

Title: An extension of Suzuki's functor to the critical level

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Abstract: Suzuki's functor relates the representation theory of the affine Lie algebra to the representation theory of the rational Cherednik algebra in type A. In this talk, we discuss an extension of this functor to the critical level, $t=0$ case. This case is special because the respective categories of representations have large centres. Our main result describes the relationship between these centres, and provides a partial geometric interpretation in terms of Calogero-Moser spaces and opers.



An extension of Suzuki's functor to the critical level

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25 June 2020



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Outline



- Schur-Weyl duality and Arakawa-Suzuki-type functors
- Affine \mathfrak{gl}_n and the Feigin-Frenkel centre
- Rational Cherednik algebra and the Calogero-Moser space
- Construction and properties of the Suzuki functor
- Relationship between the centres
- Some implications

Arakawa-Suzuki functors



Schur-Weyl duality

We have commuting actions of \mathfrak{gl}_n and S_m on the tensor product space:

$$\mathfrak{gl}_n \curvearrowright (\mathbb{C}^n)^{\otimes m} \curvearrowright S_m$$

which centralise each other. We get a functor:

$$\mathfrak{gl}_n\text{-mod} \longrightarrow S_m\text{-mod}$$

$$M \mapsto [M \otimes (\mathbb{C}^n)^{\otimes m}]_{\text{triv}} = H_0(\mathfrak{gl}_n, M \otimes (\mathbb{C}^n)^{\otimes m})$$

Generalizations


- $\mathfrak{sl}_n\text{-mod} \longrightarrow \mathbf{dAHA}\text{-mod}$ (Arakawa-Suzuki)
- $\hat{\mathfrak{sl}}_n\text{-mod} \longrightarrow \mathbf{TDAHA}\text{-mod}$ (Arakawa-Suzuki-Tsuchiya)
- $\hat{\mathfrak{gl}}_n\text{-mod} \longrightarrow \mathbf{RDAHA}\text{-mod}$ (Suzuki, Varagnolo-Vasserot)

There also exist quantum and type BC generalizations.

Critical level



A more precise statement:

- \mathcal{C}_κ = category of smooth $\hat{\mathfrak{gl}}_n$ -modules of level κ 
- \mathcal{H}_t = rational Cherednik algebra associated to S_m and parameter t

The Suzuki functor:

$$F_\kappa: \mathcal{C}_\kappa \longrightarrow \mathcal{H}_{\kappa+n}\text{-mod}$$

The non-critical case: under some mild assumptions, the functor induces an equivalence of quasi-hereditary categories between a certain subcategory of $\text{cat. } \mathcal{O}$ for $\hat{\mathfrak{gl}}_n$ and $\text{cat. } \mathcal{O}$ for the RCA.

Question: What happens as $\kappa \rightarrow -n$ (critical level) and $t \rightarrow 0$?

$$Z(\mathcal{C}_\kappa) \neq \mathbb{C} \iff \kappa \text{ is critical}$$

$$Z(\mathcal{H}_t) \neq \mathbb{C} \iff t = 0$$

Some motivations



- There are indications the two centres are closely related:
 - The connection between the Calogero-Moser integrable system and the KP hierarchy.
 - The work of Mukhin-Tarasov-Varchenko: they constructed a surjective homomorphism from the Bethe algebra of the Gaudin model associated to \mathfrak{g} to the centre of the rational Cherednik algebra. By the work of Chervov-Talalaev, the Bethe algebra can be obtained as an image of $Z(\mathcal{C}_{-n})$.
- Goal: understand, from a more algebraic point of view, how the two centres are related.
- Hope: in the cyclotomic case, the functor could perhaps tell us something about decomposition numbers for the rational Cherednik algebra for $\mathbb{Z}_l \wr S_m$ at $t = 0$.

The affine Kac-Moody algebra $\hat{\mathfrak{gl}}_n$



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- Let $\mathfrak{g} = \mathfrak{gl}_n$ and let $\hat{\mathfrak{g}}$ be the corresponding affine Lie algebra

$$0 \rightarrow \mathbb{C}\mathbf{c} \rightarrow \hat{\mathfrak{g}} \rightarrow \mathfrak{g}((t)) \rightarrow 0.$$

- Set $U_\kappa = U(\hat{\mathfrak{g}})/\langle \mathbf{c} - \kappa \rangle$ and

$$\hat{U}_\kappa = \varprojlim U_\kappa / I_r, \quad I_r = U_\kappa \cdot \mathfrak{g} \otimes t^r \mathbb{C}[[t]]$$

- We have $Z(\mathcal{C}_\kappa) = Z(\hat{U}_\kappa)$.
- By a deep theorem of Feigin-Frenkel,

$$\text{Spec } Z(\hat{U}_{-n}) \cong \text{Op}_{GL_n}(D^\times).$$

Feigin-Frenkel centre

- The Feigin-Frenkel centre also has an explicit algebraic description (due to Chervov-Molev).
- The vacuum module of $\hat{\mathfrak{g}}$ has the structure of a vertex algebra.
- Consider the matrix:

$$E := \begin{pmatrix} \tau + e_{11}[-1] & e_{12}[-1] & \cdots & e_{1n}[-1] \\ e_{21}[-1] & \tau + e_{22}[-1] & \cdots & e_{2n}[-1] \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1}[-1] & e_{n2}[-1] & \cdots & \tau + e_{nn}[-1] \end{pmatrix}$$

where $[\tau, X \otimes f(t)] = -X \otimes \partial_t f(t)$ and $e_{ij}[-1] = e_{ij} \otimes t^{-1}$.

$$\text{Tr}(E^k) = \mathbf{S}_{k;0}\tau^k + \mathbf{S}_{k;1}\tau^{k-1} + \dots + \mathbf{S}_{k;k-1}\tau + \mathbf{S}_{k;k}, \quad \mathbf{S}_k := \mathbf{S}_{k,k}$$

- The centre of the vertex algebra:

$$\mathfrak{z} = \mathbb{C}[T^l \mathbf{S}_k \mid k = 1, \dots, n, l \geq 0]$$

- The centre of the completed universal enveloping algebra:

$$Z(\widehat{U}_{-n}) = \widehat{\mathbb{C}[\mathbf{S}_{k,\langle l \rangle}]_{i=1,\dots,n}}_{l \in \mathbb{Z}}$$

- $Z(\widehat{U}_{-n})$ has a Poisson structure given by the Hayashi bracket.



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The rational Cherednik algebra

The rational Cherednik algebra \mathcal{H}_t associated to the complex reflection group S_n is the quotient of $\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle \rtimes \mathbb{C}S_n$ by the relations:

- $[x_i, x_j] = [y_i, y_j] = 0,$
- $[x_i, y_j] = s_{ij}$ if $i \neq j,$
- $[x_i, y_i] = t - \sum_{j \neq i} s_{ij}.$

Facts

- 1 There is a PBW-type isomorphism:

$$\text{gr } \mathcal{H}_t \cong \mathbb{C}[x_i, y_i]_{i=1, \dots, n} \rtimes \mathbb{C}S_n, \quad \text{gr } Z(\mathcal{H}_0) \cong \mathbb{C}[x_i, y_i]_{i=1, \dots, n}^{S_n}$$

- 2 We have $\mathbb{C}[x_i]^{S_n} \otimes \mathbb{C}[y_i]^{S_n} \subset Z(\mathcal{H}_0)$. The algebra $Z(\mathcal{H}_0)$ is a free $\mathbb{C}[x_i]^{S_n} \otimes \mathbb{C}[y_i]^{S_n}$ -module of rank $n!$.
- 3 Every simple \mathcal{H}_0 -module has dimension $n!$ and is isomorphic to $\mathbb{C}S_n$ as an S_n -module. Moreover, there is a bijection

$$\{\text{isoclasses of simple } \mathcal{H}_0\text{-modules}\} \longleftrightarrow \text{Maxspec } Z(\mathcal{H}_0).$$

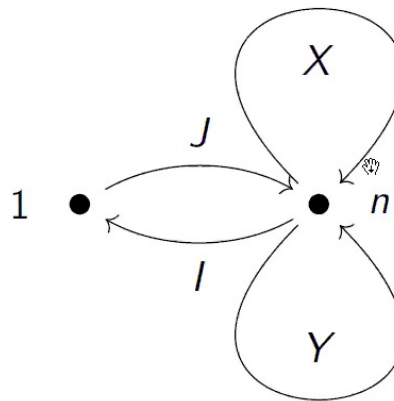
- 4 $Z(\mathcal{H}_0)$ also has a Poisson structure.



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Calogero-Moser space

- Consider the space of representations $\text{Rep}(Q, \mathbf{d})$ of the double of the framed Jordan quiver with dimension vector $\mathbf{d} = (1, n)$:



- We have the moment map:

$$\mu: \text{Rep}(Q, \mathbf{d}) \rightarrow \mathfrak{gl}_n, \quad (X, Y, I, J) \mapsto [X, Y] + JI.$$

- The Calogero-Moser space is the quotient:

$$\mathcal{CM}_n = \mu^{-1}(-\text{id}) // GL_n$$

- By a theorem of Etingof-Ginzburg: $\text{Spec } Z(\mathcal{H}_0) \cong \mathcal{CM}_n$.



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Suzuki functor

We sketch the construction of the Suzuki functor

$$F_{\kappa} : \mathcal{C}_{\kappa} \rightarrow \mathcal{H}_{\kappa+n}\text{-mod.}$$

- Let M be a smooth $\hat{\mathfrak{g}}$ -module and $V = \mathbb{C}^n$. We have an action

$$\mathfrak{g}[t] \curvearrowright (V^*)^{\otimes n} \otimes M \otimes \mathbb{C}[x_i]$$

by the rule

$$Y[s] \mapsto \sum_{i=1}^n Y^{(i)} \otimes x_i^s + (Y[-s])^{(\infty)} \otimes 1 \quad (Y \in \mathfrak{g}, s \geq 0).$$

- The coinvariants space

$$F_{\kappa}(M) = H_0(\mathfrak{g}[t], (V^*)^{\otimes n} \otimes M \otimes \mathbb{C}[x_i])$$

has a natural $\mathbb{C}[x_i] \rtimes \mathbb{C}S_n$ -module structure.

- This action is extended to an $\mathcal{H}_{\kappa+n}$ -action using the KZ connection:

$${}^{\kappa}\nabla_i := -(\kappa + n)\partial_{x_i} + \mathbf{L}_{-1}^{(i)} \quad (1 \leq i \leq n)$$



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Suzuki functor

Explicitly:

$$\mathbf{L}_{-1}^{(i)} = - \sum_{1 \leq j \neq i \leq n} \frac{\Omega^{(i,j)}}{x_i - x_j} + \sum_{p \geq 0} x_i^p \sum_{1 \leq k, l \leq n} e_{kl}^{(i)} e_{lk} [p+1]^{(\infty)}$$

$$y_i \mapsto {}^\kappa \nabla_i + \text{adjustment (Dunkl embedding)}$$

Observation: (1) this makes sense for any level $\kappa \in \mathbb{C}$.

(2) One can extend the definition to all \hat{U}_κ -modules.

Basic properties

- The functor is right-exact.

We have Verma modules $\mathbb{M}(\lambda) = \text{Ind}_{\hat{\mathfrak{b}}_+}^{\hat{\mathfrak{g}}} \mathbb{C}_\lambda$ and Weyl modules

$\mathbb{V}(\lambda) = \text{Ind}_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} L(\lambda)$ for $\hat{\mathfrak{g}}$. There are also Verma modules

$\Delta(\lambda) = \text{Ind}_{\mathbb{C}[x_i] \rtimes \mathbb{C}S_n}^{\mathcal{H}_t} S(\lambda)$ for the rational Cherednik algebra.

- $F_\kappa: \mathbb{M}(\lambda), \mathbb{V}(\lambda) \mapsto \Delta(\lambda)$



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Relationship between the centres



- **Question:** What is the relationship between the two centres when $\kappa = -n$ and $t = 0$?
- **Problem:** A functor does not in general induce a map between centres of categories.
- However, we have maps

$$Z(\mathcal{C}_{-n}) \xrightarrow{\alpha} \text{End}(F_{-n}) \xleftarrow{\beta} Z(\mathcal{H}_0)$$

- Question 1: What can we say about the “F-centre” of \mathcal{C}_{-n} :

$$Z_F(\mathcal{C}_{-n}) = \alpha^{-1}(Z(\mathcal{H}_0))?$$

Question 2: Can we find a subcategory $\mathcal{A} \subset \mathcal{C}_{-n}$ such that

$$Z(\mathcal{C}_{-n}) \xrightarrow{\alpha|_{\mathcal{A}}} \text{End}(F_{-n}|_{\mathcal{A}}) \xleftarrow{\beta} Z(\mathcal{H}_0), \quad \text{Im } \alpha|_{\mathcal{A}} = Z(\mathcal{H}_0).$$

- In practice, we want, for all $M \in \mathcal{A}$:

$$\begin{array}{ccc} Z(\mathcal{C}_{-n}) & \xrightarrow{\alpha|_{\mathcal{A}}} & Z(\mathcal{H}_0) \\ \text{can} \downarrow & & \downarrow \text{can} \\ \text{End}(M) & \xrightarrow{F_{-n}} & \text{End}(F_{-n}(M)) \end{array}$$



Relationship between the centres

Theorem (P.)

We have:

$$\mathbb{C}[\mathbf{S}_{2,\langle r+1 \rangle}, \text{id}[r]]_{r \leq 0} \subseteq Z_F(\mathcal{C}_{-n}).$$

The induced map $\mathbb{C}[\mathbf{S}_{2,\langle r+1 \rangle}, \text{id}[r]]_{r \leq 0} \rightarrow Z(\mathcal{H}_0)$ is Poisson.

- We construct a $\hat{\mathfrak{g}}$ -module \mathbb{H} such that $F_\kappa(\mathbb{H})$ is isomorphic to the regular representation of $\mathcal{H}_{\kappa+n}$.
- Take \mathcal{A} to be the category projectively generated by \mathbb{H} . It contains (generalized) Verma and Weyl modules.
- We get maps

$$Z(\mathcal{C}_{-n}) \rightarrow \text{End}(\mathbb{H}) \rightarrow \mathcal{H}_0^{\text{op}}.$$

- It is relatively easy to show the image lies in the centre of $\mathcal{H}_0^{\text{op}}$.
- To show that the image equals $Z(\mathcal{H}_0^{\text{op}})$, we explicitly compute the associated graded map with respect to a certain non-standard filtration, using the Chervov-Molev description of Segal-Sugawara operators.

Theorem (P.)

The Suzuki functor F_{-n} , restricted to the category projectively generated by \mathbb{H} , induces a surjective algebra homomorphism $Z(\mathcal{C}_{-n}) \twoheadrightarrow Z(\mathcal{H}_0)$.



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Some implications



Corollary (P.)

- The functor F_{-n} induces surjective ring homomorphisms:

$$F_{-n}: \text{End}(\mathbb{W}(\lambda)) \twoheadrightarrow \text{End}(\Delta(\lambda))$$

for $\lambda \vdash n$. The same holds for Verma modules and generalized Weyl modules.

- Every simple \mathcal{H}_0 -module is in the image of the functor F_{-n} .
- Restricted Verma modules (defined by Arakawa-Fiebig) and their simple quotients are sent to the corresponding simple modules over \mathcal{H}_0 .

Some implications



- We get an embedding $\mathcal{CM}_n \hookrightarrow \mathrm{Op}_{GL_n}(D^\times)$.
- The image lies inside of opers of singularity of order at most two.
- We have a map

$$\pi: \mathcal{CM}_n \rightarrow \mathrm{Spec} \mathbb{C}[y_1, \dots, y_n]^{S_n}$$

Each fibre $\pi^{-1}(a)$ is a disjoint union of affine cells, which can be identified with supports of generalized Verma modules.

- $\mathrm{Spec} \mathrm{End}(\Delta(\lambda)) \hookrightarrow$ opers with trivial monodromy and residue λ .

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