Title: A categorification of the Lusztigâ€"Vogan module

Speakers: Anna Romanov

Collection: Geometric Representation Theory

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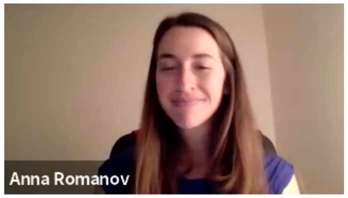
Abstract: Admissible representations of real reductive Lie groups are a key player in the world of unitary representation theory. The characters of irreducible admissible representations were described by Lustigâ€"Vogan in the 80â€™s in terms of a geometrically-defined module over the associated Hecke algebra. In this talk, Iâ€™ll describe a categorification of this module using Soergel bimodules, with a focus on examples. This is work in progress.



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A categorification of the Lusztig-Vogan module

Anna Romanov
University of Sydney

Geometric Representation Theory, June 22-26, 2020
Max Planck Institute Bonn, Perimeter Institute Waterloo

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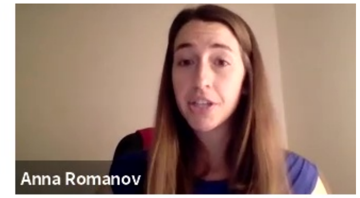
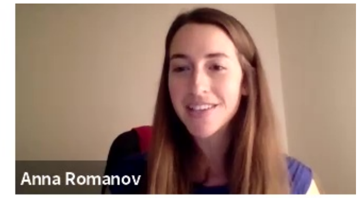


Plan:

1. Setting the scene
2. The Lusztig-Vogan module
3. A category of equivariant sheaves
4. An algebraic incarnation
5. Putting it all together



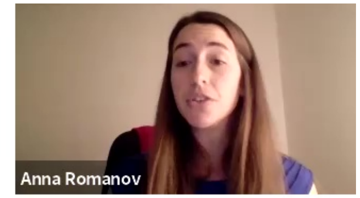
1. Setting the scene

Today's main character:

$$G_{\mathbb{R}} = \text{real reductive Lie group}$$

today's running example

e.g. $\mathrm{SL}(2,\mathbb{R}), SO(n), U(p,q),$ $Sp(2,2) = \{M \in Sp(4,\mathbb{C}) \mid M^T D M = D\}$ where $D = \mathrm{diag}(1,-1,1,-1)$



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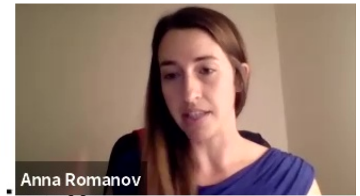
Guiding philosophy:

Understand $G_{\mathbb{R}}$ by studying its unitary representation theory.

That is, study Hilbert spaces on which $G_{\mathbb{R}}$ acts by unitary operators.



1. Setting the scene

Harish-Chandra's approach (1950's): recast the problem algebraically

unitary representation of $G_{\mathbb{R}}$ $\rightsquigarrow$ $(\mathfrak{g}, K)$-module

($\mathfrak{g}$: complexified Lie algebra; K: complexification of maximal compact subgroup)

Why this works:

- $\mathfrak{g}$ captures infinitesimal information about $G_{\mathbb{R}}$ (but loses information which distinguishes $G_{\mathbb{R}}$ from coverings).
- K captures fundamental group information about $G_{\mathbb{R}}$ (so it makes up for what $\mathfrak{g}$ loses).

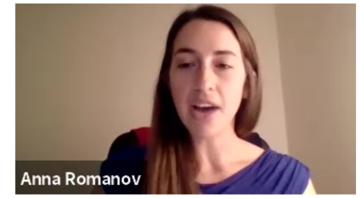
A $(\mathfrak{g}, K)$-module coming from a *unitary* representation of $G_{\mathbb{R}}$ has a nice property:

$$V = \bigoplus V_i$$

(V_i: irreducible unitary $K_{\mathbb{R}}$-representations)

each V_i has finite multiplicity

"admissibility"



1. Setting the scene

This motivates the setting we'll work in today:

$G_{\mathbb{R}} \supset K_{\mathbb{R}}$	$G \supset K$	$\mathfrak{g} \supset \mathfrak{k}$
connected reductive Lie group ⊃ maximal compact subgroup	complexifications (have structure of complex algebraic groups)	complexified Lie algebras

Example: $G_{\mathbb{R}} = Sp(2,2)$

$$\mathfrak{g} = \mathfrak{sp}(4,\mathbb{C}), \qquad K = \left\{ \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{pmatrix} \right\} \simeq \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \subset Sp(4,\mathbb{C})$$

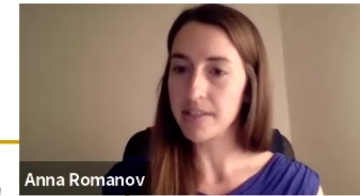
Definition: An *admissible* $(\mathfrak{g}, K)$*-module* V is both

- a finitely generated $U(\mathfrak{g})$-module, and
- an algebraic representation of K,

such that the differential of the K-action agrees with the $\mathfrak{k} \subset \mathfrak{g}$-action, and $V = \bigoplus V_i$ where V_i are irreducible K-representations occuring with finite multiplicity.

1. Setting the scene

The upshot:



Every unitary representation of $G_{\mathbb{R}}$ leads to an admissible $(\mathfrak{g}, K)$-module, so if we want to understand unitary representations, we can start by understanding admissible $(\mathfrak{g}, K)$-modules.

Classification of irreducible admissible $(\mathfrak{g}, K)$-modules:

Fix $T \subset B \subset G$. $K \circlearrowright G/B$ with finitely many orbits

$$\left\{ \begin{array}{c} \text{irreducible admissible} \\ (\mathfrak{g}, K)\text{-modules} \\ \text{with a fixed integral infinitesimal character} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{irreducible } K\text{-equivariant} \\ \text{local systems on } K\text{-orbits} \end{array} \right\}$$

Can realize this parameterization explicitly using Beilinson–Bernstein localization:

"Harish-Chandra modules" $\mathcal{M}_{\text{f.g.}}(\mathcal{U}_\theta, K) \underset{\Gamma}{\overset{\Delta_\lambda}{\rightleftarrows}} \mathcal{M}_{\text{coh}}(\mathcal{D}_\lambda, K)$ ($\sim$) K-equivariant $\mathcal{D}_\lambda$-modules on G/B

$M(Q, \tau) \longleftarrow\!\shortmid\; i_{Q!}(\tau)$ $\qquad i_Q : Q \hookrightarrow G/B$

$\downarrow \qquad\qquad \downarrow$

all irreducible admissible $(\mathfrak{g}, K)$-modules show up in this way $\rightarrow$ $L(Q, \tau) \longleftarrow\!\shortmid\; i_{Q!*}(\tau)$ $\qquad \tau$ = irreducible K-equivariant connection on Q

1. Setting the scene

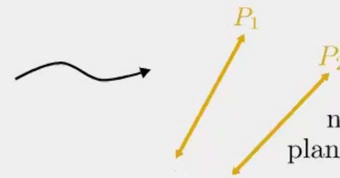
Example: $Sp(2,2)$ $\quad G/B \simeq \{\ell \subset P = P^{\perp} \subset \ell^{\perp} \subset \mathbb{C}^4\}$

point in G/B = (isotropic) line in isotropic plane

$$K = \left\{ \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{pmatrix} \right\} \circlearrowright G/B$$

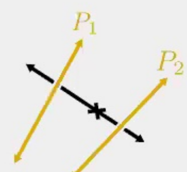
stabilizes

$$P_1 = \left\{ \begin{pmatrix} * \\ 0 \\ 0 \\ * \end{pmatrix} \right\}, P_2 = \left\{ \begin{pmatrix} 0 \\ * \\ * \\ 0 \end{pmatrix} \right\}$$

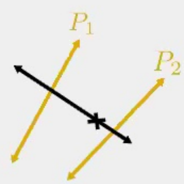


non-intersecting planes (projective lines) in $\mathbb{C}^4$ ($\mathbb{CP}^3$)

K-orbits on G/B:

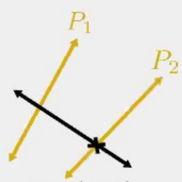


O_1 (4D)

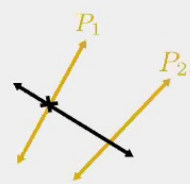


O_2 (3D)

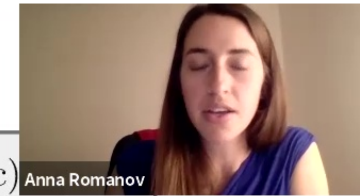
two closed orbits →



O_3 (2D)



O_4 (2D)



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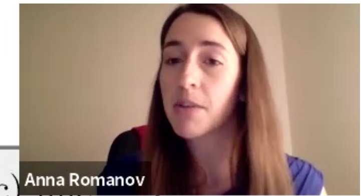
1. Setting the scene



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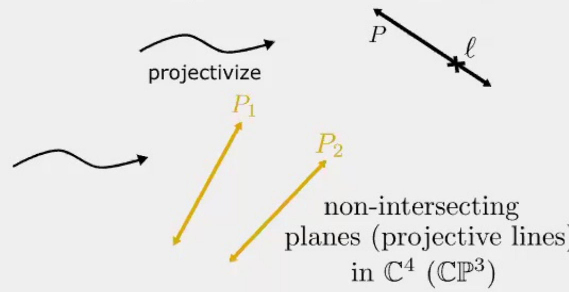
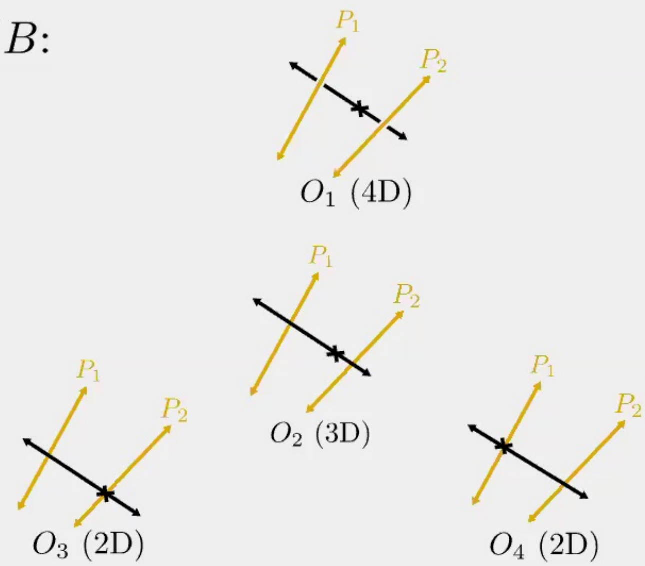


K-orbits on G/B:



two closed orbits →

Can check:

$\text{stab}_K x$ is connected for any $x \in G/B$

$\Longrightarrow$ only K-equivariant local systems/connections on orbits are trivial!

Four irreducible admissible $(\mathfrak{g}, K)$-modules:

$$L_1, L_2, L_3, L_4$$

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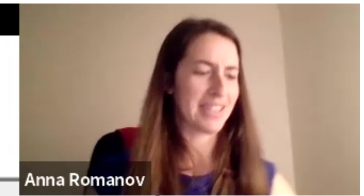

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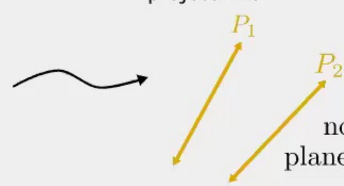

non-intersecting planes (projective lines) in $\mathbb{C}^4$ $(\mathbb{CP}^3)$

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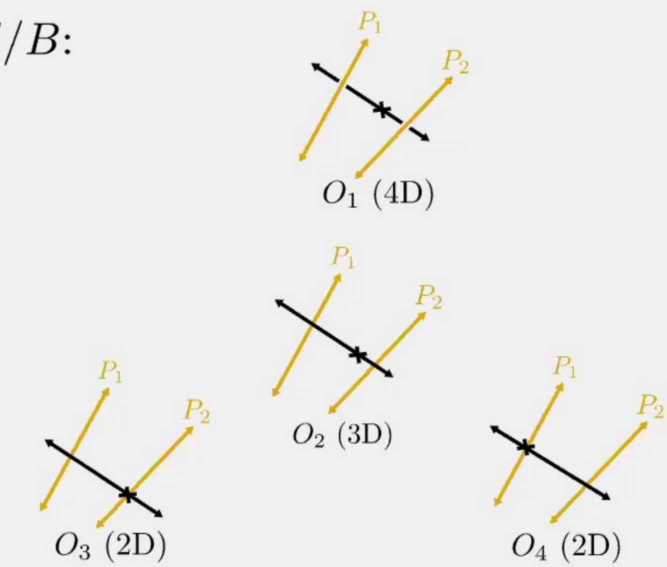

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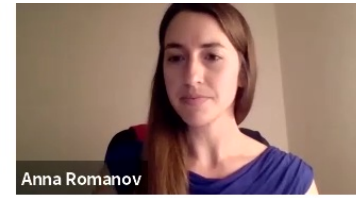


1. Setting the scene

What about character theory?

Have a classification scheme of the form $\{\text{standards}\} \twoheadrightarrow \{\text{simples}\}$ in a finite-length abelian category

$$\mathcal{A}_\lambda := \begin{array}{c}\text{admissible } (\mathfrak{g}, K)\text{-modules} \\ \text{with infinitesimal character } \chi_\lambda,\end{array}$$

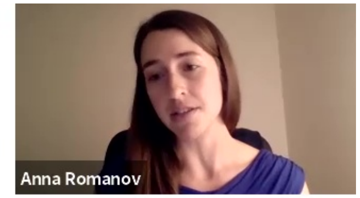
can ask about multiplicities.

In the Grothendieck group $[\mathcal{A}_\lambda]$,

$$[M(Q,\tau)] = \sum m_{Q,Q'}^{\tau,\tau'} [L(Q',\tau')].$$

Question: What are these multiplicities?

Answer: Given by the Lusztig-Vogan module of the Hecke algebra



2. The LV module

$\mathfrak{g}, K, B \rightsquigarrow (W,S)$ Coxeter system, $W_K \subset W$ reflection subgroup

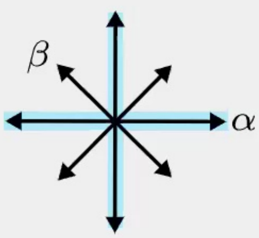
$\rightsquigarrow H = H(W,S)$ Hecke algebra

associative $\mathbb{Z}[v^{\pm 1}]$-algebra with generators $\{\delta_s\}_{s \in S}$ and relations

Kazhdan–Lusztig basis: $\{b_w\}_{w \in W}$

- $(\delta_s + v)(\delta_s - v^{-1}) = 0$ for all $s \in S$,
- $\underbrace{\delta_s \delta_t \delta_s \cdots}_{\text{order of } st} = \underbrace{\delta_t \delta_s \delta_t \cdots}_{\text{order of } st}$ for all $s, t \in S$.

Example: $Sp(2,2)$, type C_2



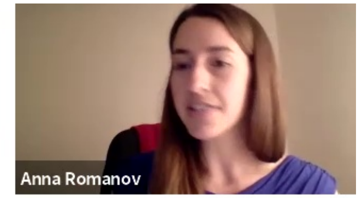
$$W = \langle s_\alpha, s_\beta \mid s_\alpha^2 = s_\beta^2 = 1, (s_\alpha s_\beta)^4 = 1 \rangle$$

$$\cup$$

$$W_K = \langle s_\alpha, s_\beta s_\alpha s_\beta \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \qquad |W_K \backslash W| = 2$$

reflection subgroup (not parabolic)

2. The LV module **Idea**: imitate Kazhdan-Lusztig theory

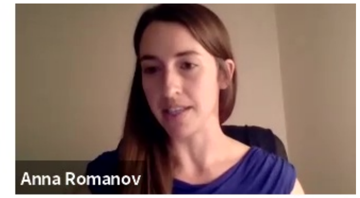


Construct an H-module M_{LV}:

- Let $\mathcal{D} = \{(Q,\tau) \mid Q \in K\backslash G/B, \tau \in \mathrm{Loc}_K(Q) \text{ irred.}\}$. $\longleftrightarrow$ W=Weyl group
- Length function on $\mathcal{D}$: $\ell(Q,\tau) = \dim Q$. $\longleftrightarrow$ $\ell(w) = \dim X_w$
- Let $M_{LV} :=$ free $\mathbb{Z}[v^{\pm 1}]$-module with basis $\mathcal{D}$. $\longleftrightarrow$ $H = \bigoplus_{w \in W} \mathbb{Z}[v^{\pm 1}]\delta_w$
- Give M_{LV} the structure of a right H-module: $\longleftrightarrow$ $H \circlearrowleft H$ by right multiplication

$$m \cdot b_s := (*)$$

- Construct a partial order on $\mathcal{D}$. $\longleftrightarrow$ Bruhat order
- There exists an involution $D : M_{LV} \to M_{LV}$ satisfying appropriate properties. $\longleftrightarrow$ Kazhdan–Lusztig involution



2. The LV module

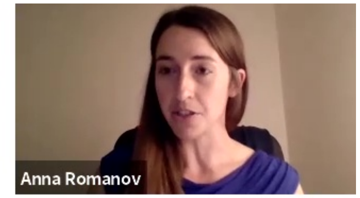
Theorem: (Lusztig–Vogan): For each $d \in \mathcal{D}$, there exists a unique self-dual element $m_d \in M_{LV}$ such that

$$m_d = \sum_{c \leq d} P_{c,d}(v) d$$

Kazhdan-Lusztig-Vogan polynomials

with $P_{c,d}(v) \in v\mathbb{Z}[v]$, $c \neq d$ and $P_{d,d}(v) = 1$.

The punchline: KLV polynomials determine multiplicities of $L(Q,\tau)$'s in $M(Q,\tau)$'s!



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But I still haven't told you how the Hecke algebra acts on M_{LV}...

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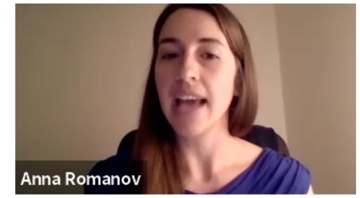
3. A category of sheaves



First step: need a categorical upgrade of the Hecke algebra

$$D^b_B(G/B) \simeq D^b_{B\times B}(G)$$

equivariant derived category (Bernstein-Lunts)

- monoidal category under convolution $*$ (Decomposition Theorem)
- contains IC sheaves:

$$\{\mathrm{IC}_w \mid w \in W\} \subset D^b_b(G/B)$$

B-equivariant intersection cohomology sheaves of Schubert varieties $\overline{X}_w$

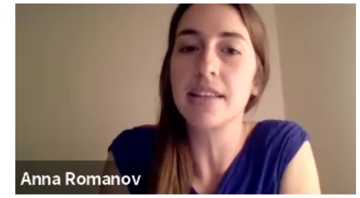
- define the geometric Hecke category

$$\mathcal{H} := \langle \mathrm{IC}_w \mid w \in W \rangle_{\oplus,[1]} \subset D^b_B(G/B)$$

  - monoidal category under $*$
  - categorifies the Hecke algebra: $[\mathcal{H}]_{\oplus} \simeq H$

split Grothendieck group

$\left(\mathbb{Z}[v^{\pm 1}]\text{-algebra via } v \cdot [\mathcal{F}] := [\mathcal{F}[1]]\right)$



3. A category of sheaves

Second step: a categorical upgrade of M_{LV}

$$\underset{\simeq D^b_{K\times B}(G)}{D^b_K(G/B)} \supset \{\mathrm{IC}(Q,\tau) := i_{Q!*}(\tau)[\dim Q] \mid \begin{matrix} Q \in K\backslash G/B, \\ \tau \in \mathrm{Loc}_K(Q) \text{ irred.} \end{matrix} \}$$

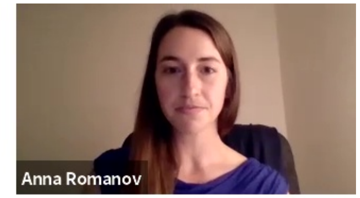
- convolution gives a right action $D^b_K(G/B) \circlearrowleft \mathcal{H}$
- define

$$\boxed{\mathcal{M}_{LV}^{\text{geom}} := \langle \mathrm{IC}(Q,\tau) * \mathcal{H} \mid Q \text{ closed} \rangle_{\oplus,\ominus,[1]}}$$

direct summands

- take split Grothendieck group $[\mathcal{M}_{LV}^{\text{geom}}]_{\oplus}$
  - $\mathbb{Z}[v^{\pm 1}]$-module via $v \cdot [\mathcal{F}] := [\mathcal{F}[1]]$
  - has basis $\{[IC(Q,\tau)]\}$ parameterized by $\mathcal{D}$

$\rightsquigarrow$ can identify $[\mathcal{M}_{LV}^{\text{geom}}]_{\oplus} \simeq M_{LV}$ as $\mathbb{Z}[v^{\pm 1}]$-modules



3. A category of sheaves

Taking stock: convolution formalism gives an action

$$D^b_K(G/B) \circlearrowleft_{*} D^b_B(G/B)$$

$$\cup \qquad\qquad \cup$$

$$\mathcal{M}^{\mathrm{geom}}_{LV} \circlearrowleft_{*} \mathcal{H}$$

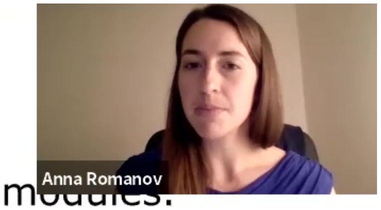
$\rightsquigarrow$ $[\mathcal{M}^{\mathrm{geom}}_{LV}]_\oplus$ has the structure of an $[\mathcal{H}]_\oplus = H$-module.

Definition: The H-module structure $(*)$ on M_{LV} is given by

$$\boxed{[\mathcal{F}] \cdot b_s := [\mathcal{F} * \mathrm{IC}_s]}$$

Kazhdan-Lusztig basis element of H

This completes the definition of the Lusztig-Vogan module



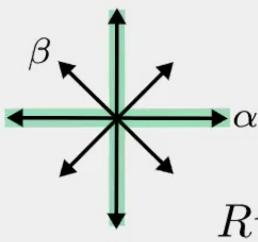
4. An algebraic incarnation

We can reformulate $\mathcal{M}_{LV}^{\mathrm{geom}}$ algebraically to place it in the world of Soergel bimodules.

Assume $G_{\mathbb{R}}$ is equal rank (i.e. G and K share a maximal torus T), let $\mathfrak{t} = \mathrm{Lie}T$

Let $R := S(\mathfrak{t}^*) \circlearrowleft W$ and $R^K := S(\mathfrak{t}^*)^{W_K}$.

Example: $Sp(2,2)$



$R = \mathbb{R}[\alpha, \beta]$

$R^K = \mathbb{R}[\alpha, \beta]^{\langle s_\alpha, s_\beta s_\alpha s_\beta \rangle}$

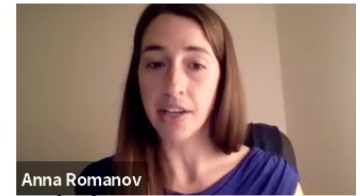
Bott-Samelson bimodules: For $s \in S$, let $R^s = s$- invariants.

- For an expression $\underline{w} = s_1, \cdots s_n$, define a graded R-bimodule

$$BS(\underline{w}) := R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \otimes \cdots \otimes_{R^{s_n}} R.$$

- These are the building blocks of **Soergel bimodules**:

$$\mathbb{S}\mathrm{Bim} := \langle BS(\underline{w}) \mid w \in W \rangle_{\oplus, \ominus, [1]}$$



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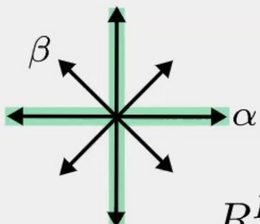


$$R = \mathbb{R}[\alpha, \beta]$$

$$R^K = \mathbb{R}[\alpha, \beta]^{\langle s_\alpha, s_\beta s_\alpha s_\beta \rangle}$$

Theorem: $\mathbb{S}\text{Bim}$ categorifies the Hecke algebra:

$$[\mathbb{S}\text{Bim}]_\oplus = H.$$

Bott-Samelson bimodules: For $s \in S$, let $R^s = s$- invariants.

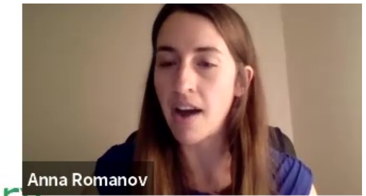
- For an expression $\underline{w} = s_1, \cdots s_n$, define a graded R-bimodule

$$BS(\underline{w}) := R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \otimes \cdots \otimes_{R^{s_n}} R.$$

- These are the building blocks of **Soergel bimodules**:

$$\mathbb{S}\text{Bim} := \langle BS(\underline{w}) \mid w \in W \rangle_{\oplus, \ominus, [1]}$$

4. An algebraic incarnation



Using $BS(\underline{w})$, we can construct an algebraic Lusztig-Vogan category.

Definition: Let $x \in W$ be a minimal coset representative of $\overline{x} \in W_K \backslash W$. Define a (R^K, R)-bimodule structure on R (denoted R_x) by

$$(f, g) \cdot r = fr(xg).$$

"standard bimodules"

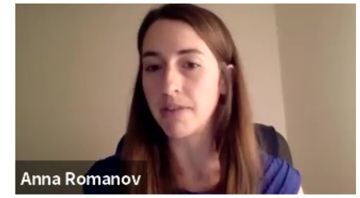
left action is multiplication — right action is twisted by x

Definition: (Romanov) The *Lusztig–Vogan category* is

$$\mathcal{M}_{LV} := \langle R_x \otimes_R BS(\underline{w}) \mid \overline{x} \in W_K \backslash W, w \in W \rangle_{\oplus, \ominus, (1)} \circlearrowleft \mathbb{S}\mathrm{Bim}$$

objects are (R^K, R)-bimodules $\rightsquigarrow$ $[\mathcal{M}_{LV}]_{\oplus}$ is a right H-module

Remark: We could define $\mathcal{M}_{LV}$ for *any* reflection subgroup of W, and in general these categories are rather mysterious. Do they have finitely many indecomposible objects? Can their Grothendieck groups be described combinatorially?



5. Putting it all together

Where we've come:

- We have a module for the Hecke algebra which tells us about the character theory of irreducible admissible representations of a real reductive Lie group.

$$\boxed{M_{LV}}$$

- It is defined as the Grothendieck group of a geometric category.

$$\boxed{M_{LV} := [\mathcal{M}_{LV}^{\text{geom}}]}$$

- We have defined an algebraic category whose Grothendieck group has the structure of an H-module.

$$\boxed{[\mathcal{M}_{LV}] \circlearrowleft H}$$

The hope: $\mathcal{M}_{LV}$ provides an algebraic categorification of M_{LV}.

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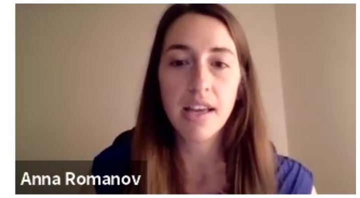
5. Putting it all together



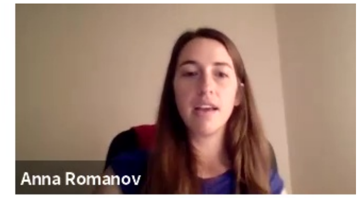
Why this would be interesting:

- Could provide a setting for an algebraic proof that the KLV polynomials describe characters of irreducible admissible representations of a real reductive group.

- Gives a geometric description of a class of interesting module categories over the (algebraic) Hecke category ($\mathbb{S}\mathrm{Bim}$).

- Unlocks a new toolbox (Soergel bimodules) in the study of real reductive groups.

- Gives a potential algebraic setting for a proof of Soergel's conjecture on Vogan character duality.



5. Putting it all together

Equivariant hypercohomology provides a functor:

$$\mathbb{H}^\bullet_{K\times B} : \mathcal{M}^{\mathrm{geom}}_{LV} \to \mathcal{M}_{LV}.$$

- $\mathbb{H}^\bullet_{K\times B}$ intertwines $\mathcal{H}$-action with $\mathbb{S}$Bim-action.
- $\mathbb{H}^\bullet_{K\times B}$ sends generating objects to generating objects:

$$\{\mathrm{IC}(Q,\tau) \mid Q \text{ closed}\} \mapsto \{R_x \mid \overline{x} \in W_K \backslash W\}.$$

- **Work in progress:** $\mathbb{H}^\bullet_{K\times B}$ is fully faithful.

Remark: This is a special case of related work by Bezrukavnikov-Vilonen, "Koszul duality for quasi-split real groups", in which a categorified Lusztig-Vogan module provides the combinatorial setting for a proof of Soergel's conjecture in the quasi-split case.

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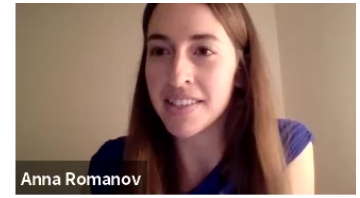

Thanks for listening!

These slides can be found on my website *www.maths.usyd.edu.au/u/romanova*