

Title: Relative critical loci, quiver moduli, and new lagrangian subvarieties

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Collection: Geometric Representation Theory

Date: June 25, 2020 - 11:15 AM

URL: <http://pirsa.org/20060045>

Abstract: The preprojective algebra of a quiver naturally appears when computing the cotangent to the quiver moduli, via the moment map. When considering the derived setting, it is replaced by its differential graded (dg) variant, introduced by Ginzburg. This construction can be generalized using potentials, so that one retrieves critical loci when considering moduli of perfect modules.

Our idea is to consider some relative, or constrained critical loci, deformations of the above, and study Calabi--Yau structures on the underlying relative versions of Ginzburg's dg-algebras. It yields for instance some new lagrangian subvarieties of the Hilbert schemes of points on the plane.

This reports a joint work with Damien Calaque and Sarah Scherotzke
arxiv.org/abs/2006.01069

Relative critical loci, quiver moduli, and new lagrangian subvarieties

Tristan Bozec (joint work with Damien Calaque and Sarah Scherotzke)

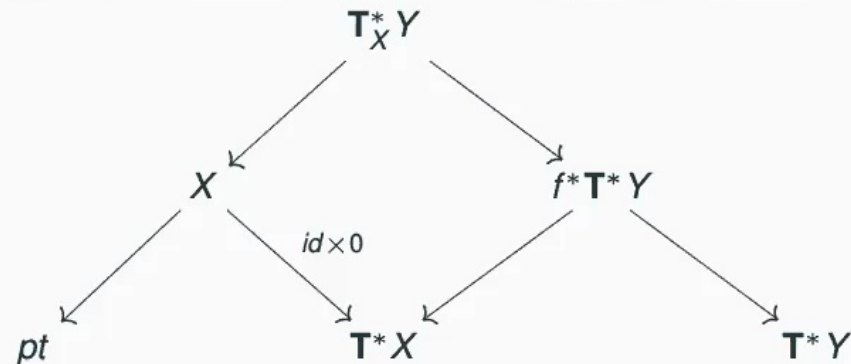
June 25, 2020

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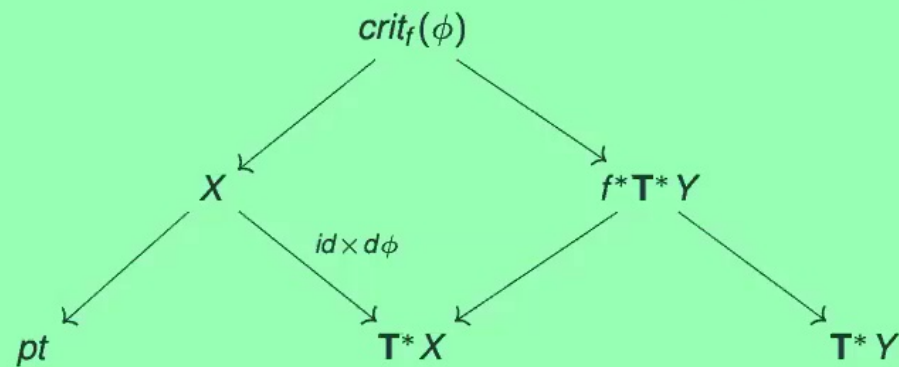
Symplectic motivation

$f : X \rightarrow Y$ morphism (of derived Artin k -stacks).

The conormal stack is the following composition of lagrangian correspondences :

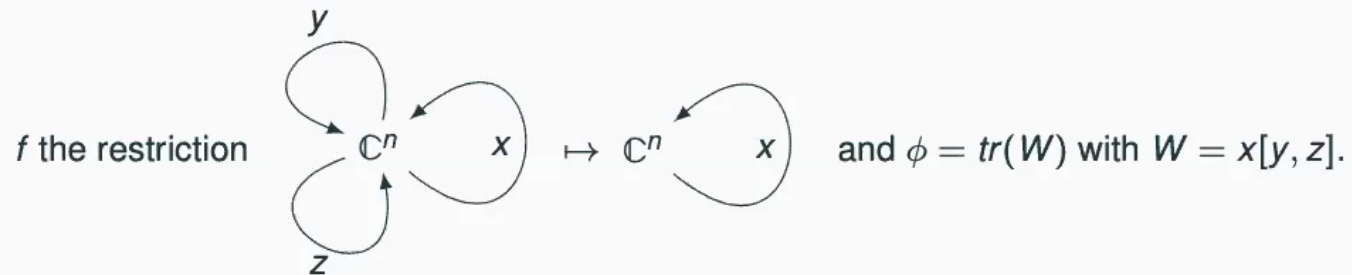


Definition : Relative (constrained) critical locus of $\phi : X \rightarrow \mathbb{A}_k^1$:



Example : $Y = pt$, usual $crit(\phi)$.

Example I



Then

$$\text{crit}_f(\phi) = \left\{ (x, x^*, y, z) \in \text{End}(\mathbb{C}^n)^4 \left| \begin{array}{l} [y, z] = x^* \\ [z, x] = 0 \\ [x, y] = 0 \end{array} \right. \right\}.$$

Image in T^*Y

$$\Lambda_n = \left\{ (x, [y, z]) \in \text{End}(\mathbb{C}^n)^2 \left| \begin{array}{l} [z, x] = 0 \\ [x, y] = 0 \end{array} \right. \right\}$$

is lagrangian !

Remark : differs from $T_{\mathcal{N}}^* \text{End}(\mathbb{C}^n)$ although irreducible components are parametrized by partitions of n .

Example II : Yang-Mills

$$f : \begin{array}{c} y \\ \curvearrowright \\ \mathbb{C}^n \end{array} \begin{array}{c} \curvearrowright \\ x \end{array} \mapsto \begin{array}{c} \curvearrowright \\ \mathbb{C}^n \end{array} x \text{ and } \phi = \text{tr}(W) \text{ with } W = x[y, [y, x]].$$

Then

$$\text{crit}_f(\phi) = \left\{ (x, x^*, y) \in \text{End}(\mathbb{C}^n)^3 \left| \begin{array}{l} [y, [y, x]] = x^* \\ [x, [x, y]] = 0 \end{array} \right. \right\}$$

and its image in T^*Y

$$\Lambda_n = \left\{ (x, [y, [y, x]]) \in \text{End}(\mathbb{C}^n)^2 \mid [x, [x, y]] = 0 \right\}$$

is lagrangian (?)

Question : what about quotient stacks ?

Ginzburg algebras

$Q = (Q_0, Q_1)$ quiver

Proposition (B-Calaque-Scherotzke)

If $X = \mathbf{Perf}_{kQ}$, then $T^*X = \mathbf{Perf}_{\mathcal{G}_2(kQ)}$ where $\mathcal{G}_2(kQ) := T_{kQ}(kQ^\vee[1])$ is generated by t in degree -1 , $k\bar{Q}$ in degree 0 , with differential $\partial(t) = \sum_{h \in Q_1} [h, h^*]$. The symplectic structure on $\mathbf{Perf}_{\mathcal{G}_2(kQ)}$ induced by the 2CY one on $\mathcal{G}_2(kQ)$ matches the standard one on T^*X .

A potential defines a function $X \xrightarrow{\text{tr}(W)} \mathbb{A}^1$, as well as a deformed (3)CY completion [Ginzburg/Keller] $\mathcal{G}_3(kQ, W)$ defined as the pushout

$$\begin{array}{ccc} \mathcal{G}_3(kQ, W) & \longleftarrow & kQ \\ \uparrow & & \uparrow [id, 0] \\ kQ & \xleftarrow{[id, dW]} & \mathcal{G}_2(kQ) \end{array}$$

where $dW(h^*) = \partial_h W$, $h \in Q_1$.

Fact

It is generated by t in degree -2 , Q_1^* in degree -1 , and Q_1 in degree 0 . We have $\text{crit}(\text{tr}(W)) = \mathbf{Perf}_{\mathcal{G}_3(kQ, W)}$.

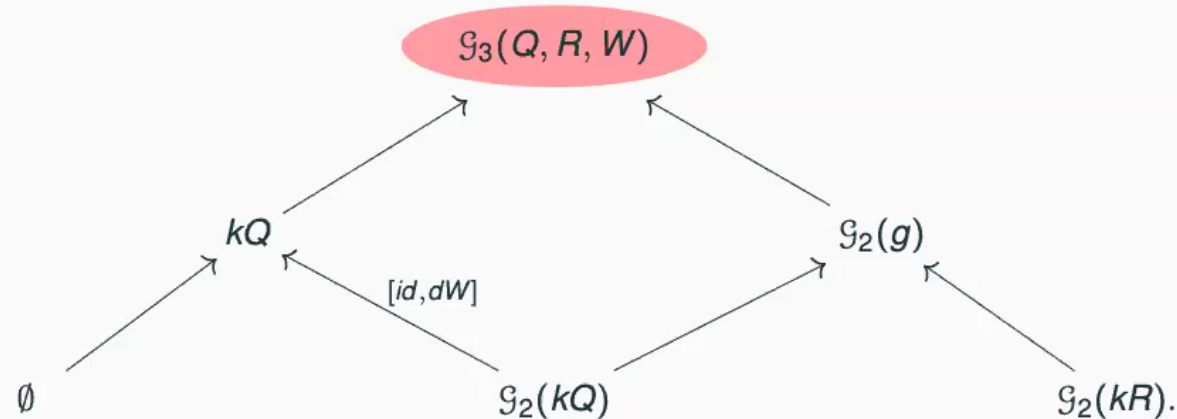
From CY cospan to lagrangian correspondences

To $g : kR \rightarrow kQ$ we associate a 2CY structure on the cospan $\mathcal{G}_2(kR) \rightarrow \mathcal{G}_2(g) := T_{kQ}(kR^\vee[1] \otimes_{kR^e} kQ^e) \leftarrow \mathcal{G}_2(kQ)$.

Theorem (BCS)

The induced lagrangian structure on **Perfs** matches the standard one on $\mathbf{T}^*\mathbf{Perf}_{kR} \leftarrow f^*\mathbf{T}^*\mathbf{Perf}_{kQ} \rightarrow \mathbf{T}^*\mathbf{Perf}_{kQ}$, where $f = g^*$.

Putting things together, we get $\text{crit}_f(\text{tr}(W))$ as the **Perf** of the composition of cospan



Corollary

The induced lagrangian structure on $\text{crit}_f(\text{tr}(W)) \rightarrow \mathbf{T}^*\mathbf{Perf}_{kR}$ is the standard one.

From derived to nonderived : example

Consider g the inclusion $R = \bullet \xrightarrow{v} \bullet \xrightarrow{x} \bullet \subset \bullet \xrightarrow{v} \bullet \xrightarrow{x} \bullet \xrightarrow{y} \bullet \xrightarrow{z} \bullet = Q$

together with $W = x[y, z]$.

Set $Y = [\mathbf{dRep}(kR, (n, 1)) / \mathrm{PGL}_{n,1}(k)]$, open substack of $\mathbf{Perf}_{kR}$.

Fact

The Hilbert scheme of n points on the plane

$$(\mathbb{C}^2)^{[n]} = \left\{ \mathbb{C} \xrightarrow{v} \mathbb{C}^n \begin{array}{l} \xrightarrow{y} \bullet \\ \xrightarrow{x} \bullet \end{array} \mid \begin{array}{l} [x, y] = 0 \\ \mathbb{C}[x, y] \cdot v = \mathbb{C}^n \end{array} \right\} / \mathrm{GL}_n(\mathbb{C}) \simeq (\tau_0 \mathbf{T}^* Y)^{st}$$

Write $f = g^*$, and $\Lambda_{n,1} = \tau_0 \left(\mathrm{crit}_f(\mathrm{tr}(W)) \times_{\mathbf{T}^* Y} (\mathbb{C}^2)^{[n]} \right)$.

Corollary

$\Lambda_{n,1} = \{[x, y, v] \mid y \in [\mathfrak{g}_x, \mathfrak{g}_x]\} \subset (\mathbb{C}^2)^{[n]}$ is lagrangian.

More lagrangian subvarieties

Problem

Better behaved lagrangian ? (factorization structure, coha etc)

Idea : “saturate” with respect to the Hilbert-Chow morphism $\rho : (\mathbb{C}^2)^{[n]} \rightarrow (\mathbb{C}^2)^{(n)}$ mapping $[x, y, v]$ to the joint spectrum of (x, y) .

Proposition

Set $L_{n,1} = \rho^{-1}(\rho(\Lambda_{n,1}))$. Then

$$L_{n,1} = \left\{ [x, y, v] \mid \text{tr}(y_E) = 0 \quad \forall \text{ generalized eigenspace } E \text{ of } x \right\} \subset (\mathbb{C}^2)^{[n]}$$

is lagrangian.

▷ more irreducible components

▷ factorization property : consider disjoint open subsets $\mathcal{U}, \mathcal{V} \subset \mathbb{C}^2$ and set $L_{n,1}(\mathcal{U}) = L_{n,1} \cap \rho^{-1}(\mathcal{U})$. Up to some principal bundles we have

$$L_{n,1}(\mathcal{U} \sqcup \mathcal{V}) \simeq L_{n,1}(\mathcal{U}) \times L_{n,1}(\mathcal{V}).$$

▷ COHA structure ? characteristic cycle ?