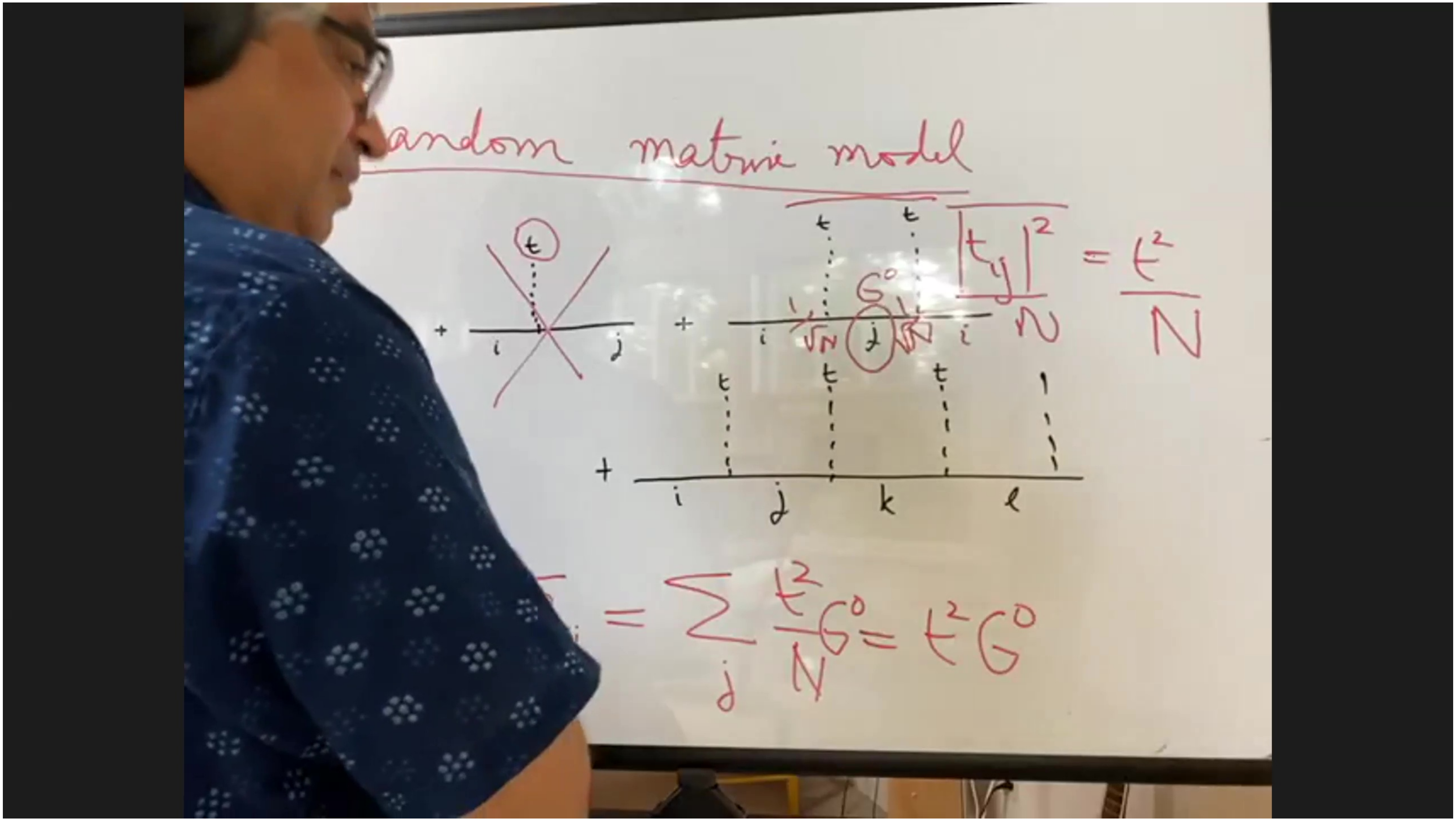
$\rho(\omega) = -\frac{1}{\pi} \text{Im}\, G(\omega)$

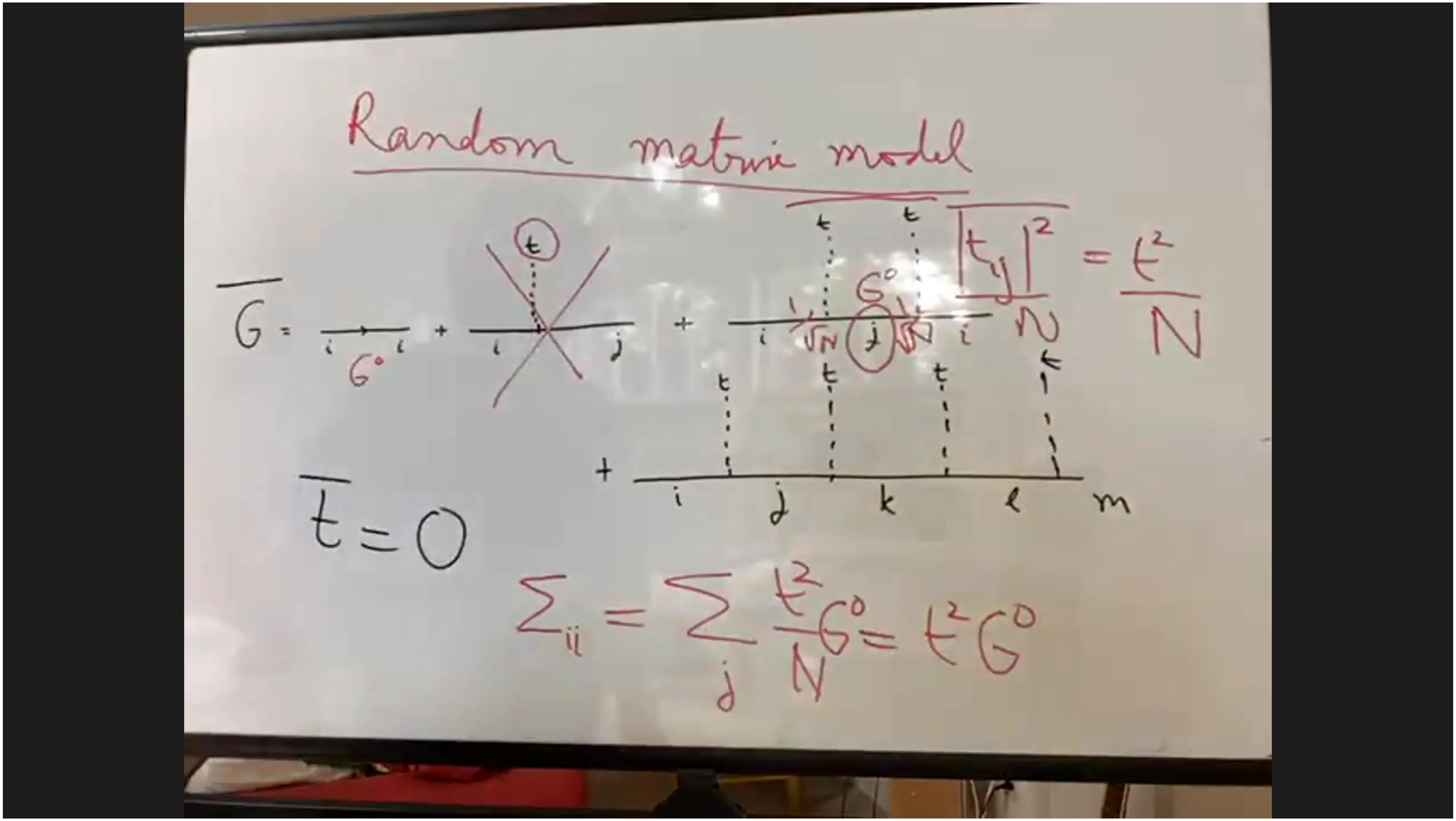


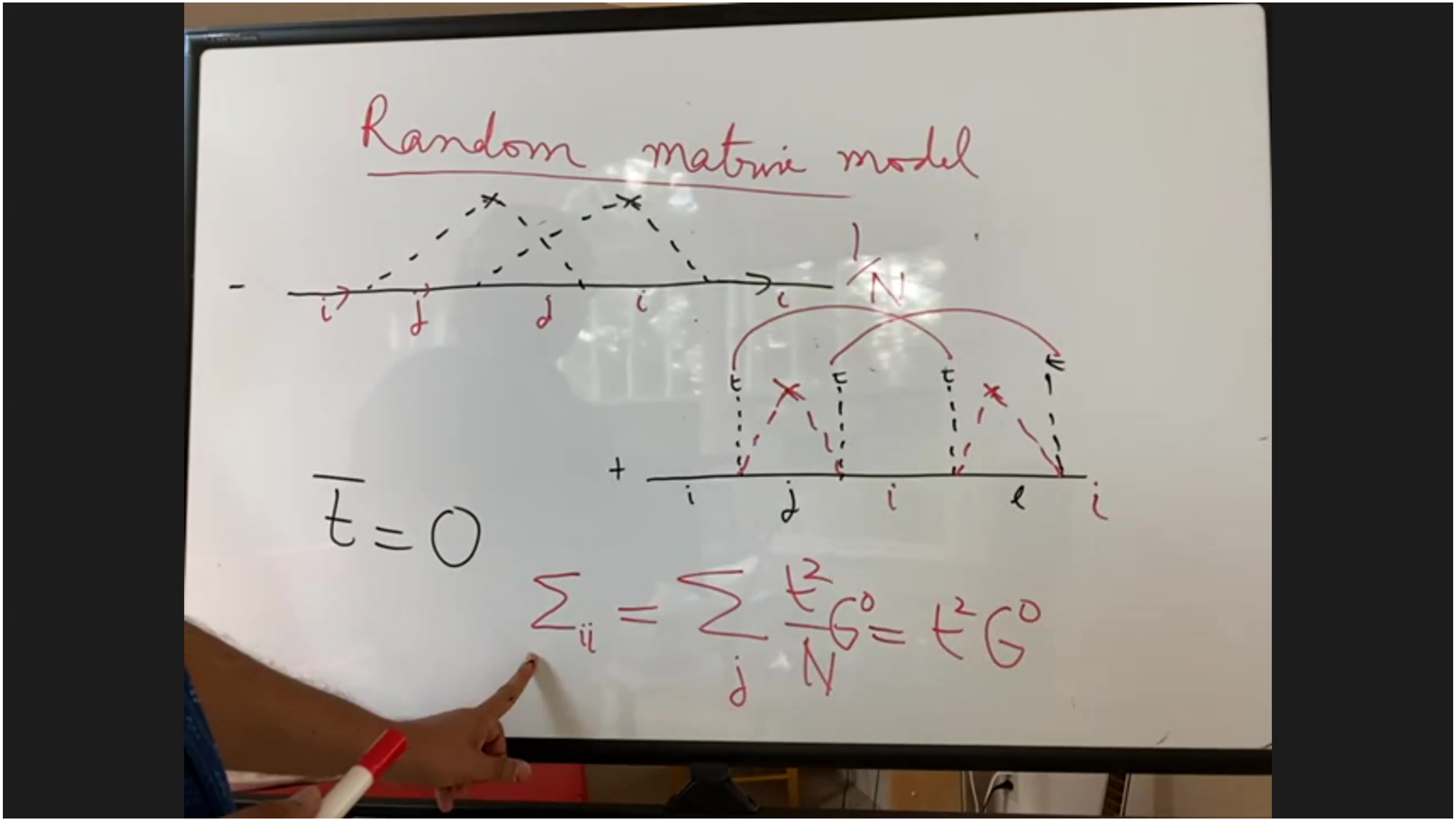

For details: see whiteboard

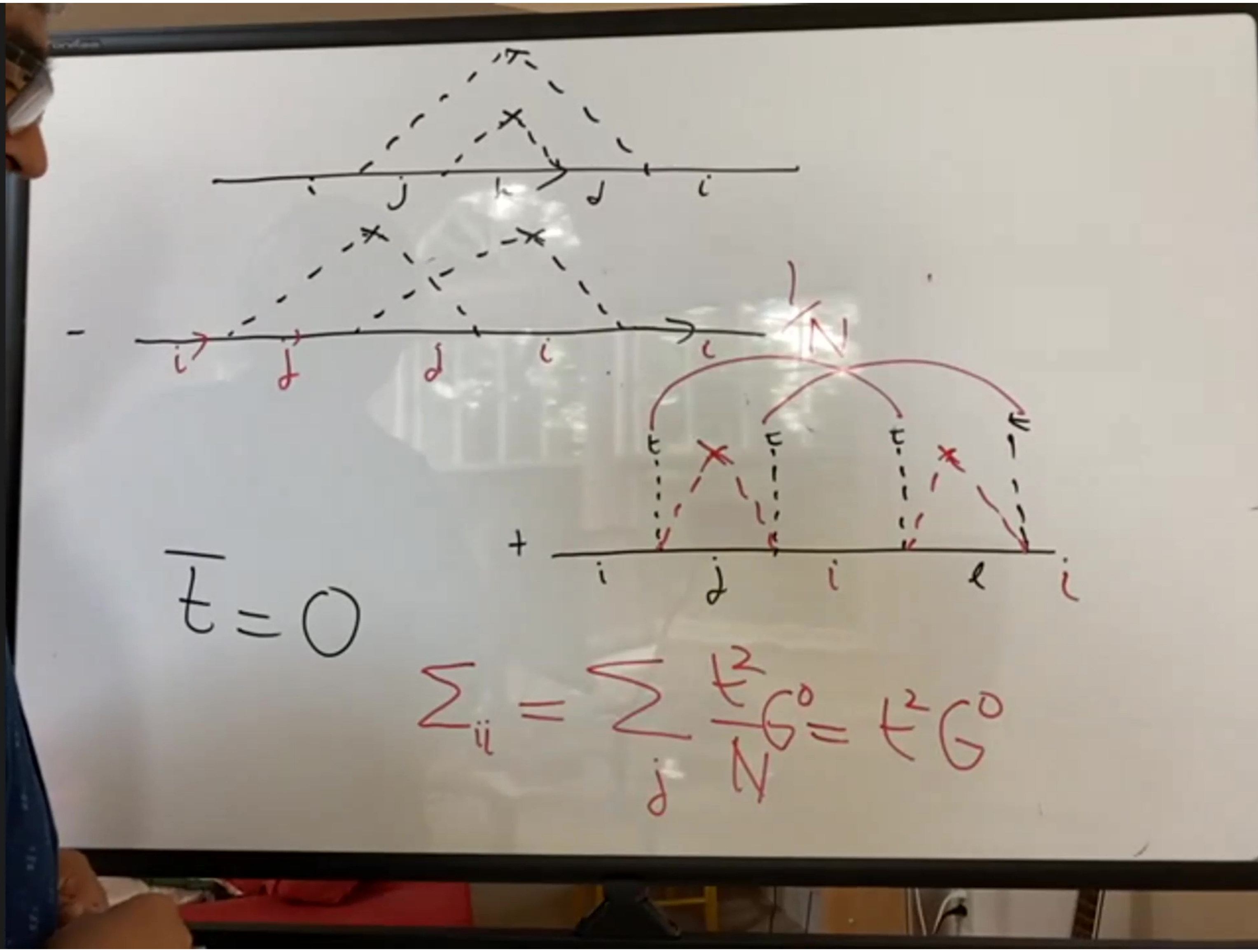

Random matrix model
G =
$G^0$
i
j
k
l
t


Random matrix model
$\overline{|t_{ij}|^2} = \frac{t^2}{N}$
$\overline{t} = 0$
$\Sigma_{ii} = \sum_j \frac{t^2}{N} G^0 = t^2 G^0$


andom matrix model
$\overline{|t_{ij}|^2} = \frac{t^2}{N}$
$= \sum_j \frac{t^2}{N} G^0 = t^2 G^0$


Random matrix model
$\overline{|t_{ij}|^2} = \frac{t^2}{N}$
$\overline{G} =$
$G^0$
$\frac{1}{\sqrt{N}}$
i j k l m
$\overline{t} = 0$
$\Sigma_{ii} = \sum_j \frac{t^2}{N} G^0 = t^2 G^0$


Random matrix model
1/N
t̄ = 0
Σ_ii = Σ_j t²/N G⁰ = t² G⁰


j k j i
i j j i i
1/N
i j i l i
$\overline{t} = 0$
$\Sigma_{ii} = \sum_j \frac{t^2}{N} G^0 = t^2 G^0$

# A simple model of a metal with quasiparticles

Feynman graph expansion in $t_{ij..}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

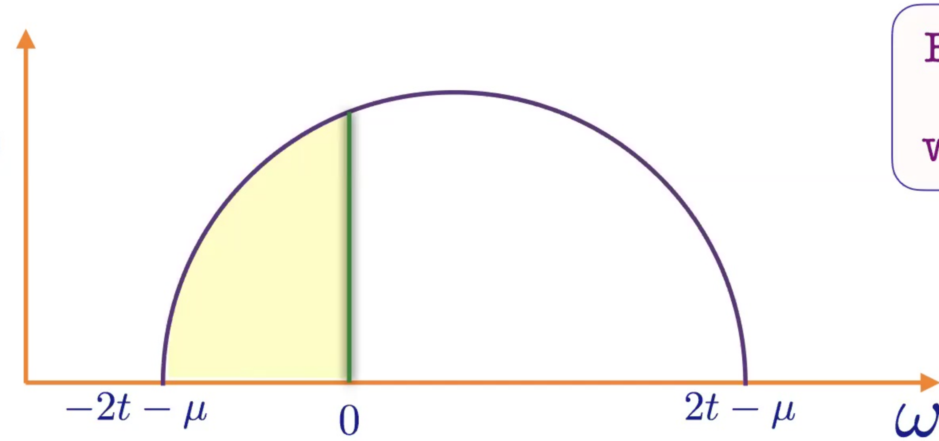
$$G(\tau) \equiv -T_\tau \left\langle c_i(\tau) c_i^\dagger(0) \right\rangle$$

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = t^2 G(\tau)$$

$$G(\tau = 0^-) = \mathcal{Q}.$$

$G(\omega)$ can be determined by solving a quadratic equation: yields $G(\tau) \sim 1/\tau$.

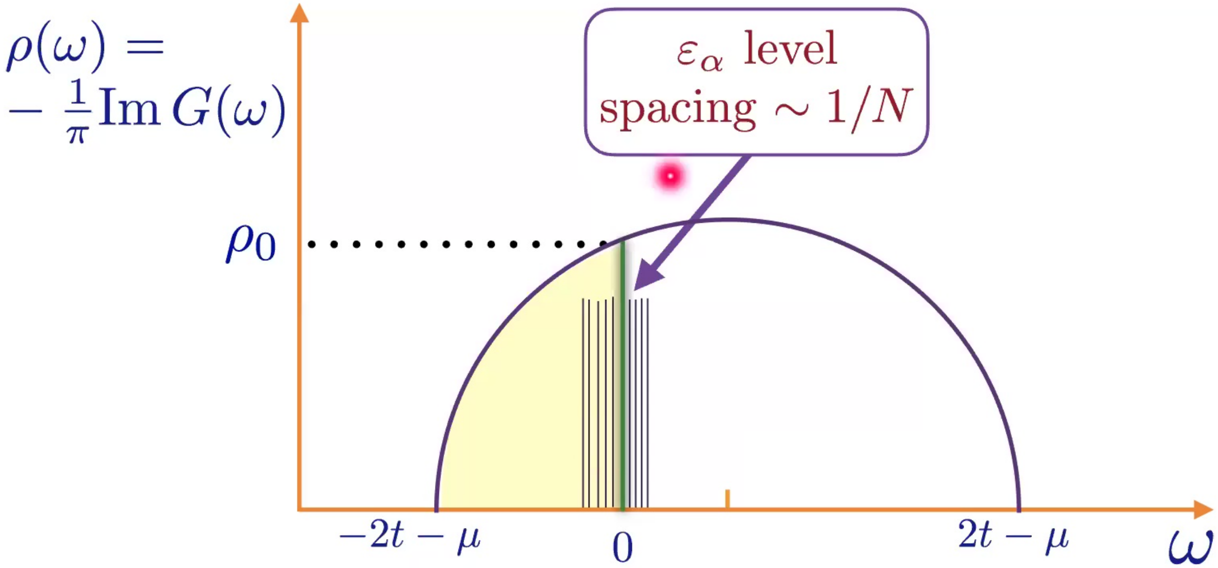
$\rho(\omega) = -\frac{1}{\pi} \text{Im}\, G(\omega)$



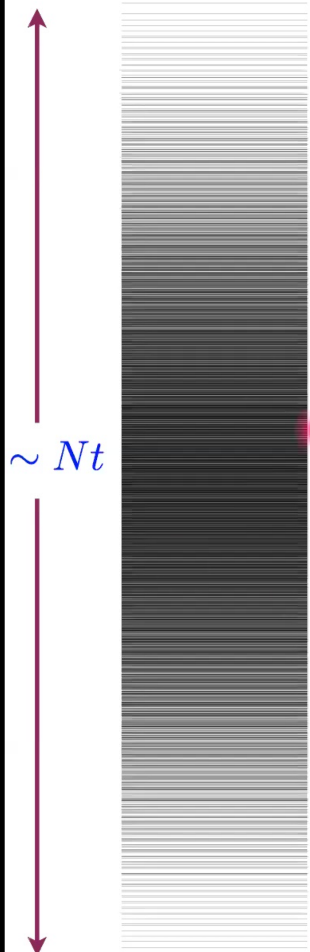

For details: see whiteboard

# A simple model of a metal with quasiparticles

Let $\varepsilon_\alpha$ be the eigenvalues of the matrix $t_{ij}/\sqrt{N}$. The fermions will occupy the lowest $NQ$ eigenvalues, upto the Fermi energy $E_F$. The single-particle density of states is
$\rho(\omega) = (1/N) \sum_\alpha \delta(\omega - \varepsilon_\alpha)$, and $\rho_0 \equiv \rho(\omega = 0)$.




# A simple model of a metal with quasiparticles




Many-body level spacing $\sim 2^{-N}$

Quasiparticle excitations with spacing $\sim 1/N$

There are $2^N$ many body levels with energy

$$E = \sum_{\alpha=1}^{N} n_\alpha \varepsilon_\alpha,$$

where $n_\alpha = 0, 1$. Shown are all values of $E$ for a single cluster of size $N = 12$. The $\varepsilon_\alpha$ have a level spacing $\sim 1/N$.

1. Quantum matter with quasiparticles:
   random matrix model

2. Quantum matter without quasiparticles:
   the complex SYK model

No "Mottness"

3. Random *J* model (insulator)
   *RG analysis and exact exponent*

4. Random Hubbard and *t-J* models
   *Numerical results*

5. Random t-*J* model (metals): *exact exponents*

# The complex SYK model

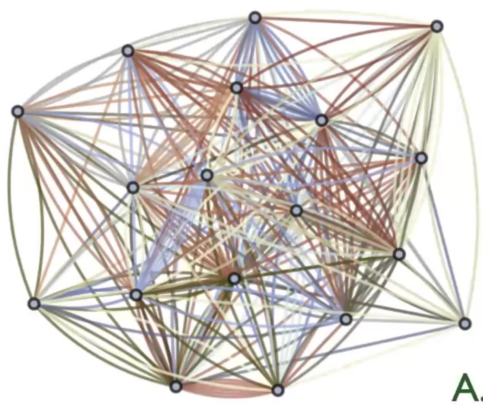
(See also: the "2-Body Random Ensemble" in nuclear physics; did not obtain the large $N$ limit;
T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981))

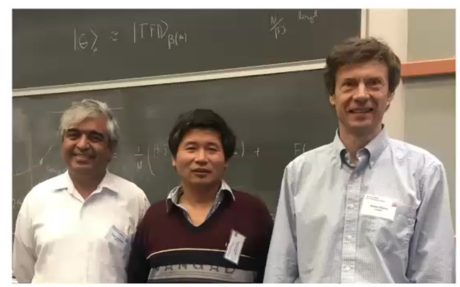
$$H = \frac{1}{(2N)^{3/2}} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}\, c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta - \mu \sum_\alpha c_\alpha^\dagger c_\alpha$$

$$c_\alpha c_\beta + c_\beta c_\alpha = 0 \quad , \quad c_\alpha c_\beta^\dagger + c_\beta^\dagger c_\alpha = \delta_{\alpha\beta}$$

$$\mathcal{Q} = \frac{1}{N} \sum_\alpha c_\alpha^\dagger c_\alpha$$

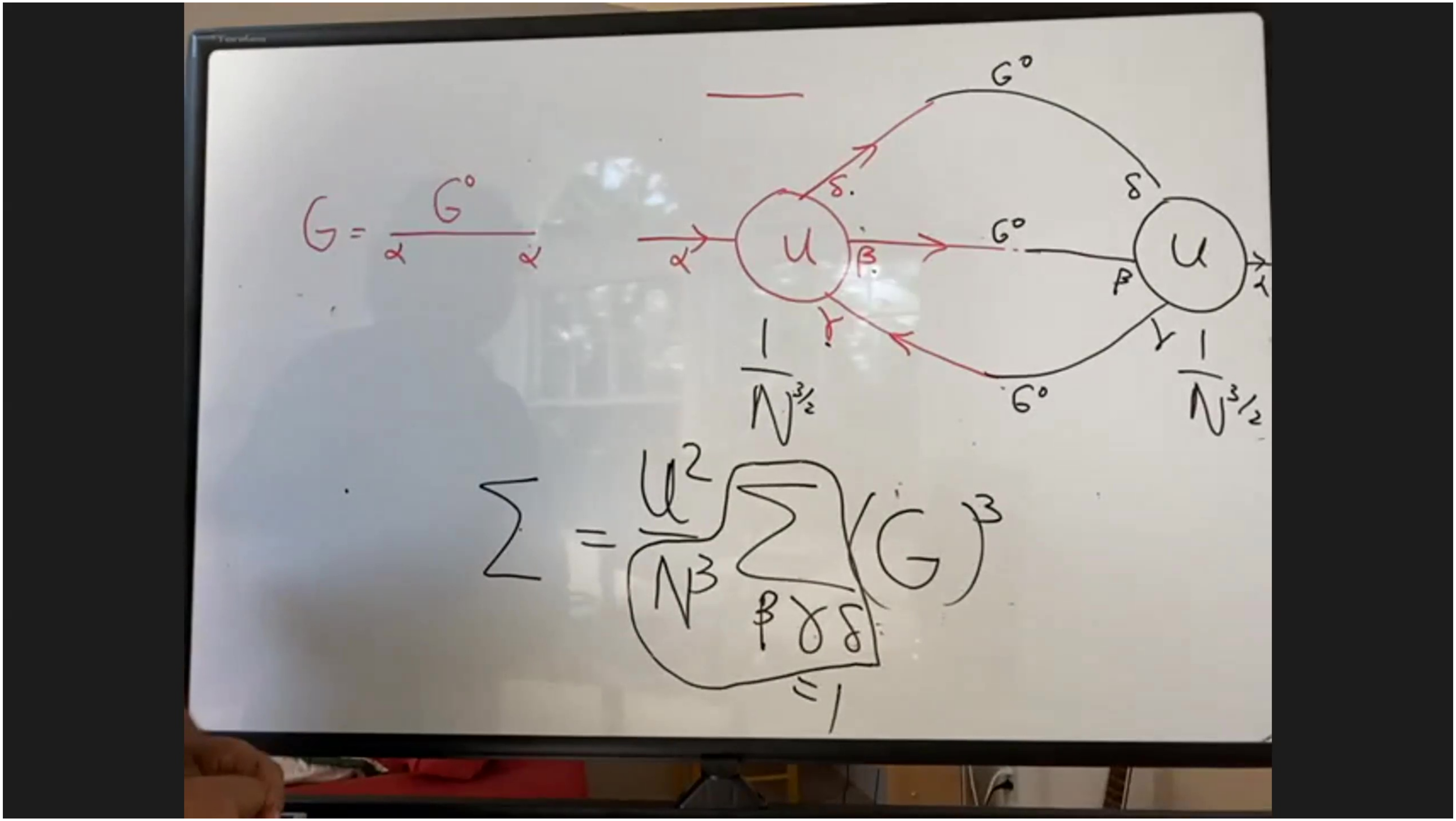
$U_{\alpha\beta;\gamma\delta}$ are independent random variables with $\overline{U_{\alpha\beta;\gamma\delta}} = 0$ and $\overline{|U_{\alpha\beta;\gamma\delta}|^2} = U^2$
$N \to \infty$ yields critical strange metal.





S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

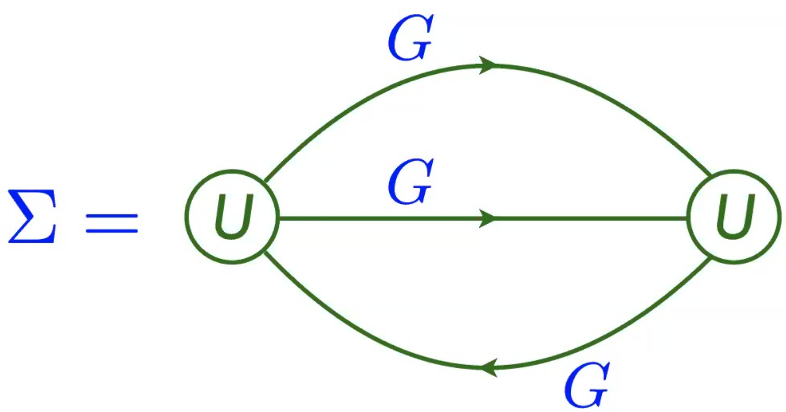
A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)


G = G°
α α
α
U
U
β
γ
δ
G°
G°
G°
1/N^{3/2}
1/N^{3/2}
Σ = U²/N³ Σ_{βγδ} (G)³
= 1

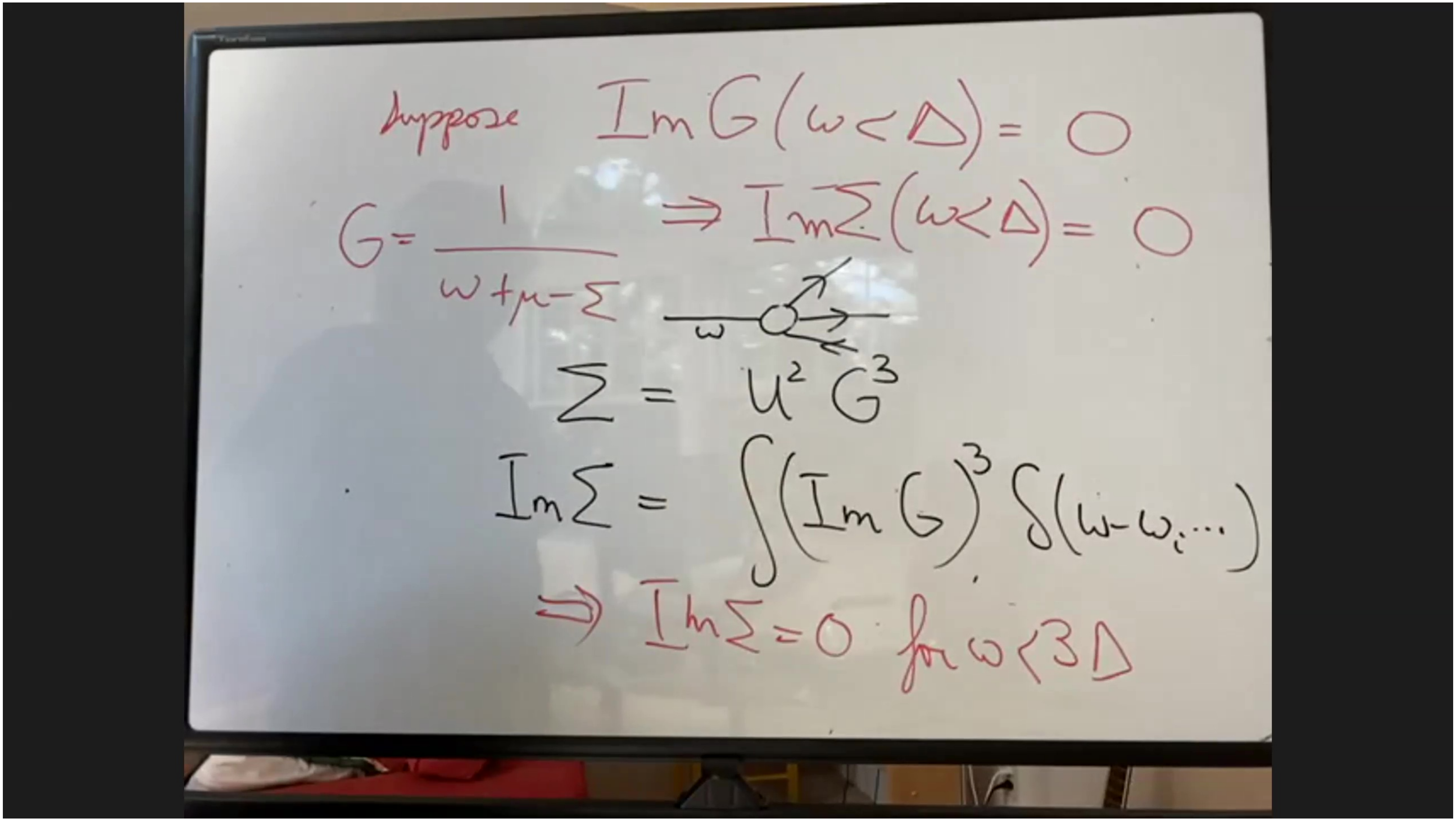
# The complex SYK model

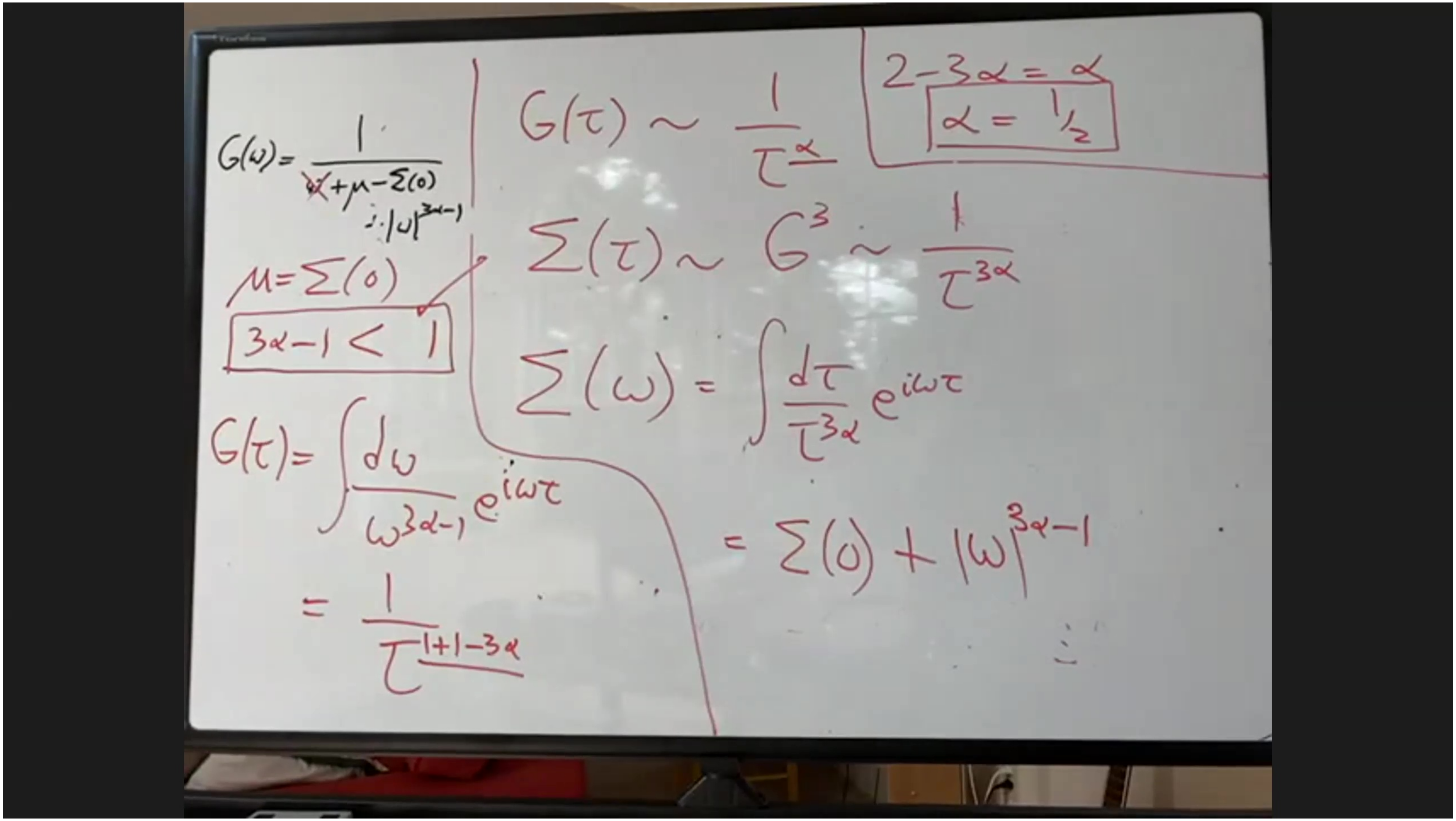
Feynman graph expansion in $U_{\alpha\beta;\gamma\delta}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$
$$G(\tau = 0^-) = \mathcal{Q}.$$




S. Sachdev and J. Ye, PRL **70**, 3339 (1993)


Suppose Im G(ω<Δ) = 0
G = 1/(ω+μ−Σ) ⇒ Im Σ(ω<Δ) = 0
ω
Σ = U² G³
Im Σ = ∫(Im G)³ δ(ω−ωᵢ⋯)
⇒ Im Σ = 0 for ω<3Δ



G(ω) = 1 / (ω + μ − Σ(0) − |ω|^{3α−1})
μ = Σ(0)
3α − 1 < 1
G(τ) = ∫ dω / ω^{3α−1} e^{iωτ}
= 1 / τ^{1+1−3α}
G(τ) ~ 1 / τ^α
2 − 3α = α
α = 1/2
Σ(τ) ~ G³ ~ 1 / τ^{3α}
Σ(ω) = ∫ dτ / τ^{3α} e^{iωτ}
= Σ(0) + |ω|^{3α−1}

# The complex SYK model

The large $N$ limit is given by the sum of "melon" Feynman graphs

For long times $\tau > 0$

$$\langle c_\alpha(\tau) c_\alpha^\dagger(0) \rangle = \frac{A}{\sqrt{\tau}}$$

$$\langle c_\alpha^\dagger(\tau) c_\alpha(0) \rangle = e^{-2\pi\mathcal{E}} \frac{A}{\sqrt{\tau}}$$

The parameter $\mathcal{E} = \mathbb{C}\,(\epsilon/U)$ determines the particle-hole asymmetry, and has a universal "Luttinger" relation to $\mathcal{Q}$.

For details: see whiteboard and Sarosi, Sec. 4.4

In a Fermi liquid,

$$\langle c_\alpha(\tau) c_\alpha^\dagger(0) \rangle = \langle c_\alpha^\dagger(\tau) c_\alpha(0) \rangle = \widetilde{A}/\tau$$

# The complex SYK model

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

$$\Sigma(z) = \mu - \frac{1}{A}\sqrt{z} + \ldots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

At frequencies $\ll U$, the $i\omega + \mu$ can be dropped, and without it equations are invariant under the reparametrization and gauge transformations. The singular part of the self-energy and the Green's function obey

$$\int_0^\beta d\tau_2 \, \Sigma_{\text{sing}}(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma_{\text{sing}}(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1)$$

A. Kitaev, 2015

S. Sachdev, PRX **5,** 041025 (2015)

# The complex SYK model

$$\int_0^\beta d\tau_2\, \Sigma_{(}\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1)$$

These equations are invariant under

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} \widetilde{G}(\sigma_1, \sigma_2)$$

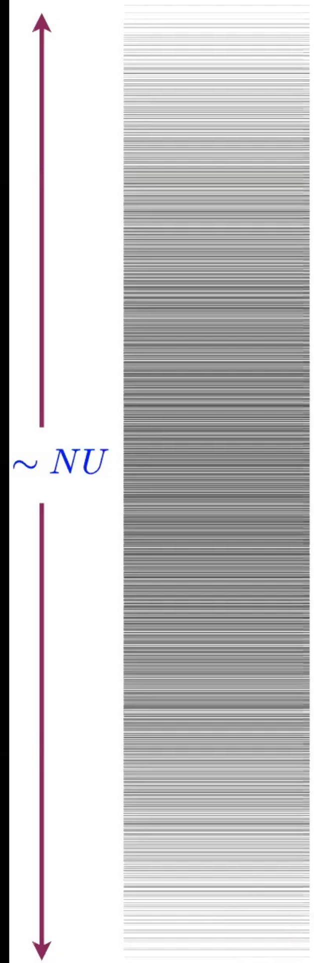
$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \widetilde{\Sigma}(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.
By using $f(\sigma) = \tan(\pi T \sigma)/(\pi T)$ and
$g(\sigma) = e^{-2\pi \mathcal{E} T \sigma}$, we can now obtain
the $T > 0$ solution from the $T = 0$ solution.

A. Kitaev, 2015

S. Sachdev, PRX **5,** 041025 (2015)

# The complex SYK model




Many-body level spacing $\sim 2^{-N} = e^{-N \ln 2}$

Non-quasiparticle excitations with spacing $\sim e^{-N s_0}$

W. Fu and S. Sachdev, PRB **94**, 035135 (2016)

There are $2^N$ many body levels with energy $E$. Shown are all values of $E$ for a single cluster of size $N = 12$. The $T \to 0$ state has an entropy $S_{GPS} = N s_0$, where $s_0 < \ln 2$ is determined by integrating

$$\frac{ds_0}{d\mathcal{Q}} = 2\pi\mathcal{E}\,.$$

At $\mathcal{Q} = 1/2$,

$$s_0 \;=\; \frac{G}{\pi} + \frac{\ln(2)}{4} = 0.464848\ldots$$

where $G$ is Catalan's constant.

GPS: A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

# SYK criticality

**Key properties**

1. There is a quantum critical state, without quasiparticle excitations, for a range of charge densities around $\mathcal{Q} = 1/2$.

2. There is a non-zero extensive entropy as $T \to 0$

$$\lim_{T \to 0} \lim_{N \to \infty} \frac{S}{N} = \mathcal{S}_0(\mathcal{Q}) \neq 0$$

   This entropy is *not* due to an exponentially large ground degeneracy. Instead, it reflects an exponentially small many-body level spacing $\sim e^{-N\mathcal{S}_0}$ down to the ground state.

3. Thermal equilibration in a 'Planckian time' $\sim \hbar/(k_B T)$

# SYK criticality

**Key properties**

4. The leading low temperature behavior of many observables is controlled by a time reparameterization soft mode. A Schwarzian action for this soft mode is implied by an emergent SL(2,R) symmetry. Specifically, the entropy is $S(T)/N = \mathcal{S}_0(\mathcal{Q}) + \gamma T$, where $\gamma$ propotional to the co-efficient of the Schwarzian.

5. Maximal quantum Lyapunov exponent for the out-of-time-order correlator (OTOC):

$$\left\langle c_a^\dagger(t) c_b(0) c_a(t) c_b^\dagger(0) \right\rangle = C_0 + C_1 \left( \frac{e^{\lambda t}}{N} \right) + \ldots$$

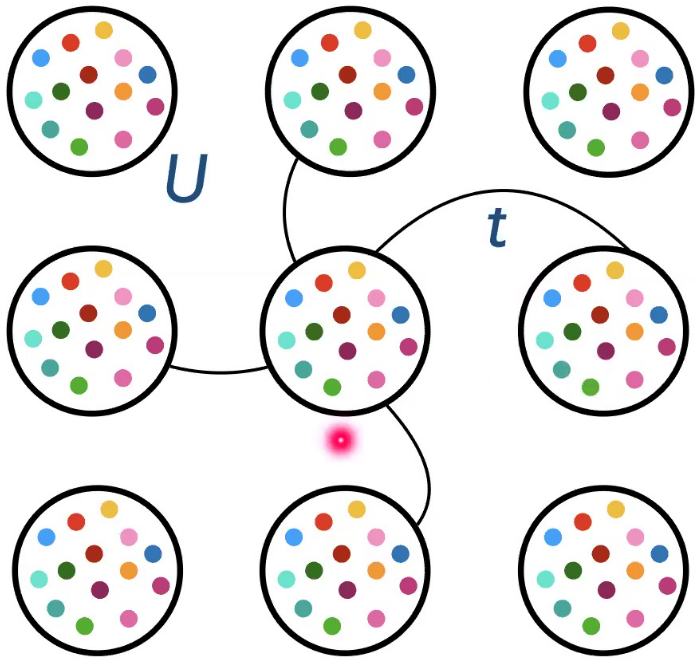
with $\lambda = 2\pi k_B T/\hbar$.

6. For spinful fermions, spin correlations decay as

$$\left\langle \vec{S}(\tau) \cdot \vec{S}(0) \right\rangle \sim 1/|\tau|$$

# A lattice SYK model

$$H = \frac{1}{(2N)^{3/2}} \sum_i \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}\, c^\dagger_{i\alpha} c^\dagger_{i\beta} c_{i\gamma} c_{i\delta} - t \sum_{\langle ij \rangle} \sum_\alpha c^\dagger_{i\alpha} c_{j\alpha}$$



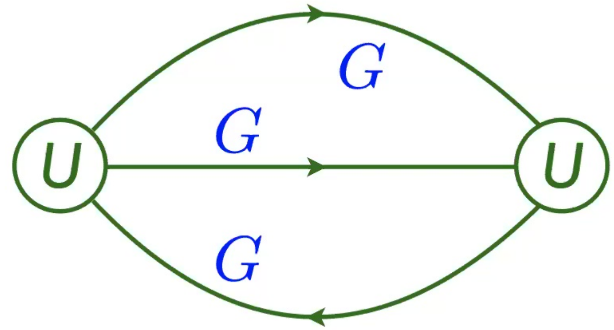

Xue-Yang Song, Chao-Ming Jian, and L. Balents, PRL **119**, 216601 (2017); Pengfei Zhang, PRB **96**, 205138 (2017); Debanjan Chowdhury, Yochai Werman, Erez Berg, T. Senthil, PRX **8**, 031024 (2018); Aavishkar A. Patel, John McGreevy, Daniel P. Arovas, Subir Sachdev, PRX **8**, 021049 (2018)

See also Antoine Georges and Olivier Parcollet PRB **59**, 5341 (1999)

# A lattice SYK model

The large $N$ limit is still given by the sum of "melon" diagrams.

$$G(k, i\omega) = \frac{1}{i\omega - \epsilon_k - \Sigma(k, i\omega)}$$

$\Sigma =$ 


For $t^2/U \ll k_B T \ll U$ we obtain an 'incoherent metal' with no Fermi surface or quasiparticles with

$$G(\mathbf{k}, \omega) = G_{\rm SYK}(\epsilon, \hbar\omega/(k_B T))$$

independent of $\mathbf{k}$. In this regime, there is a linear-in-$T$ resistivity but only with bad metal behavior with $\rho > h/e^2$, and co-efficient dependent upon $U$:

$$\rho \sim \frac{h}{e^2} \frac{k_B T}{t^2/U}$$

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# Random J model (insulator)

$$H = \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij}\, \vec{S}_i \cdot \vec{S}_j$$

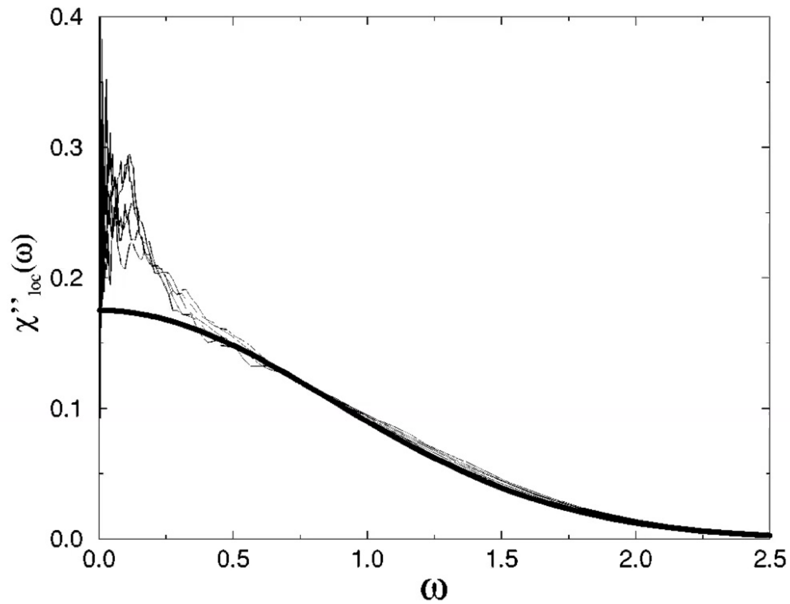
Numerical studies for SU(2) spin-1/2 show spin-glass order!

Spin-glass order parameter:

$$q = \lim_{t\to\infty} \overline{\left\langle \vec{S}_i(t)\vec{S}_i(0) \right\rangle}.$$

Exact diagonalization results for $\chi''_{\rm loc}(\omega)$
Analytic continuation of

$$\chi_{\rm loc}(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n\tau}\, \overline{\left\langle \vec{S}_i(\tau)\vec{S}_i(0) \right\rangle}$$




L. Arrachea and M. J. Rozenberg, PRB **65**, 224430 (2002)

# Random *J* model (insulator)

$$H = \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij}\, \vec{S}_i \cdot \vec{S}_j$$

Numerical studies for SU(2) spin-1/2 show spin-glass order!

Spin-glass order parameter:

$$q = \lim_{t\to\infty} \overline{\left\langle \vec{S}_i(t)\vec{S}_i(0)\right\rangle}.$$

Exact diagonalization results for $\chi''_{\mathrm{loc}}(\omega)$
Analytic continuation of

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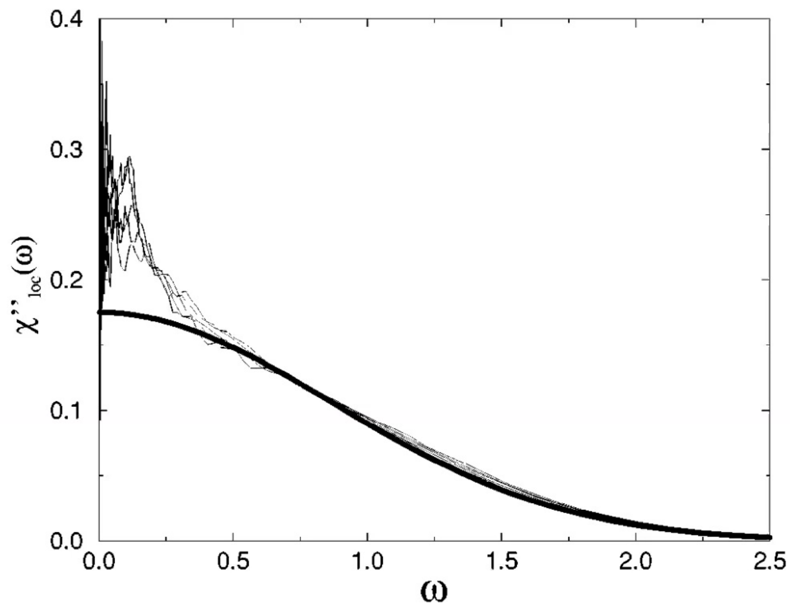


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$$\alpha = \uparrow, \downarrow, \quad \vec{S}_i = \frac{1}{2} f_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} f_{i\beta}, \quad \sum_\alpha f_{i\alpha}^\dagger f_{i\alpha} = 1$$

$$J_{ij} \text{ random, } \overline{J_{ij}} = 0, \ \overline{J_{ij}^2} = J^2$$

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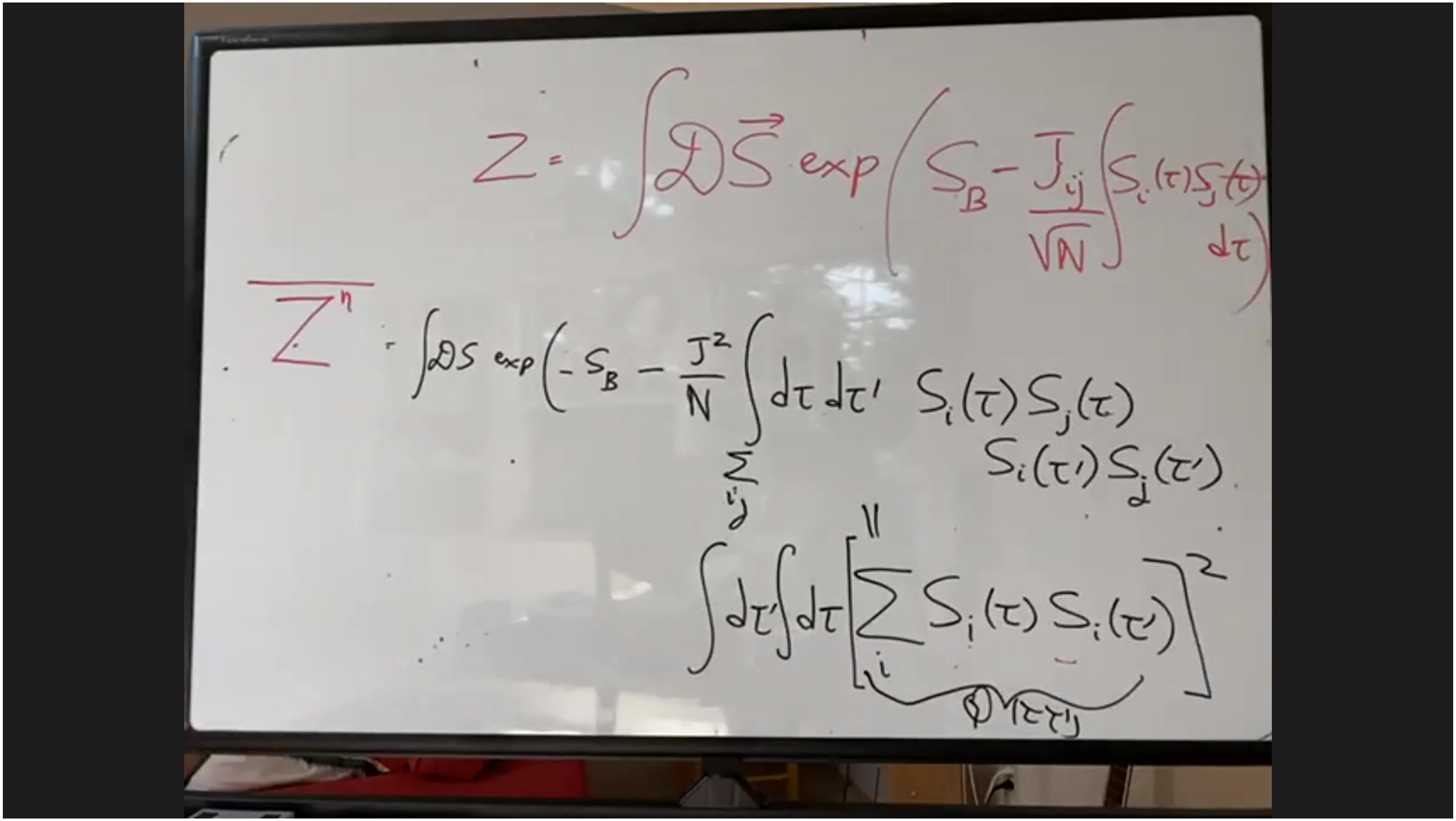

$Z = \int \mathcal{D}\vec{S} \exp\left( S_B - \frac{J_{ij}}{\sqrt{N}} \int S_i(\tau) S_j(\tau) d\tau \right)$
$\overline{Z^n} = \int \mathcal{D}S \exp\left( - S_B - \frac{J^2}{N} \sum_{ij} \int d\tau d\tau' \, S_i(\tau) S_j(\tau) S_i(\tau') S_j(\tau') \right)$
$= \int d\tau' \int d\tau \left[ \sum_i S_i(\tau) S_i(\tau') \right]^2$
$Q(\tau\tau')$

# Random *J* model (insulator)

$$\mathcal{Z} = \int \mathcal{D}\vec{S}(\tau)\delta(\vec{S}^2 - 1)e^{-\mathcal{S}_B - \mathcal{S}_J}$$

$$\mathcal{S}_B = \frac{i}{2}\int_0^1 du \int d\tau \vec{S}\cdot\left(\frac{\partial \vec{S}}{\partial \tau}\times\frac{\partial \vec{S}}{\partial u}\right)$$

$$\mathcal{S}_J = -\frac{J^2}{2}\int d\tau d\tau' Q(\tau-\tau')\vec{S}(\tau)\cdot\vec{S}(\tau')\,.$$

From this action we compute

$$\overline{Q}(\tau-\tau') = \frac{1}{3}\left\langle \vec{S}(\tau)\cdot\vec{S}(\tau')\right\rangle_{\mathcal{Z}}$$

and then impose the self-consistency condition

$$Q(\tau) = \overline{Q}(\tau).$$

S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

# Random $J$ model (insulator): large $M$

Express the spin operator in terms of fermions $\vec{S} = (1/2) f_\alpha^\dagger \vec{\sigma}_{\alpha\beta} f_\beta$, and let $\alpha = 1 \dots M$. The fermions obey the constraint

$$\sum_{\alpha=1}^{M} f_\alpha^\dagger f_\alpha = \frac{M}{2}$$

In the large $M$ limit we obtain for the fermion Green's function $G$ and self energy $\Sigma$ (same as the SYK equations)

$$G(i\omega) = \frac{1}{i\omega - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -J^2 G^2(\tau) G(-\tau)$$

The solution is

$$G(\tau) \sim \frac{\text{sgn}(\tau)}{\sqrt{\tau}} \quad , \quad \left\langle \vec{S}(\tau) \cdot \vec{S}(0) \right\rangle \sim \frac{1}{|\tau|}$$

S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

# Random J model (insulator):RG

We assume a power-law decay

$$Q(\tau) \sim \frac{\gamma^2}{|\tau|^\alpha}\,.$$

Ignore the self-consistency condition for now. We decouple the $\vec{S}(\tau)\cdot\vec{S}(0)$ interaction by introducing a bosonic ($\phi_a$, $a = 1\ldots 3$) bath. Then the problem reduces to the Hamiltonian

$$H_{\rm imp} \quad = \quad \gamma S_a\, \phi_a(0) + \frac{1}{2}\int d^dx \left[\pi_a^2 + (\partial_x \phi_a)^2\right]$$

where $\pi_a$ is canonically conjugate to the field $\phi_a$, and $\phi_a(0) \equiv \phi_a(x=0)$. We identify $Q(\tau)$ with temporal correlator of $\phi_a(0)$, and then we need $\alpha = d - 1$.

M. Vojta, C. Buragohain, and S. Sachdev, PRB **61**, 15152 (2000)
S. Sachdev, Physica C **357**, 78 (2001)

# Random J model (insulator):RG

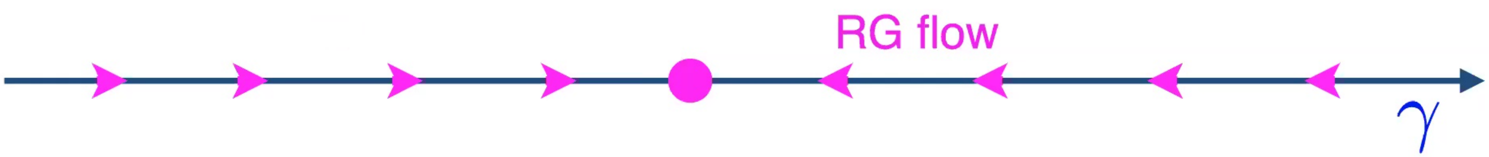
- The $\beta$-function of $\gamma$ can by computed order-by-order in $\epsilon = 2 - \alpha$

$$\frac{d\gamma}{d\ell} = \epsilon \frac{\gamma}{2} - \gamma^3 \,.$$

For details: see whiteboard and Whitsitt, Sec. III

- There is an attractive fixed point at $\gamma = \gamma^* = \mathcal{O}(\sqrt{\epsilon})$.

- Because of the quantized Berry phase (Wess-Zumion-Witten) term, the renormalization of the coupling $\gamma$ is given only by the wavefunction renormalization. We can then prove that at this fixed point $\overline{Q}(\tau) \sim 1/|\tau|^{2-\alpha}$ to all orders in $\epsilon$.




M. Vojta, C. Buragohain, and S. Sachdev, PRB **61**, 15152 (2000)
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# Random J model (insulator)

$$
\begin{aligned}
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\mathcal{S}_B &= \frac{i}{2}\int_0^1 du \int d\tau \vec{S} \cdot \left( \frac{\partial \vec{S}}{\partial \tau} \times \frac{\partial \vec{S}}{\partial u} \right) \\
\mathcal{S}_J &= -\frac{J^2}{2}\int d\tau d\tau' Q(\tau - \tau')\vec{S}(\tau) \cdot \vec{S}(\tau') \,.
\end{aligned}
$$

From this action we compute

$$
\overline{Q}(\tau - \tau') = \frac{1}{3}\left\langle \vec{S}(\tau) \cdot \vec{S}(\tau') \right\rangle_{\mathcal{Z}}
$$

and then impose the self-consistency condition

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Q(\tau) = \overline{Q}(\tau).
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- The self-consistency condition therefore yields

$$\left\langle \vec{S}(\tau) \cdot \vec{S}(0) \right\rangle \sim \frac{1}{|\tau|} .$$

to all orders in $\epsilon$.

M. Vojta, C. Buragohain, and S. Sachdev, PRB **61**, 15152 (2000)
S. Sachdev, Physica C **357**, 78 (2001)

1. Quantum matter with quasiparticles:
   random matrix model

2. Quantum matter without quasiparticles:
   the complex SYK model

No "Mottness"

3. Random *J* model (insulator)
   *RG analysis and exact exponent*

4. Random Hubbard and *t-J* models
   *Numerical results*

5. Random t-*J* model (metals): *exact exponents*

# Random $t$-$J$-$U_H$ model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij}\, c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij}\, \vec{S}_i \cdot \vec{S}_j + U_H \sum_{i=1}^{N} n_{i\uparrow} n_{i\downarrow}$$

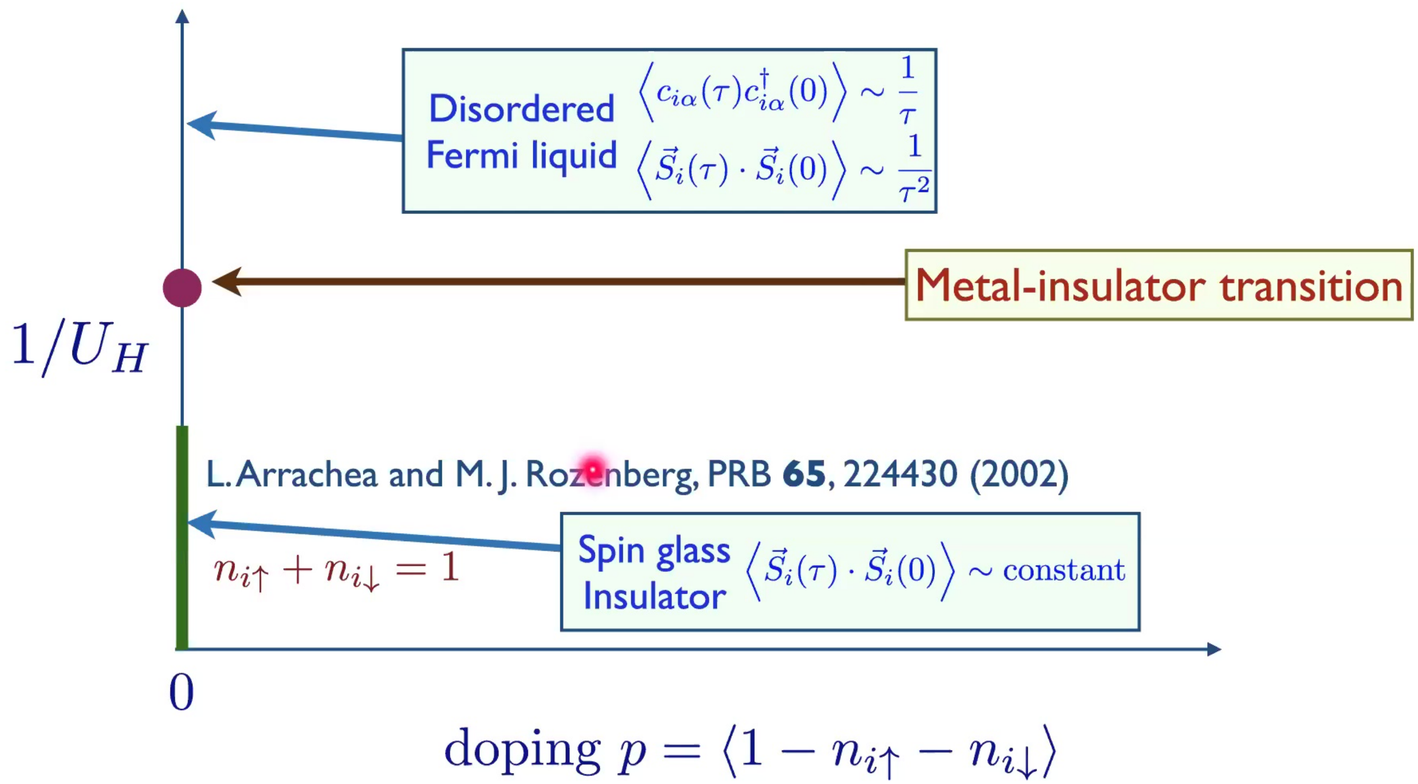
$$\alpha = \uparrow, \downarrow, \quad \vec{S}_i = \frac{1}{2} c_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{i\beta}, \quad n_{i\alpha} = c_{i\alpha}^\dagger c_{i\alpha},$$

$$J_{ij} \text{ random}, \ \overline{J_{ij}} = 0, \ \overline{J_{ij}^2} = J^2$$
$$t_{ij} \text{ random}, \ \overline{t_{ij}} = 0, \ \overline{t_{ij}^2} = t^2$$

$$U_H > 0 \text{ non-random}$$

# Random $t$-$J$-$U_H$ model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij}\, c_{i\alpha}^{\dagger} c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij}\, \vec{S}_i \cdot \vec{S}_j + U_H \sum_{i=1}^{N} n_{i\uparrow} n_{i\downarrow}$$

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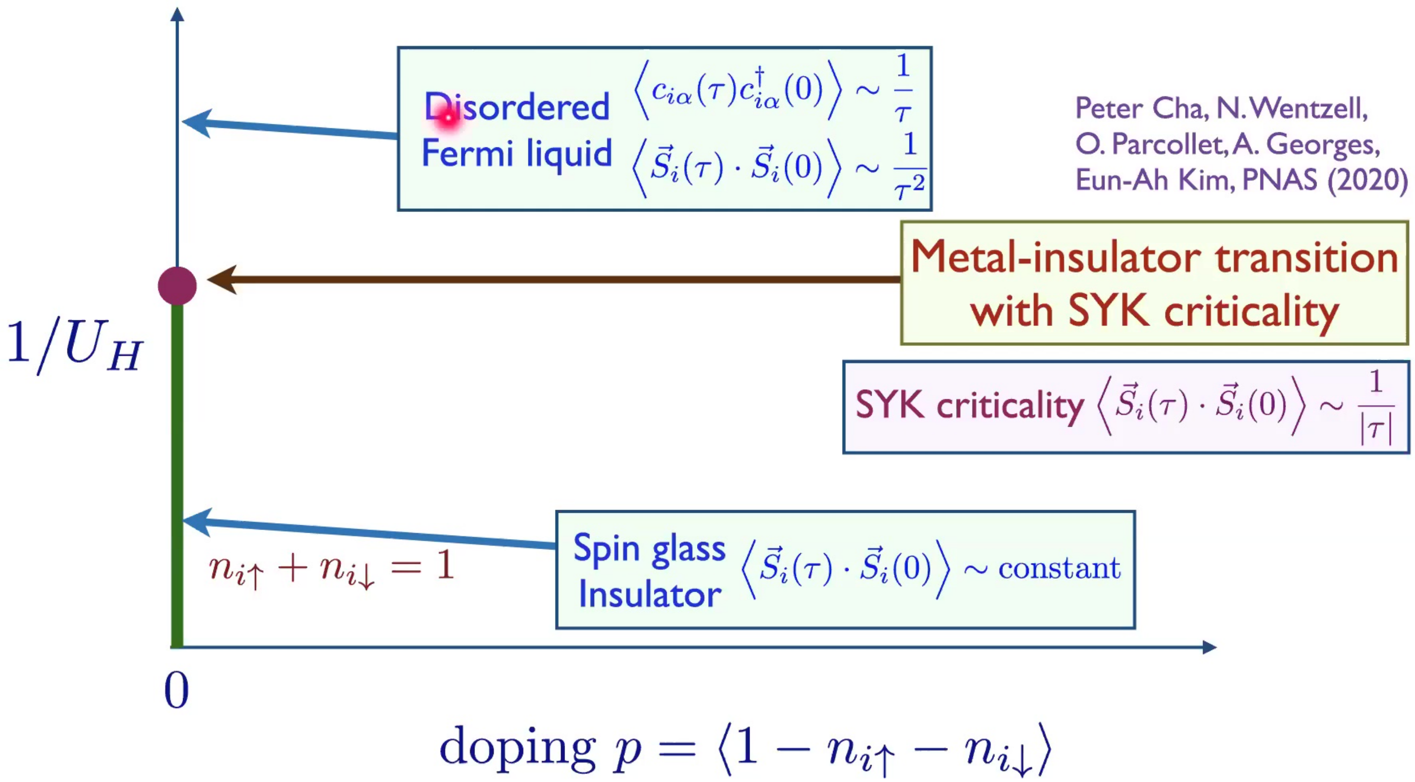
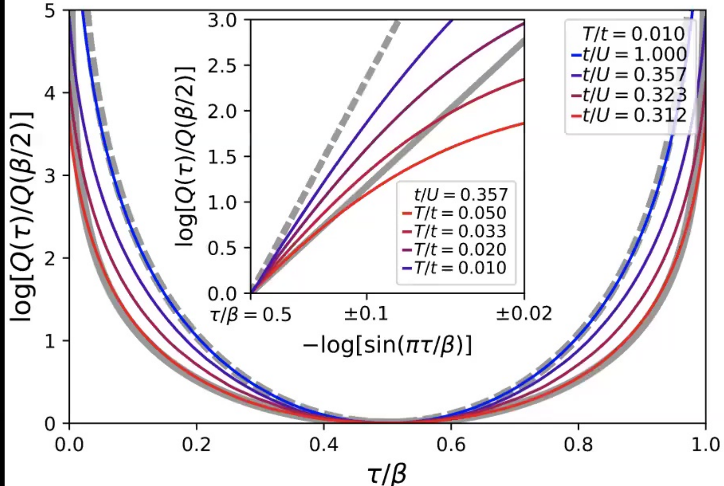
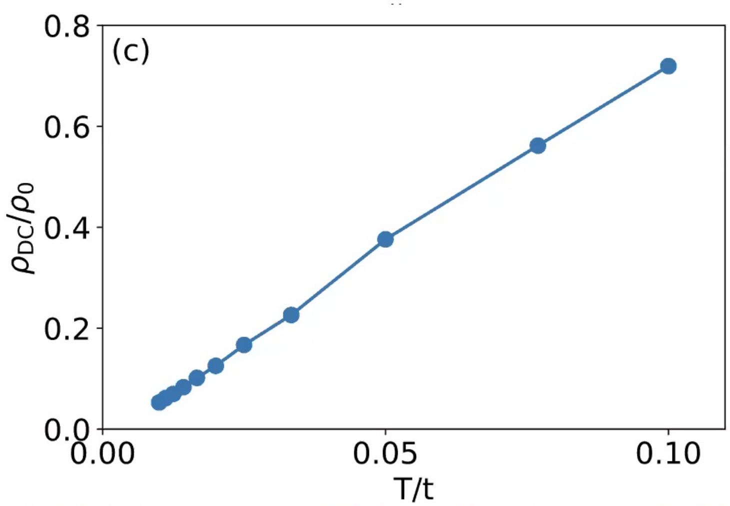



Peter Cha, N. Wentzell, O. Parcollet, A. Georges, Eun-Ah Kim, PNAS (2020)

# Linear resistivity and Sachdev-Ye-Kitaev (SYK) spin liquid behavior in a quantum critical metal with spin-1/2 fermions




Critical spin correlations:
$$\left\langle \vec{S}(\tau) \cdot \vec{S}(0) \right\rangle \sim \frac{1}{|\tau|}$$

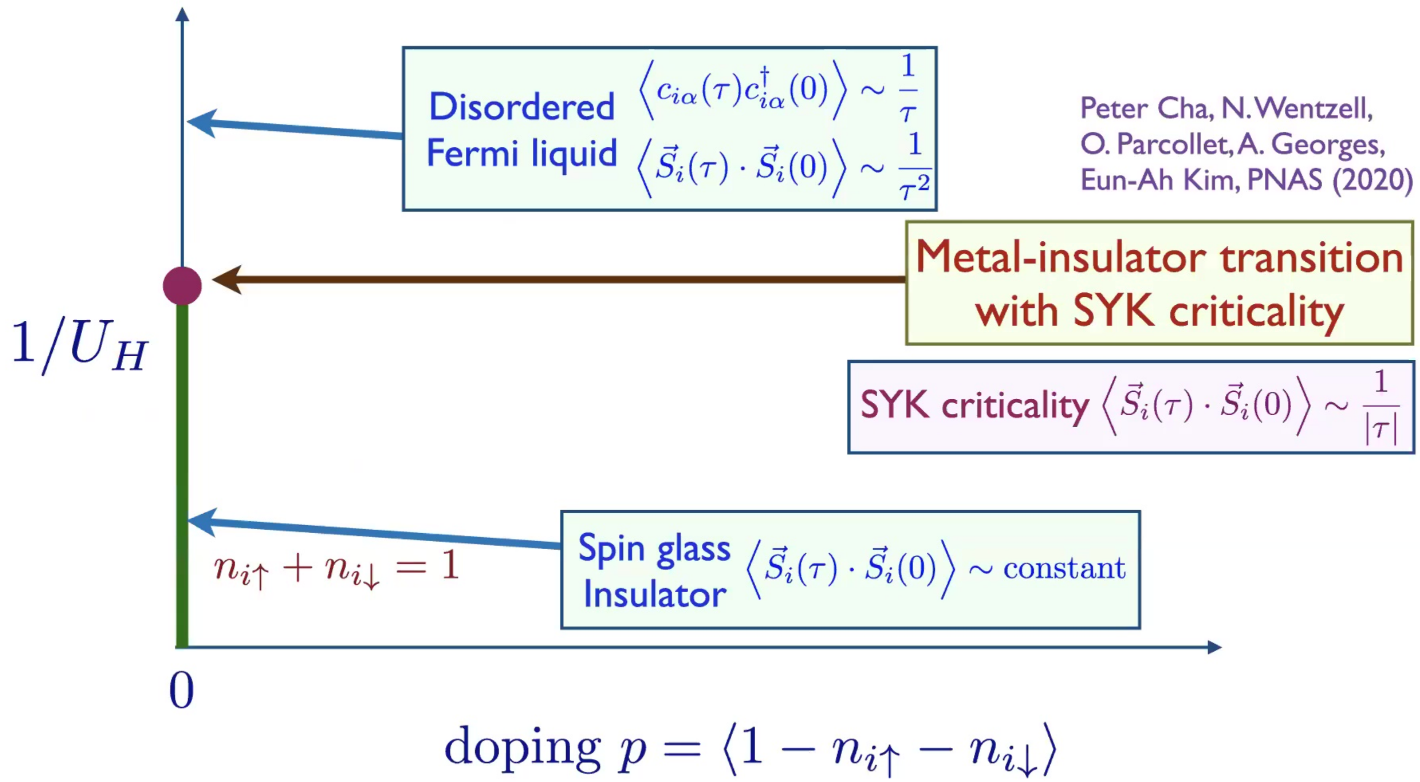
Resisitivity $\rho \sim T$ to the lowest $T$ at the critical point

**Onset of insulating gap and spin glass order co-incide.**

Peter Cha, N. Wentzell, O. Parcollet, A. Georges, Eun-Ah Kim, PNAS (2020)

# Random t-J-$U_H$ model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij}\, c_{i\alpha}^{\dagger} c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij}\, \vec{S}_i \cdot \vec{S}_j + U_H \sum_{i=1}^{N} n_{i\uparrow} n_{i\downarrow}$$




# Random $t$-$J$-$U_H$ model

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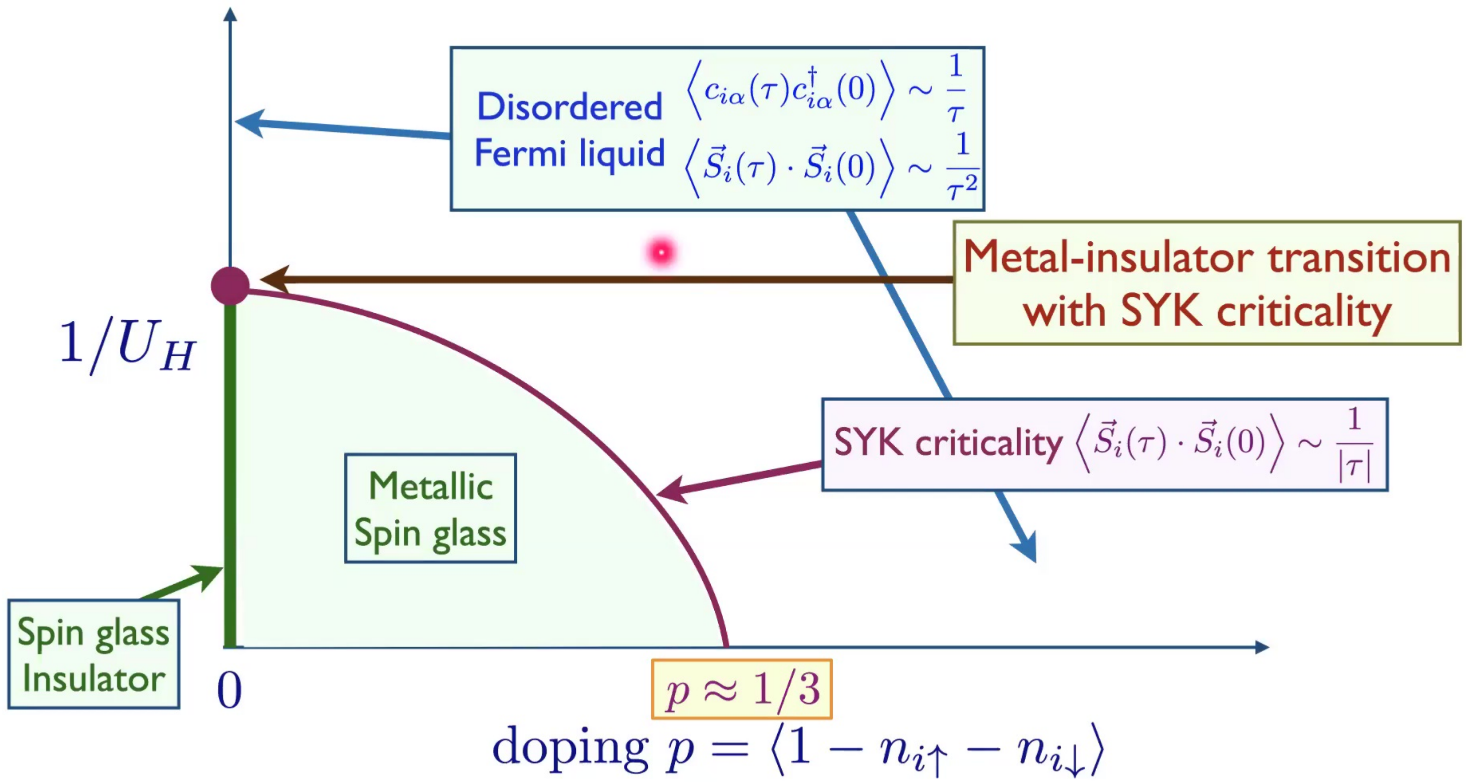



At the critical point/phase of the $t$-$J$ model, the Fermi liquid-like behavior of the electron Green's function

$$\left\langle c_{i\alpha}(\tau) c_{i\alpha}^{\dagger}(0) \right\rangle \sim \frac{1}{\tau}$$

leads to a non-zero *residual resitivity*, $\rho(0) \neq 0$.

However, the critical state is *not* a Fermi liquid, as indicated by the slow decay of the spin correlations

$$\left\langle \vec{S}_i(\tau) \cdot \vec{S}_i(0) \right\rangle \sim \frac{1}{|\tau|}$$

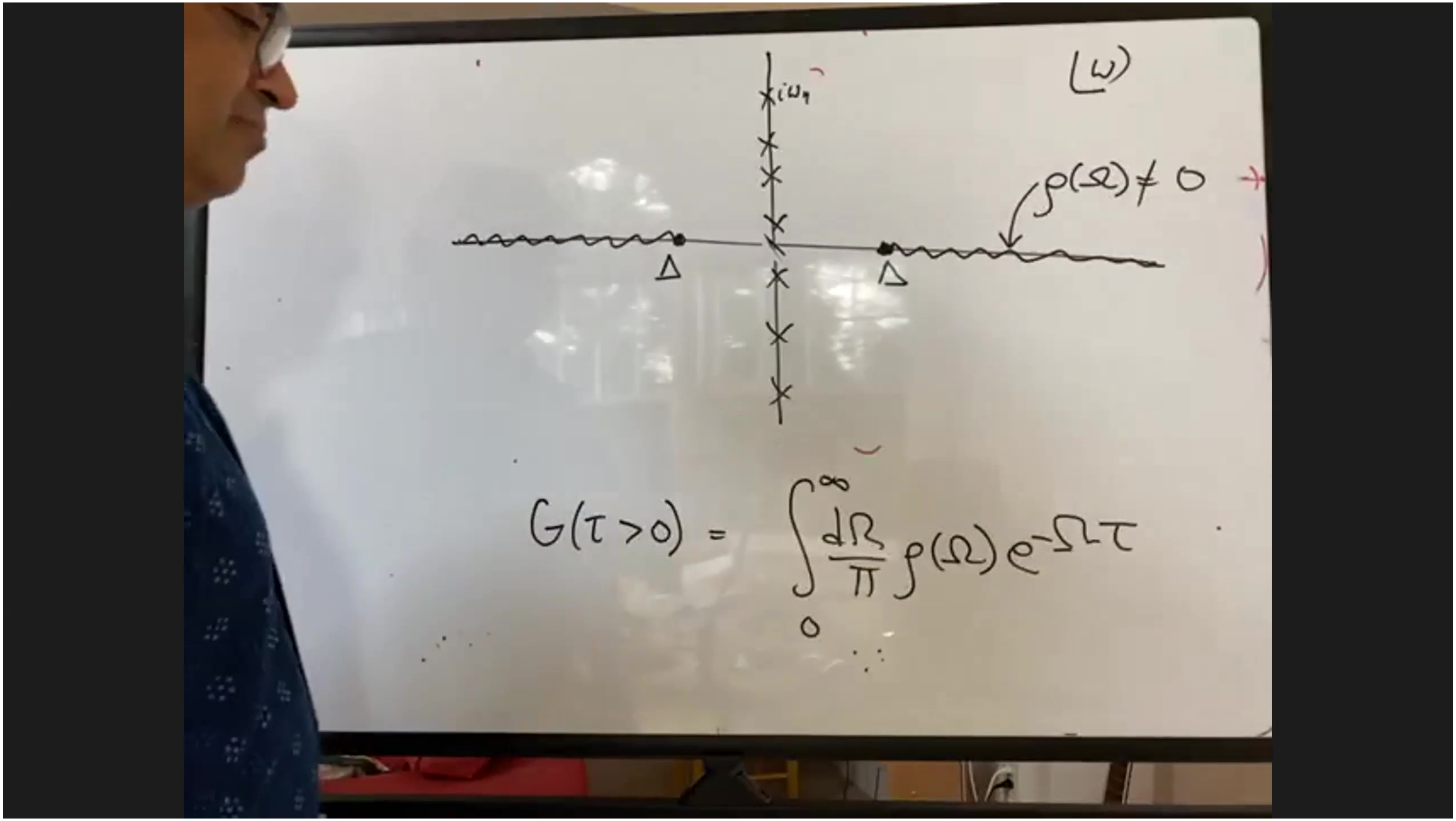
Moreover, in a Fermi liquid, we expect $\rho(T) - \rho(0) \sim T^2$, which also does not hold here.

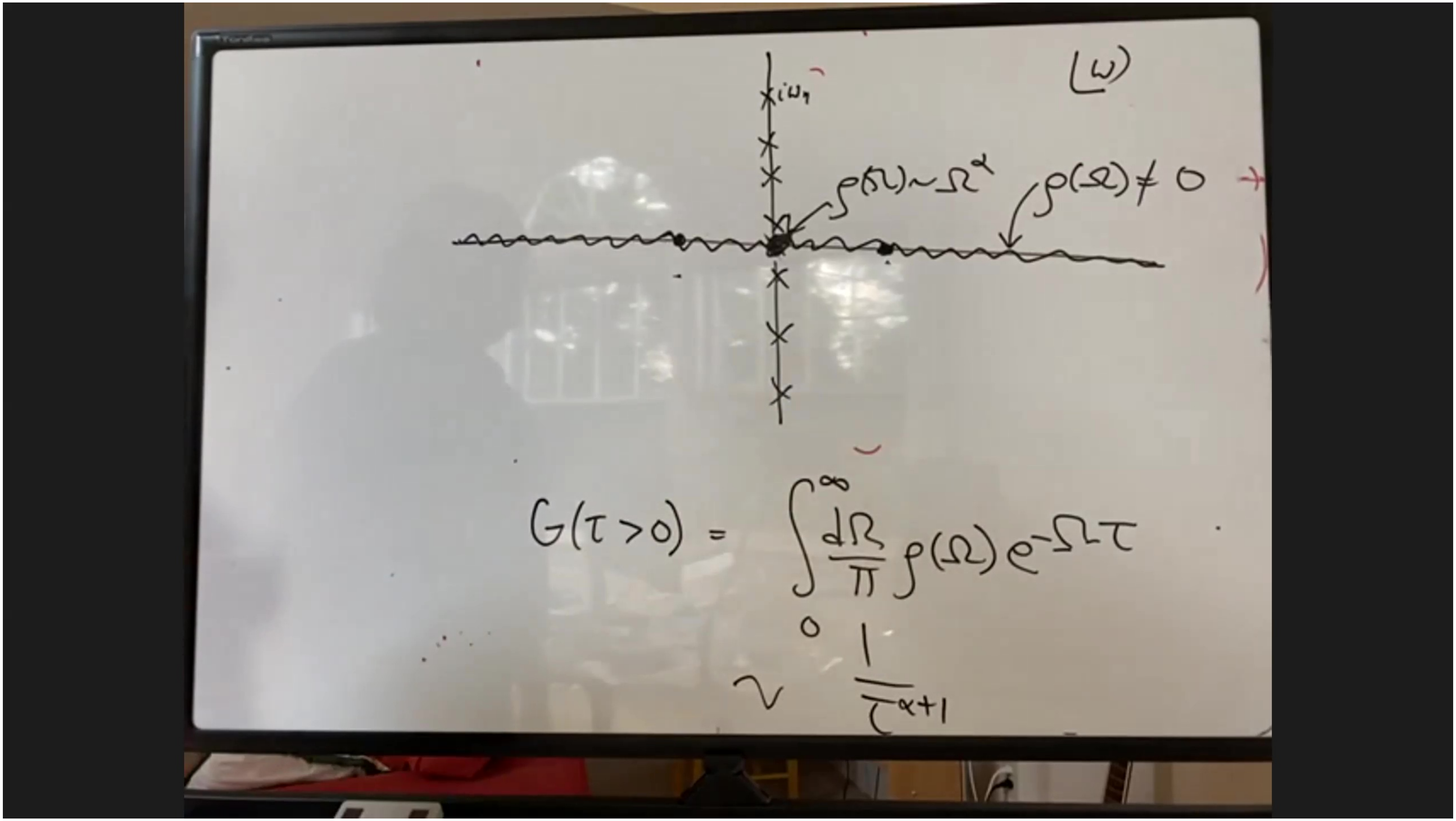
## Summary: Presented a series of increasingly realistic (partly) solvable random models of correlated metals

5. Random SU($M$) magnets: spin glass ground state for small $M$, spin liquid ground state for large $M$ with $\chi(\tau) \sim 1/|\tau|$.

6. Random $t$-$J$ model: transition at a doping density $p$ with many attractive features:

   - Critical correlations $G(\tau) \sim 1/\tau$ and $\chi(\tau) \sim 1/|\tau|$.
   - Can be interpreted in terms of fractionalization with spinon and holon correlators $\sim 1/\sqrt{\tau}$ (deconfined criticality).
   - Linear-in-$T$ resistivity down to $T = 0$ at the critical point from the time reparameterization soft mode.
   - Carrier density $p$ for $p < p_c$, and $1 + p$ for $p > p_c$.


(ω)
iω_n
ρ(Ω) ≠ 0
Δ
Δ
G(τ > 0) = ∫_0^∞ dΩ/π ρ(Ω) e^{-Ωτ}


$i\omega_n$
$(\omega)$
$\rho(\Omega) \sim \Omega^\alpha$
$\rho(\Omega) \neq 0$
$G(\tau>0) = \int_0^\infty \frac{d\Omega}{\pi} \rho(\Omega) e^{-\Omega\tau}$
$\sim \frac{1}{\tau^{\alpha+1}}$