Title: Analyticity and unitarity's offspring: anomalous dimensions and the space of EFTs

Speakers: Marc Riembau

Series: Particle Physics

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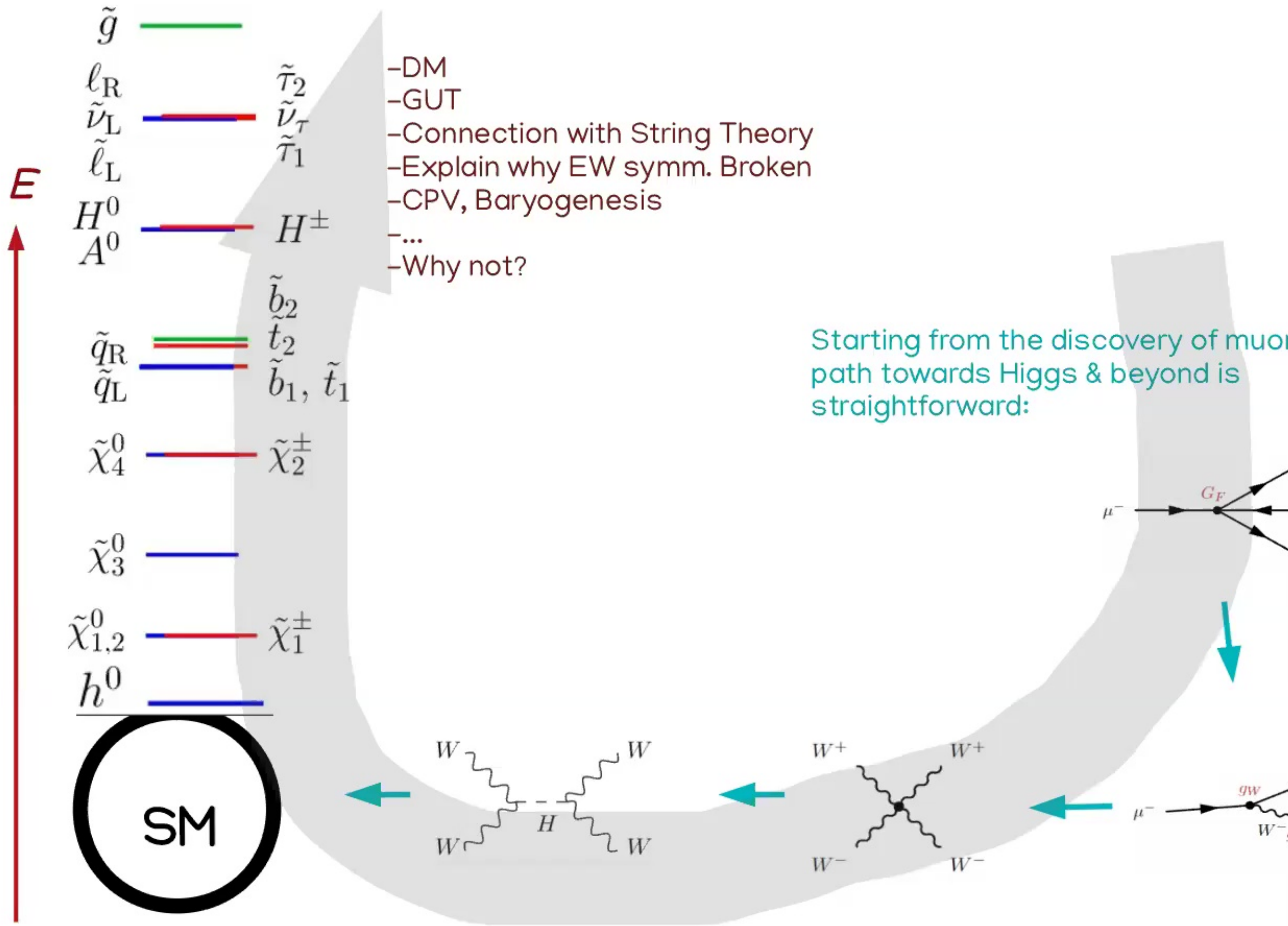
Abstract: In the first part of the talk I will present how to compute anomalous dimensions of EFT operators using on-shell scattering amplitudes. The method is used to compute some two loop transitions, which are important to provide a complete characterisation of the dynamics affecting some low energy precision experiments. In the second part, I show how unitarity, analycity and locality impose stringent non-trivial constraints to the space of possible EFTs, invisible at the Lagrangian level by only considering the symmetries of the IR theory.

Anomalous dimensions from the S-matrix & the space of EFTs

Marc Riembau
Université de Genève

14th July 2020

XXth Century particle physics from a XXIst Century perspective:



$$\mathcal{L}\,?$$

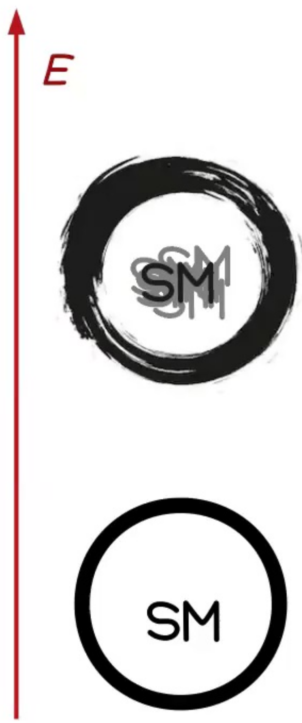
E

Particle Physics is back to the origin, is again the exploration of the unknown.

SM

$$\mathcal{L} = \mathcal{L}_{\rm SM}$$

$\mathcal{L}\,?$

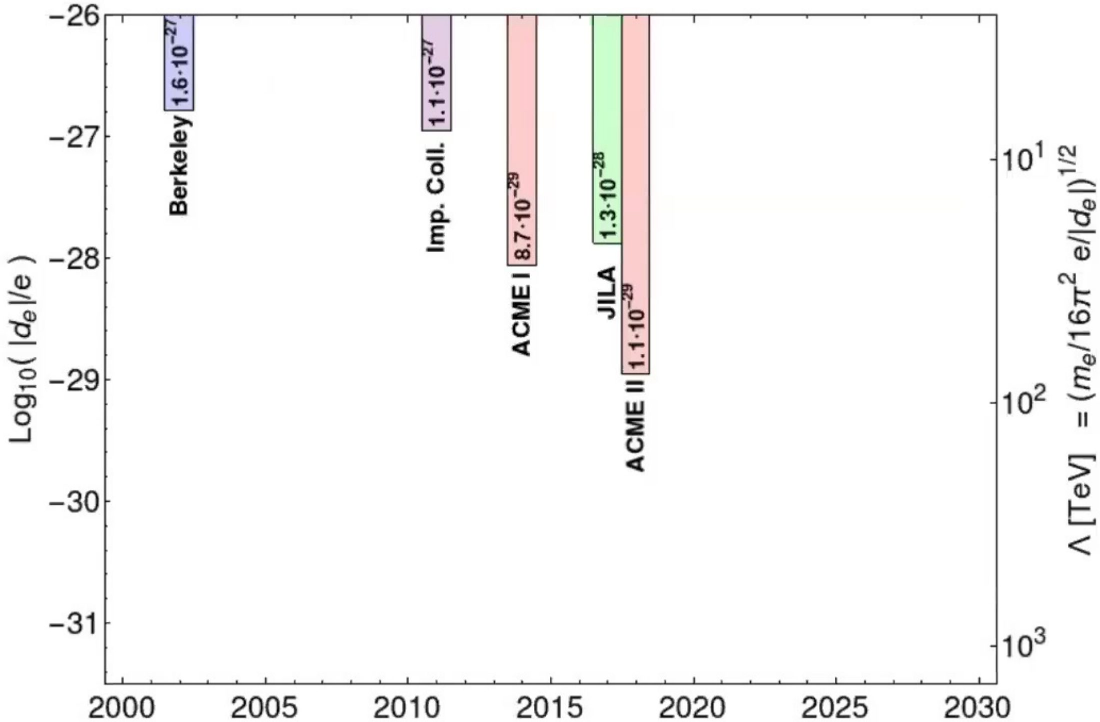


$$\mathcal{L} = \mathcal{L}_{\rm SM} + \sum_i \frac{c_i}{\Lambda}\mathcal{O}_i$$

EFT operators encode in
about the heavy dynamic
and tells us in which way
the SM is deformed.

$$\mathcal{L} = \mathcal{L}_{\rm SM}$$

Evolution of electron EDM constraints



Current: ACME II $|d_e| < 1.1 \cdot 10^{-29}$ e cm

Translation of ACME constraints to particle physics:

$$\frac{d_e}{e} \sim \frac{1}{(16\pi^2)^2}\frac{m_e}{\Lambda^2} \qquad \to \Lambda > 3\,\mathrm{TeV}$$

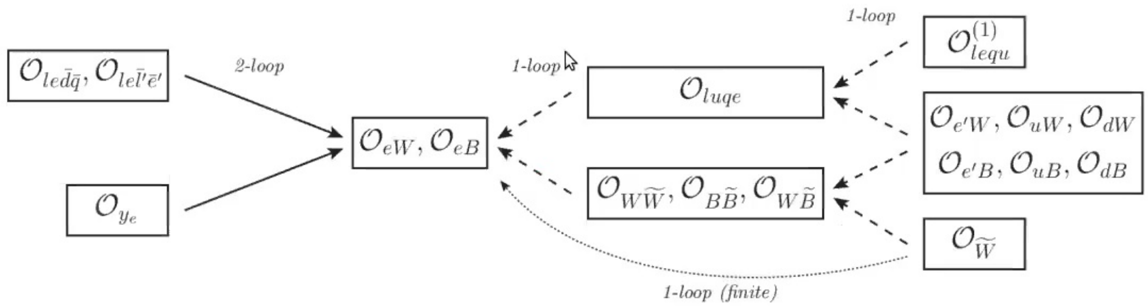
Relevant constraints even at two loops.

We want to characterize all effects that enter with

Two loops   Chirality flip   log enhanced

This is the key to help organize the contributions

G. Panico, A. Pomarol, MR [181



ACME-II implications for BSM:

Fix $\Lambda = 10\ TeV$

tree-level

C_{eW}	$5.5 \times 10^{-5}\, y_e g$
C_{eB}	$5.5 \times 10^{-5}\, y_e g'$

one-loop

C_{luqe}	$1.0 \times 10^{-3}\, y_e y_t$
$C_{W\widetilde{W}}$	$4.7 \times 10^{-3}\, g^2$
$C_{B\widetilde{B}}$	$5.2 \times 10^{-3}\, g'^2$
$C_{W\widetilde{B}}$	$2.4 \times 10^{-3}\, gg'$
$C_{\widetilde{W}}$	$6.4 \times 10^{-2}\, g^3$

two-loops

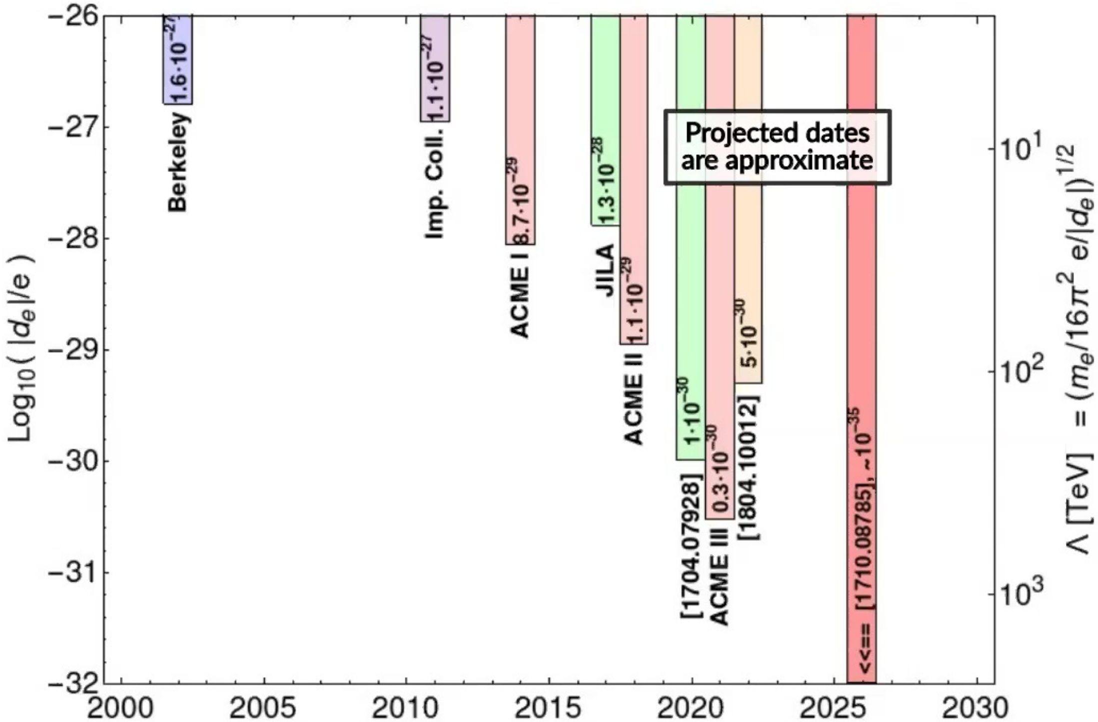
C_{lequ}	$3.8 \times 10^{-2}\, y_e y_t$
$C_{\tau W}$	$260\, y_\tau g$
$C_{\tau B}$	$380\, y_\tau g'$
C_{tW}	$6.9 \times 10^{-3}\, y_t g$
C_{tB}	$1.2 \times 10^{-2}\, y_t g'$
C_{bW}	$64\, y_b g$
C_{bB}	$47\, y_b g'$
$C_{led\bar{q}}$	$10\, y_e y_t (y_t/y_b)$
$C_{le\bar{e}'\bar{l}'}$	$0.63\, y_e y_t (y_t/y_\tau)$

two-loops finite

C_{y_e}	$14\, y_e \lambda_h$
C_{y_t}	$14\, y_t \lambda_h$
C_{y_b}	$2.9 \times 10^3\, y_b \lambda_h$
C_{y_τ}	$3.1 \times 10^4\, y_\tau \lambda_h$

G. Panico, A. Pomarol, MR [181

Evolution of electron EDM constraints

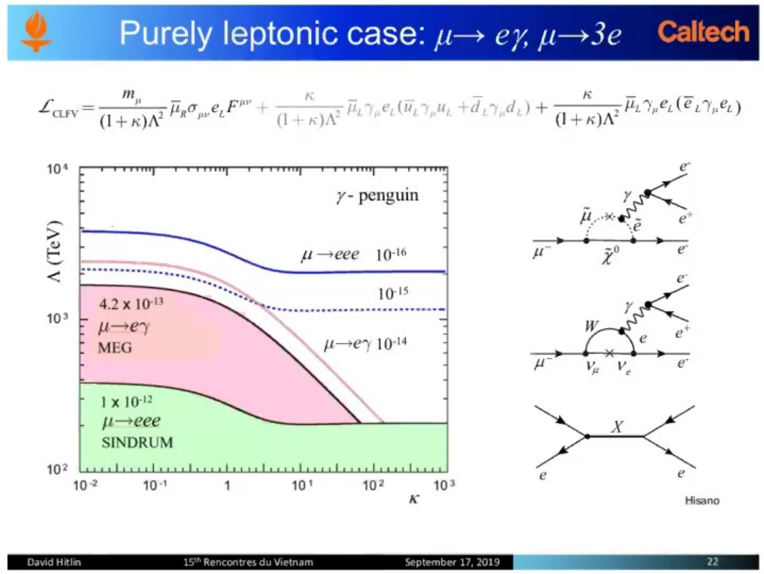


- After some time of promises of improvements with nothing happening, it seems that there will [redacted]
further progress in a short time scale.
- If there is a positive signal, we'll have confirmation very quickly.
- There are some proposals for a total breakthrough.

Electron EDM searches are not the only low energy probes to get a boost in the next years,

Muon to electron conversion searches (+photon, +ee, conversion in hadrons) are expected to improve by orders of magnitude

Slide from **David Hitlin**, Rencontres du Vietnam, Sept 17th 2019



$$\mathrm{Br}(\mu \to e\gamma) < 6 \cdot 10^{-14} \quad \Longrightarrow \quad \Lambda \geq 10\mathrm{TeV}, \text{ for } \mathcal{L} \supset \frac{\sqrt{y_e y_\mu}}{\Lambda^2}\left(\frac{g^4}{(16\pi^2)^2}\log\frac{m_w^2}{\Lambda^2}\right)\bar{\mu}\sigma_{\mu\nu}e$$

Similar for the other operators. So two-loop RGE relevant.

Only if there was a way to compute anomalous dimensions easily...

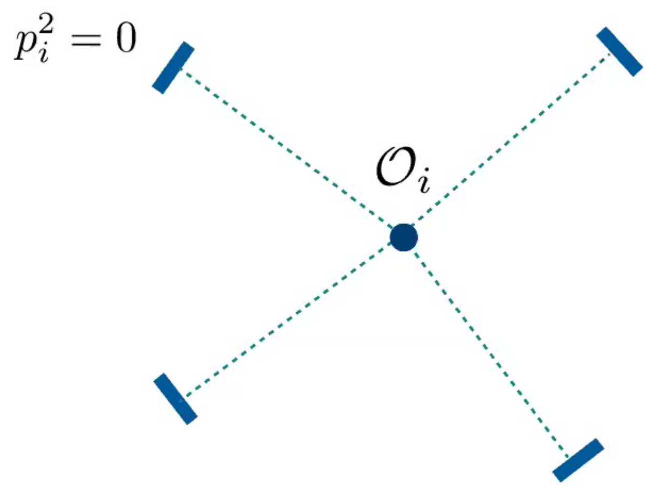
Part I

SMEFT anomalous dimensions from the S-matrix

Joan Elias Miró, James Ingoldby, MR [2005.06§

Form Factors of specific operators give the response of a given state once we insert that operator

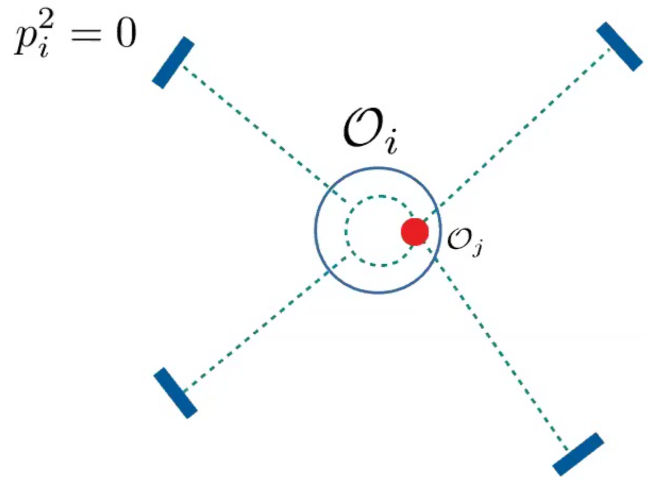
$$F_{\mathcal{O}_i} = \langle p_1, p_2, p_3, p_4 | \mathcal{O}_i | 0 \rangle \overset{\text{e.g.}}{=} \delta_{ab}(s+t)$$



$$\sum_i p_i \neq 0$$

They receive, of course, loop corrections.

$$F_{\mathcal{O}_i} = \langle p_1, p_2, p_3, p_4 | \mathcal{O}_i | 0 \rangle \overset{\text{e.g.}}{=} \delta_{ab}(s+t)\left(1 + c_j \frac{g^2}{16\pi^2} \log \frac{s}{\Lambda}\right.$$

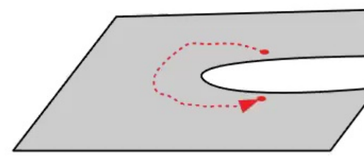


$$\sum_i p_i \neq 0$$

In perturbation theory (and perhaps beyond), a Form Factor is related across both sides [redacted]
of the cut by the **reality condition**

$$F(s_{ij}+i\epsilon) = F^*(s_{ij}-i\epsilon)$$

This is generated by a complex rotation of the momenta,



$$F = e^{-i\pi\sum_i p_i \frac{\partial}{\partial p_i}} F^* = e^{-i\pi D} F^*$$

On the other hand, **unitarity** implies

CPT

$$F = {}_{out}\langle\alpha|\mathcal{O}|0\rangle = \sum_\beta {}_{out}\langle\alpha|\beta\rangle_{in}\, {}_{in}\langle\beta|\mathcal{O}|0\rangle = SF^*$$

So,

CPT
Unitarity
Analyticity

$$e^{-i\pi D}F^* = SF^*$$

(This is a gener[redacted]
of the Watson [redacted]

The dilatation operator is proportional to the phase of the S-matrix

see [Caron-Huot, Wilhelm '16] for alternative [redacted]

[Caron-Huot, Wil[redacted]

$$e^{-i\pi D}F = SF$$

Dilatation operator related to anomalous dimensions by RG equation:

$$DF \sim \mu\frac{\partial}{\partial\mu}F \sim (\gamma_{UV} - \gamma_{IR} + \beta(g^2)\frac{\partial}{\partial g})F$$

Convolution of FF with S-matrix:

$$S = 1 + i\mathcal{M}$$

At LO, dependence on beta ignored,

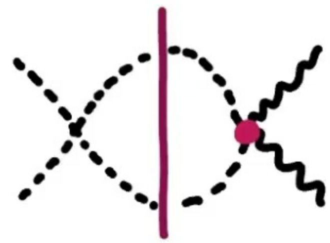
$$(\gamma_{UV} - \gamma_{IR})\langle\alpha|\mathcal{O}|0\rangle = -\frac{1}{\pi}\langle\alpha|\mathcal{M}\otimes\mathcal{O}|0\rangle$$

For this talk, we focus on elements with no IR divergences

$$\gamma_{i\leftarrow j} = -\frac{1}{\pi}\frac{\langle\alpha|\mathcal{M}\otimes\mathcal{O}_j|0\rangle}{\langle\alpha|\mathcal{O}_i|0\rangle}$$

Example 1: Self renormalization of $\mathcal{O}_{FF} = |H|^2 F_{\mu\nu}^2$

$$\langle \alpha | \mathcal{M} \otimes \mathcal{O}_{FF} | 0 \rangle =$$



$$\langle 1_i 2_j^* | \mathcal{M} | 3_k 4_l^* \rangle = 2\lambda(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj})$$

$$\langle 1_{\phi_i} 2_{\phi_j^*} 3^- 4^- | H^\dagger H F_{\mu\nu} F_{\mu\nu} | 0 \rangle = 2\delta_{ij}$$

$$= \int [\mathrm{d}p_{1'}][\mathrm{d}p_{2'}] 4\lambda \langle 34 \rangle^2 (\delta_{ij}\delta_{i'j'} + \delta_{ii'}\delta_{jj'})\delta_{i'j'}$$

$$= (2\delta_{ij}\langle 34 \rangle^2)\, 4\lambda(n_s + 1) \int [\mathrm{d}p_{1'}][\mathrm{d}p_{2'}]$$

next slide I'l
Not importan

$$= (2\delta_{ij}\langle 34 \rangle^2)\, 4\lambda(n_s + 1) \frac{1}{16\pi}$$

So, $\gamma_{FF \leftarrow FF} = \dfrac{\lambda}{16\pi^2} 4(n_s + 1)$

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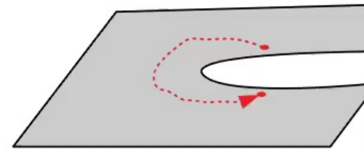

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CPT

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So,

CPT
Unitarity
Analyticity

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see [Caron-Huot, Wilhelm '16] for alternative

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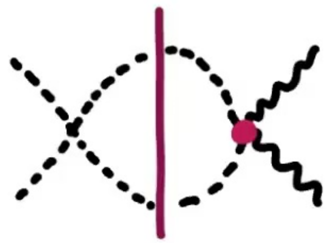

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Spinor notation for massless momenta:

See [Elvang-Huang] fo[redacted]

$$p_\mu \sigma^\mu_{a\dot{b}} = p_{a\dot{b}} = -|p]_a \langle p|_{\dot{b}} \qquad\qquad |p\rangle^{\dot{a}} = \sqrt{2E} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

The important point is that angle and square brackets carry opposite little group weigh[redacted]
or helicity.
Example:

$$\bar{f}\gamma_\mu f \phi \overleftrightarrow{D}_\mu \phi \;\to\; \bar{u}_\pm(p_1)(\not{p}_3 - \not{p}_4) v_\mp(p_2) \begin{array}{l} \xrightarrow{+-} [13]\langle 32\rangle + [14]\langle 42\rangle \\ \xrightarrow[-+]{} \langle 13\rangle[32] + \langle 14\rangle[42] \end{array}$$

Phase space integral:

The previous integral was trivial, but in general it is not.
It will be useful to write it as

$$\int [\mathrm{d}p][\mathrm{d}q] = \frac{1}{16\pi} \int_0^{\pi/2} 2\cos\theta \sin\theta \mathrm{d}\theta \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi}$$

The angles parametrize the rotation to base spinors,

$$\begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta e^{i\phi} \\ s_\theta e^{-i\phi} & c_\theta \end{pmatrix} \begin{pmatrix} |p\rangle \\ |q\rangle \end{pmatrix}$$

Non-renormalization theorems

This language trivializes the non-renormalization theorems of [Alonso, Jenkins, Ma[redacted]
[Elias Miro, Espinos[redacted]
[Cheung, Shen]

See also [Bern, Parra-Martine[redacted]

The spinor language naturally splits the dim-6 operators into 5 categories:

$$\langle\,\cdot\,\rangle^3,\quad \langle\,\cdot\,\rangle^2,\quad \langle\,\cdot\,\rangle,\quad 1\quad \text{and}\quad \langle\,\cdot\,\rangle[\,\cdot\,]$$

	Operator	MFF	
O_{3F}	$\frac{f^{ABC}}{2\cdot 3!}F^{\mu}_{A\,\nu}F^{\nu}_{B\,\rho}\bar{F}^{\rho}_{C\,\mu}$	$F_3(1^-_A 2^-_B 3^-_C)$	$\langle 12\rangle\langle 23\rangle\langle 31\rangle$
O_{FF}	$\frac{1}{2}H^\dagger H\, F^A_{\mu\nu}\bar{F}^{A\,\mu\nu}$	$F_{FF}(1_i 2^*_j 3^-_A 4^-_B)$	$\langle 34\rangle\langle 34\rangle$
O_{qF}	$\bar{Q}\sigma^{\mu\nu}T^A qHF^A_{\mu\nu}$	$F_{qF}(1^-_i 2^- 3_k 4^-_A)$	$\langle 14\rangle\langle 42\rangle$
O_{4F_1}	$(\bar{Q}_i u)\epsilon_{ij}(\bar{Q}_j d)$	$F_{4F_1}(1^-_i 2^- 3^-_j 4^-)$	$\langle 12\rangle\langle 34\rangle$
O_y	$\lvert H\rvert^2\bar{Q}qH$	$F_y(1_i 2^*_j 3_k 4^-_l 5^-)$	$\langle 45\rangle$
O_6	$\lvert H\rvert^6$	$F_6(1_a 2_b 3_c 4^*_d 5^*_e 6^*_f)$	1
O_{4F_2}	$(\bar{Q}T^A\gamma^\mu Q)(\bar{Q}T^A\gamma_\mu Q)$	$F_{4F_2}(1^-_i 2^-_j 3^+_k 4^+_l)$	$\langle 12\rangle[34]$
O_{QH}	$(\bar{Q}T^A\gamma^\mu Q)\,(iH^\dagger T^A \overleftrightarrow{D}_\mu H)$	$F_{QH}(1_i 2^*_j 3^-_k 4^+_l)$	$\langle 31\rangle\,[14]$
$O_\perp$	$(H^\dagger D_\mu H)(D^\mu H)^\dagger H$	$F_\perp(1_i 2_j 3^*_k 4^*_l)$	$\langle 13\rangle[13]$
$O_\parallel$	$\lvert H\rvert^2(D^\mu H)^\dagger(D_\mu H)$	$F_\parallel(1_i 2_j 3^*_k 4^*_l)$	$\langle 14\rangle[14]$

similar to [Henn[redacted]

Example 2: 4-fermions

- There are two fundamentally different types of 4-fermions, with and without net helicity

$$(\bar{\ell}\gamma_\mu \ell)(\bar{\ell}\gamma_\mu \ell) = \langle 1|\gamma_\mu|2]\langle 3|\gamma_\mu|4] = 2 \cdot \langle$$

$$(\bar{\ell}u)(\bar{q}e) = \langle 12\rangle\langle 34\rangle$$

- We'll compute anomalous dimensions of the second type due to a U(1):

$$\mathcal{A}(f^- f^+ f^- f^+) \;=\; g^2 \langle 13\rangle[24]\left(\frac{1}{s}+\frac{1}{u}\right)$$

- Only two flavour structures will be independent due to Schouten identity:

$$0 = \langle \ell u\rangle\langle qe\rangle + \langle \ell q\rangle\langle eu\rangle + \langle \ell e\rangle\langle uq\rangle$$

$$f_1 f_2 \in \{\ell u, eq, \ell e, eu, \ell q, qu\}$$

$$\gamma = \sum_{f_1 f_2} \langle f_1 f_2|\mathcal{M}|xy\rangle\langle f_3 f_4 xy|\mathcal{O}_{\ell equ}|0\rangle$$

$$= \int \mathrm{d}\Omega g^2 \left(Y_\ell Y_u \frac{\langle \ell u\rangle[xy]\langle xe\rangle\langle qy\rangle}{s_{\ell x}} + Y_e Y_q \frac{\langle eq\rangle[xy]\langle \ell x\rangle\langle yu\rangle}{s_{ex}} + \dots \right)$$

$$= \int \mathrm{d}\Omega\, g^2\, (Y_\ell Y_u + Y_\ell Y_q + Y_e Y_u + Y_e Y_q)\langle \ell u\rangle\langle qe\rangle + \;\; \langle \ell equ\rangle \dots$$

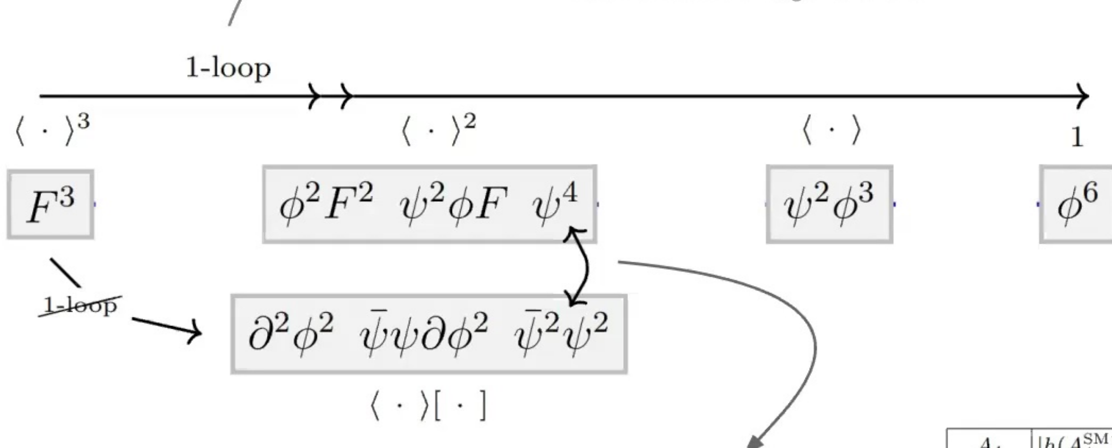
$$= \frac{g^2}{16\pi^2}(Y_\ell + Y_e)(Y_q + Y_u)\langle \ell u\rangle\langle qe\rangle$$

Non-renormalization theorems

One loop structure:

A two-particle cut
removes two legs of the operator and
the S-matrix inertion adds, at least, two legs.

So operators can only renormalize operators w
same number of legs or more.



Between the two 4-particle class (and from F^3 to <>[]) there is another effect.

Transitions between them require 4-particle amplitudes to violate helicity.
but there is only one of them in the SM.

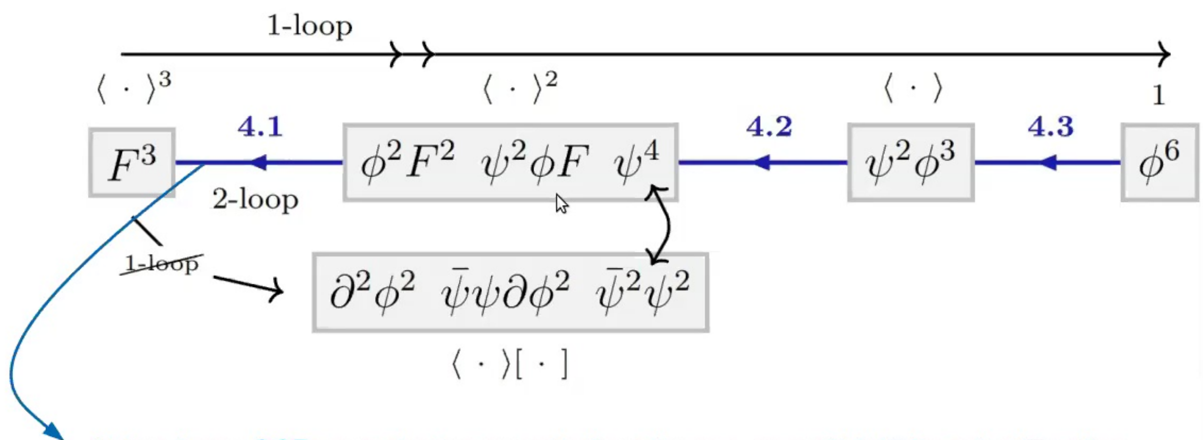
The only non-vanishing transition is between 4-fermion operators

A_4	$\lvert h(A_4^{\text{SM}})\rvert$
$VVVV$	0
$VV\phi\phi$	0
$VV\psi\psi$	0
$V\psi\psi\phi$	0
$\psi\psi\psi\psi$	2,0
$\psi\psi\phi\phi$	0
$\phi\phi\phi\phi$	0

[Azatov, Contino, Machado, Riva]

Non-renormalization theorems

Two loop structure:

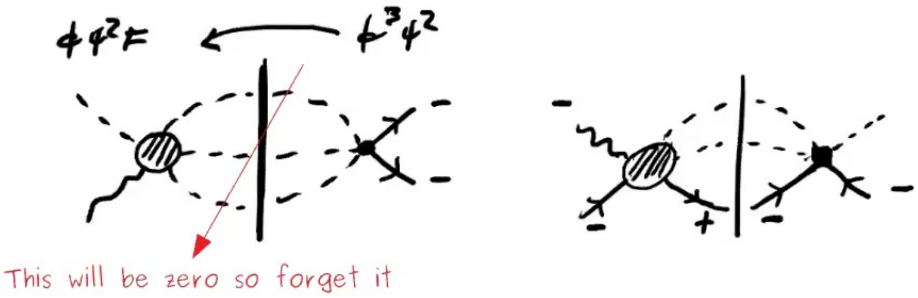


At two loops, RGEs can shorten operator legs by one, or avoid helicity selection rules.

The former type are particularly easy to compute with this method!

Example 3: A 2-loop example, Yukawa to dipole

Two types of diagrams, involving 3-particle cut and 5-point amplitudes:



The entire difficulty of this calculation is to write down the 5-point amplitude in a simple
so that the integral is easily doable. Let's focus on the pure gauge part:

$$\mathcal{A}(f^- f^+ \phi\phi\gamma^-) = g^3 \left(Q_f Q_\phi^2 \frac{[23][24]}{[12][35][45]} - Q_f^2 Q_\phi \frac{[23][24]}{[15][25][34]} \right)$$

Example 3: A 2-loop example, Yukawa to dipole

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→ 0

Since integral sy
under 3, 4 exch
and this term is

$$\text{integral} = s_{14} \int d\mu \frac{[xy][xz]}{[1x][y4][z4]} \langle x2 \rangle = \langle 14 \rangle \langle 42 \rangle \cdot N$$

→ dipole FF

with

$$N = \int_0^{\pi/2} 2 s_{\theta_1} c_{\theta_1} \mathrm{d}\theta_1 \int_0^{\pi/2} 4 s_{\theta_2}^3 c_{\theta_2} \mathrm{d}\theta_2 \int_0^{\pi/2} 2 s_{\theta_3} c_{\theta_3} \mathrm{d}\theta_3 \, \frac{c_{\theta_1}^2}{s_{\theta_2}^2} = 1$$

Adding the flavour structure, one gets the result in the literature,

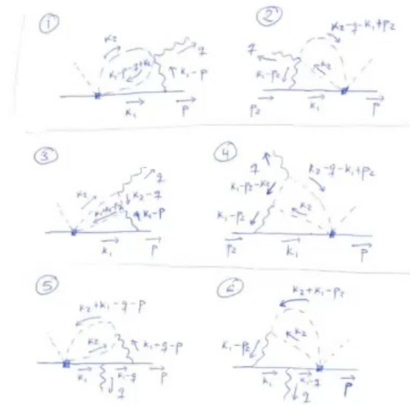
$$\frac{d}{d\ln\mu} \begin{pmatrix} C_{eB} \\ C_{eW} \end{pmatrix} = \frac{g^3}{(16\pi^2)^2} \frac{3}{4} \begin{pmatrix} t_{\theta_W} Y_H + 4 t_{\theta_W}^3 Y_H^2 (Y_L + Y_e) \\ \frac{1}{2} + \frac{2}{3} t_{\theta_W}^2 Y_H (Y_L + Y_e) \end{pmatrix} C_{y_e}$$

from [Pomarol, Panico, M

Example 3:

My brain is not a standard candle, but to get an idea...

The Verdict

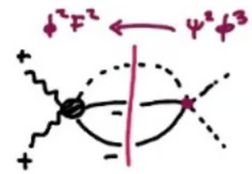


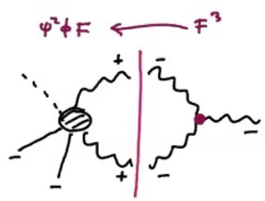
VS

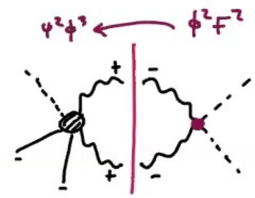
One month+ of struggle,
of sign-chasing, of looking
for factors of two and comparing
with collaborators...

One day to get the amplitude
in a nice form.
30min of writing the rotation a[redacted]
do the integrals in Mathematic[redacted]

Last but not least, an obvious but important aspect of the method is that it is GREEN







Same amplitude appears in several computations, in this case $\mathcal{A}(f^- f^- \phi\, g^+ g^+)$

Part II

Positivity and the space of EFTs

B. Bellazzini, J. Elias Miro, R. Rattazzi, MR, F. Riva, to appear....

We know know that not all IR dynamics can be UV-completed, e.g.

$$\mathcal{A}_{\rm tree}(s) = g \sum_{n=1}^{\infty} c_n \left(\frac{s^2}{\Lambda^4} \right)^n \qquad \text{must have} \qquad c_n > 0$$

[A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis, R. Ratt

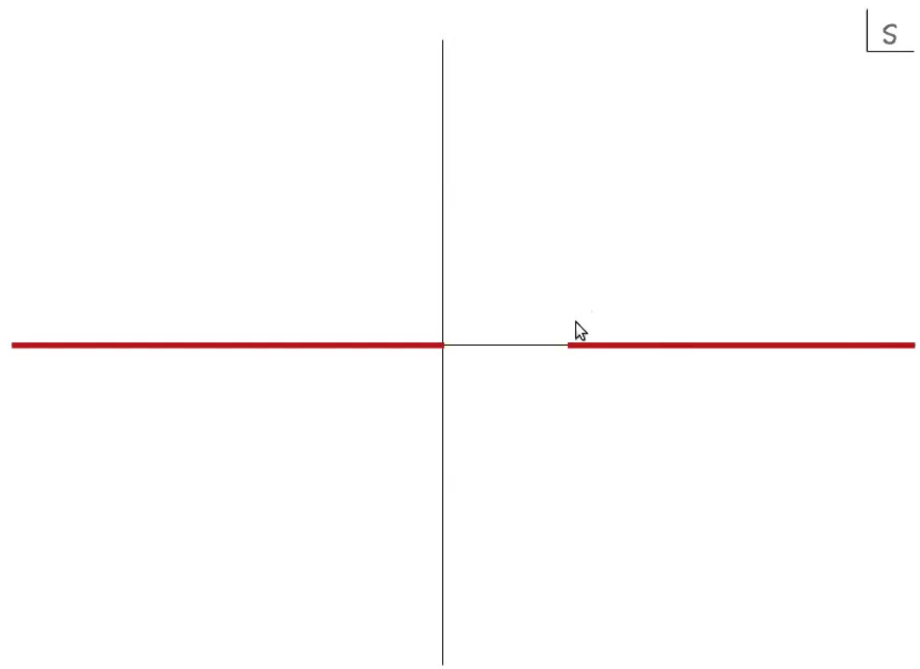
The amplitude and IR dynamics seem fine,

but negative coefficients violate unitarity of the UV theory.

Actually, this is just the tip of the iceberg and the coefficients,
using exactly the same assumptions as the ones used for the result above,
are in fact much more constrained.

- **Analyticity:** the (forward) amplitude consist solely on the unitarity cut at s>4m^2, plus the crossing symmetric one.
- **Unitarity:** imaginary part of the amplitude related to the total cross section.
- **Polynomial boundedness.** Froissart bound guarantees $\mathcal{M}(s)/s^2 \to 0 \text{ for } s \to \infty$

$$\mathcal{M}(s) = \lim_{t \to 0} \mathcal{M}(s,t)$$



- **Analyticity:** the (forward) amplitude consist solely on the unitarity cut at s>4m^2, plus the crossing symmetric one.
- **Unitarity:** imaginary part of the amplitude related to the total cross section.

- **Polynomial boundedness.** Froissart bound guarantees $\mathcal{M}(s)/s^2 \to 0 \text{ for } s \to \infty$

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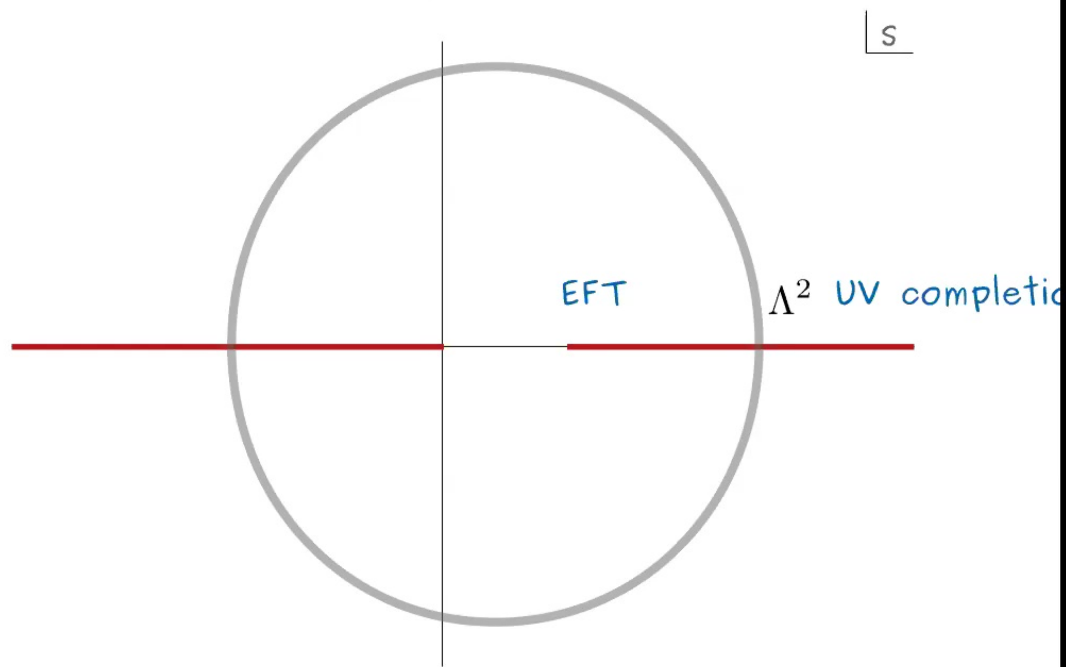
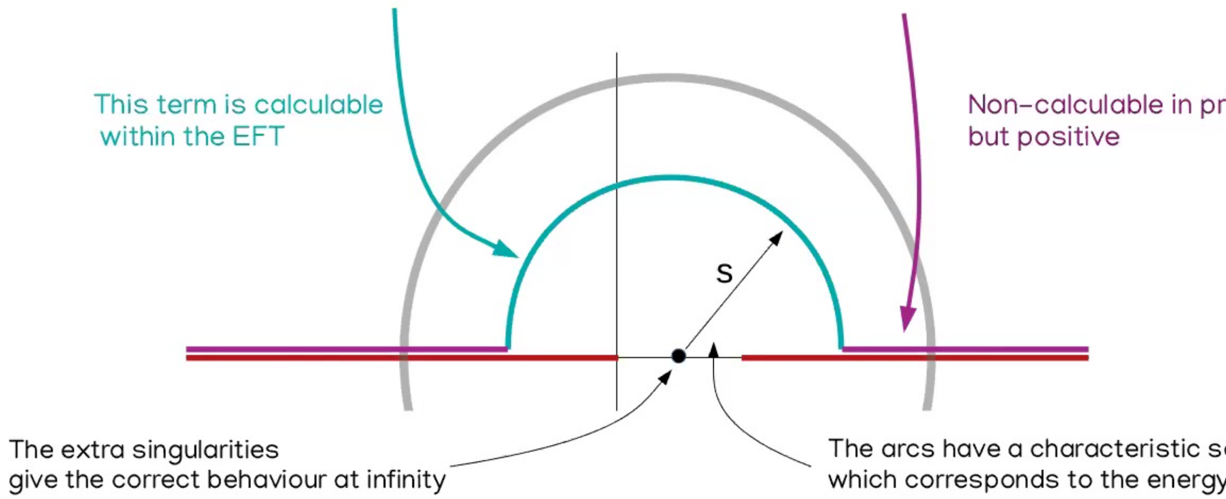


The arcs

- Using the Cauchy theorem on $\mathcal{M}(s)/s^{2n+3}$, $n \geq 0$, so that the integral at infinity vanishes,[redacted] and using crossing symmetry to relate the s- and u-channel cuts, we have the relation

$$a_n(s) \equiv \int_{\cap} \frac{ds'}{i\pi} \frac{\mathcal{M}(s')}{(s'-2m^2)^{2n+3}} = \frac{2}{\pi} \int_{s+2m^2}^{\infty} ds' \frac{\mathrm{Im}\mathcal{M}(s')}{(s'-2m^2)^{2n+3}}$$



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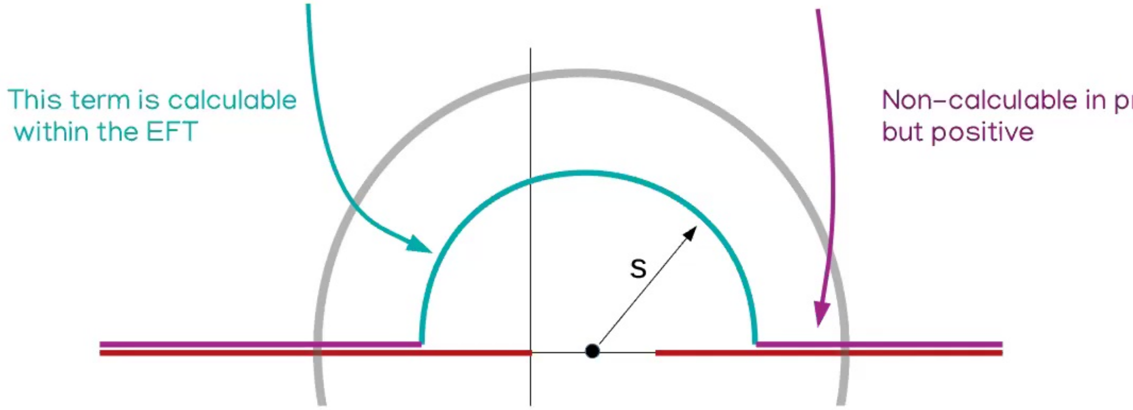
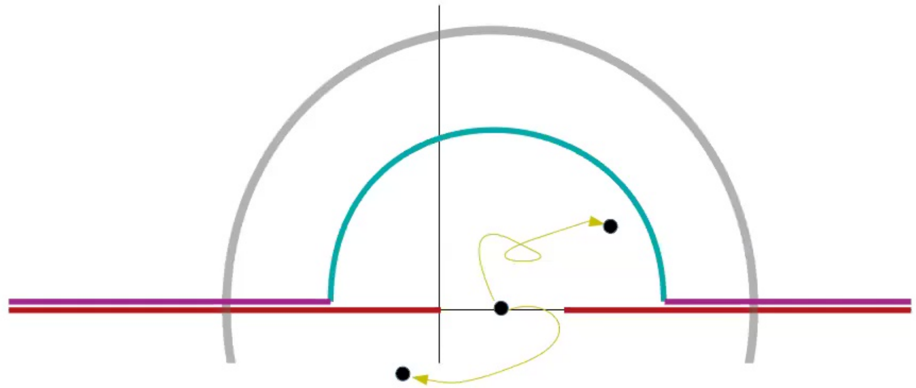


So the arcs are calculable quantities that inherit positivity conditions from the UV

For instance, $\mathcal{M}(s) = \sum_{n=0} c_{2n} s^{2n} \quad \rightarrow \quad a_n(s) = c_{2n+2}$

And we recover the known result of $c_n > 0$

But there is more...



– More generally, we can consider $\mathcal{M}(s)/P(s)$ as long as the integral is positive

$$\frac{2}{\pi}\int_s^\infty ds' \frac{\mathrm{Im}\mathcal{M}(s')}{P(s')} > 0$$

One can see that all zeros of the polynomial are within z < |s|. So,

$$\frac{2}{\pi}\int_s^\infty ds' \frac{\mathrm{Im}\mathcal{M}(s')}{P(s')} = \sum_{n=0} \alpha_n \frac{2}{\pi}\int_s^\infty ds' \frac{\mathrm{Im}\mathcal{M}(s')}{s'^{2n+3}} s^{2n} = \sum_{n=0} \alpha_n s^{2n} a_n(s$$

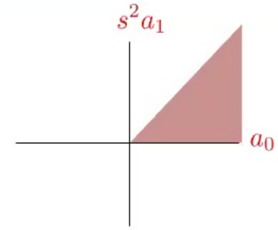
i.e. a **positive polynomial is mapped into a linear inequality between arcs.**

What is the space of allowed arcs? ⟷ What is the space of positive polynomials?

Before moving to the general result, let me give you a simple example:

$$\frac{2}{\pi}\int_s^\infty ds' \frac{\mathrm{Im}\mathcal{M}(s')}{s'^3}\left(1-\frac{s^2}{s'^2}\right) = a_0 - s^2 a_1 > 0$$

For obvious reasons, let think of this
polynomial as a derivative D acting on the arcs



So the polynomial D, which is positive, implies an inequality between the arcs.

Moreover,

$$\frac{2}{\pi}\int_s^\infty ds' \frac{\mathrm{Im}\mathcal{M}(s')}{s'^3}\left(\frac{s^2}{s'^2}\right)^n\left(1-\frac{s^2}{s'^2}\right) = a_n - s^2 a_{n+1} > 0$$

Similarly, this is a shift operator S

So the (tree-level) Wilson coefficients must decrease.

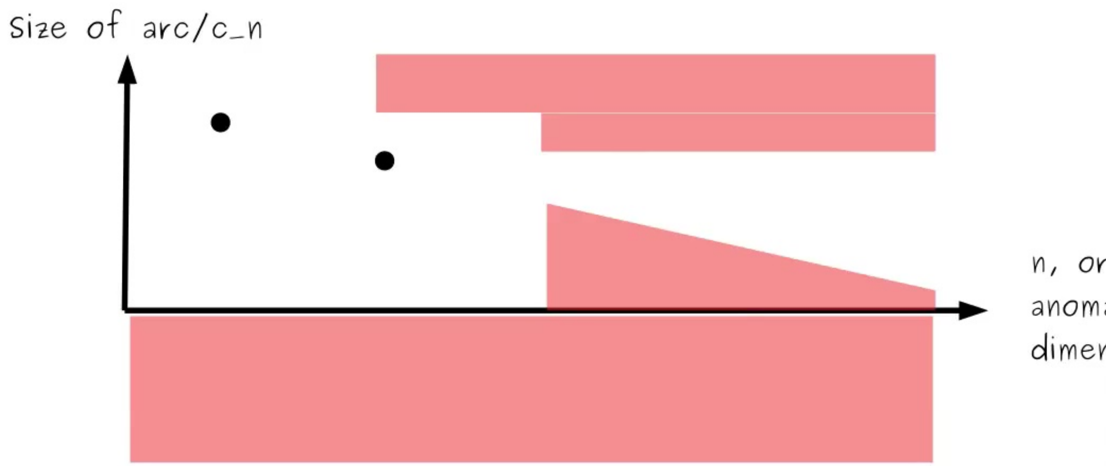
i.e., super-soft theories (dominant s^4 etc, behaviours) cannot be UV-completed by unitary the

Moreover, squaring the previous polynomial leads to a further constraint:

$$\left(1 - s^2/s'^2\right)^2 \quad \rightarrow \quad \frac{a_0 + a_2}{2} > a_1$$

i.e. an arc is always smaller than the average of its neighbours.

Pictorially,



Back to the generic result:

With the following change of variables

$$x \equiv (\hat{s}/s')^2 \quad d\mu(x) = \frac{dx}{\pi}\,\mathrm{Im}\,\mathcal{M}(\hat{s}/\sqrt{x} + 2m^2)$$

The arcs can be written as

$$s^{2n+2} a_n = \int_0^1 x^n \, d\mu(x)\,, \qquad \text{with } d\mu(x) \text{ a positive measure}$$

So the arcs are a sequence of moments of some positive distribution.

This maps our question directly into the **Hausdorff moment problem.**
The idea is that any positive function can be written as a sum of Bernstein polynomials:

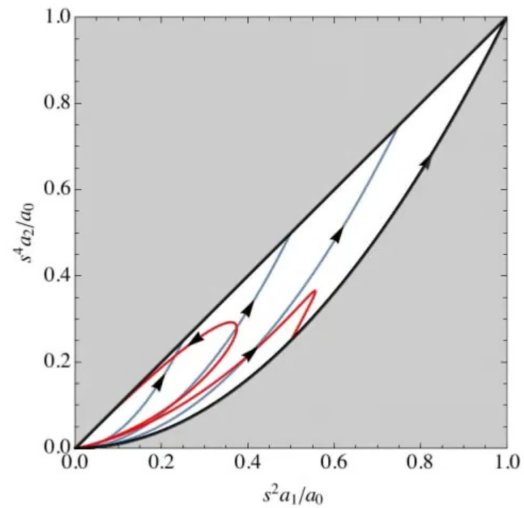
$$p(x) > 0 \text{ for } x \in [0,1] \text{ iif } p(x) = \sum_{i,j} c_{i,j} x^i (1-x)^j \text{ with } c_{i,j} > 0$$

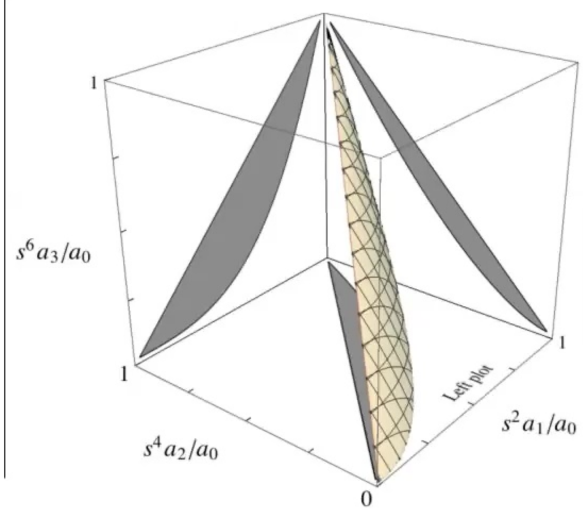
Which is nothing but an arbitrary positive combination of shifts S and derivatives D.
Hausdorff showed that

$$\{a_n\} \text{ is an } \textit{infinite} \text{ sequence of moments iif } S^i D^j a_n \geq 0$$

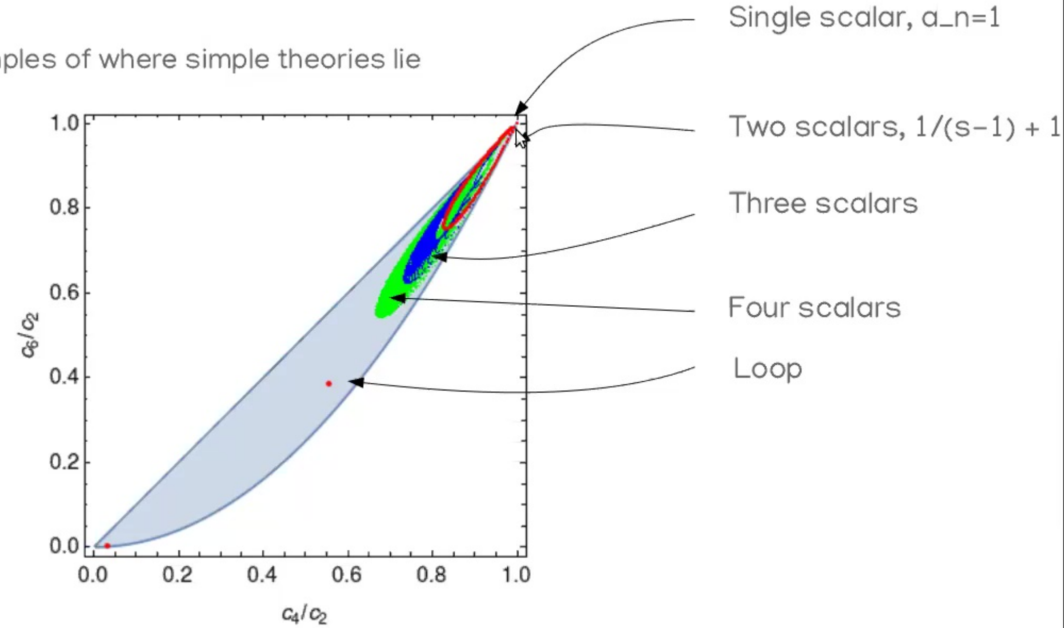
If n was a continuous variable, $c(n)>0$, $c'(n)<0$, $c''(n)>0$, $c'''(n)<0$, ...

For example,


1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
$s^4 a_2/a_0$
$s^2 a_1/a_0$


$s^6 a_3/a_0$
$s^4 a_2/a_0$
$s^2 a_1/a_0$
Left plot
1
1
1
0

Some examples of where simple theories lie



Forward limit of Veneziano amplitude:

$$A(s) \sim s\tan(s\pi/2) \sim s^2 + \frac{\pi^2}{12}s^4 + \frac{\pi^4}{120}s^6 + \frac{17\pi^6}{20160}s^8 + \frac{31\pi^8}{362880}s^{10} + \dots$$

$$a_0 - a_1 = 1 - \frac{\pi^2}{12} \simeq 0.177$$

$$a_1 - a_2 = \frac{\pi^2}{120}(10-\pi^2) \simeq 0.0107$$

$$a_2 - a_3 = \frac{\pi^4}{20160}(168 - 17\pi^2) \simeq 0.001$$

$$a_0a_2 - a_1^2 = \pi^4/720 \simeq 0.14$$

$$a_1a_3 - a_2^2 = \pi^8/1209600 \simeq 0.0079$$

Conclusions:

Analyticity and unitarity give us
new ways to compute relevant quantities for interpreting experimental data and tech us which dynamics may or may not be UV completed.

Thank you!