Title: K-Motives and Koszul Duality

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Abstract: Koszul duality, as conceived by Beilinson-Ginzburg-Soergel, describes a remarkable symmetry in the representation theory of Langlands dual reductive groups. Geometrically, Koszul duality can be stated as an equivalence of categories of mixed (motivic) sheaves on flag varieties. In this talk, I will argue that there should be an an 'ungraded' version of Koszul duality between monodromic constructible sheaves and equivariant K-motives on flag varieties. For this, I will explain what K-motives are and present preliminary results.

$pt = \mathrm{Spec}\,\overline{\mathbb{F}}_p$, coeff. $= \mathbb{Q}$

K-motives and Koszul duality

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split alg. torus $\boxed{T \xleftrightarrow{\text{duality}} T^\vee}$ dual torus

character lattice $\boxed{X(T) = Y(T^\vee)}$ cocharacter lattice

$R := \mathbb{Q}[X(T)]$

T-equivariant alg. K-theory $\boxed{K_0^T(pt) = R = \mathbb{Q}[\pi_1(T^\vee(\mathbb{C}))]}$ group algebra of fundamental group

constant T-eq. K-motives $\boxed{DK^T_{const}(pt) = D^b(R\text{-mod}) = D^b(LC(T^\vee(\mathbb{C})))}$ der. cat. of locally const. sheaves

Today:
* K-motives ??
* $T \rightsquigarrow G$

II K-motives

(constructible sheaves : singular cohomology)

$=$

(K-motives : alg. K-theory)

variety $\boxed{X \longmapsto DK(X)}$ triangulated category of K-motives on X

* 6 functor formalism $(f^*, f_*, f_!, f^!, \otimes, \mathcal{H}om)$
* equivariant version $DK^T(X)$, for $T \curvearrowright X$

for X regular $\boxed{\mathrm{Hom}_{DK^T(X)}(\mathbb{Q}, \mathbb{Q}[n]) = K_n^T(X)}$ T-equivariant (higher) alg. K-theory

III "Intersection K-theory complexes"

$$\boxed{DK^T_{const}(X) \overset{def.}{=} \langle \mathbb{Q} \rangle_\Delta \subset DK^T(X)}$$

$$\boxed{\begin{array}{c} DK^T_{const}(pt) = D^b(R\text{-mod}) \\ \mathbb{Q} \longleftrightarrow R \end{array}}$$

$\operatorname{Hom}_{DK^T(pt)}(\mathbb{Q}, \mathbb{Q}[n]) = K^T_n(pt) = \begin{cases} R & n=0 \\ 0 & \text{else} \end{cases}$

$$\boxed{DK^T_{\mathcal{J}}(X) \overset{def}{=} \{ M \mid M|_{X_s} \text{ constant} \}}$$

$X = \biguplus_{s \in \mathcal{J}} X_s$, $X_s \cong \mathbb{A}^n$

affinely stratified

+ assumptions

$\Rightarrow$

Thm (E.)

* $DK^T_{\mathcal{J}}(X)$ has "weight structure"
* $DK^T_{\mathcal{J}}(X)_{w=0}$ pure K-motives
* $DK^T_{\mathcal{J}}(X) \xrightarrow{\sim} K^b(DK^T_{\mathcal{J}}(X)_{w=0})$

analogous to

* intersection coh. complexes
* parity sheaves

"Intersection K-th. complexes"

IV Koszul duality à la Beilinson–Ginzburg–Soergel

$$T \subset G \xleftrightarrow[\text{dual}]{\text{Langlands}} T^\vee \subset G^\vee$$

mixed sheaves: * Tate twist (1)
* "graded version" of constr. sheaves

$D^b(\mathcal{O}_0^{\mathbb{Z}}(\mathfrak{g}))$
graded cat. $\mathcal{O}$ of Lie($G(\mathbb{C})$)

$- \otimes_{\mathbb{Q}} \mathbb{C}$

$$D^{mix}_{(B)}(G/B) \xleftrightarrow[\text{duality}]{\text{Koszul}} D^{mix}_{(B^\vee)}(G^\vee/B^\vee)$$

IC-complex $\longleftrightarrow$ perverse tilting sheaf

$(1)[2] \longleftrightarrow (1)$

constant along B-orbits (Bruhat cells)

Not true without mix!!

V Ungraded Koszul duality (E.)

$$
\begin{tikzcd}
 & (1)[2] \arrow[r, leftrightarrow] & (1) & \\
(1)[2] \arrow[d] & D^{mix}_{(B)}(\mathcal{G}/B) \arrow[r, leftrightarrow] \arrow[d, "\text{real}"] & D^{mix}_{(B^\vee)}(\mathcal{G}^\vee/B^\vee) \arrow[d, "\text{real}"] & (1) \arrow[d] \\
\mathrm{id} & ??? \arrow[r, leftrightarrow] & D_{(B^\vee)}(\mathcal{G}^\vee/B^\vee) \arrow[d, dash] & \mathrm{id} \\
 & & \boxed{\begin{array}{c}\text{constructible sheaves} \\ \text{on } (\mathcal{G}^\vee/B^\vee)(\mathbb{C})^{an}\end{array}} &
\end{tikzcd}
$$

V Ungraded Koszul duality (E.)

Thm (E.)

$$
\begin{tikzcd}
(1)[2] \arrow[d] & D^{mix}_{(B)}(\mathcal{G}/B) \arrow[r, leftrightarrow, "{(1)[2] \leftrightarrow (1)}"] \arrow[d, "real"] & D^{mix}_{(B^\vee)}(\mathcal{G}^\vee/B^\vee) \arrow[d, "real"] & (1) \arrow[d] \\
id & DK_{(B)}(\mathcal{G}/B) \arrow[r, leftrightarrow] & D_{(B^\vee)}(\mathcal{G}^\vee/B^\vee) & id
\end{tikzcd}
$$

$H^{i,j}(\mathbb{P}^1) = \mathbb{Q}, \mathbb{Q}$

$K_i(\mathbb{P}^1) = \mathbb{Q} \oplus \mathbb{Q}$

constructible sheaves on $(\mathcal{G}^\vee/B^\vee)(\mathbb{C})^{an}$

VI Work in progress

THANK YOU

Thm/Conj. [E.]

$$\begin{tikzcd}
DK^{T}_{(B)}(G/B) \arrow[rr, leftrightarrow] \arrow[dr, "\sim", "\text{weight complex functor}"'] & & D_{T^\vee\text{-mon}}(G^\vee/U^\vee) \\
& K^b(R\text{-SBim}) \arrow[ur, "\sim"', "\text{realisation functor}"] &
\end{tikzcd}$$

K-theory Soergel bimodules

etc . . .

THANK YOU