Title: Perverse sheaves and the cohomology of regular Hessenberg varieties

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Collection: Geometric Representation Theory

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Abstract: Hessenberg varieties are a distinguished family of projective varieties associated to a semisimple complex algebraic group. We use the formalism of perverse sheaves to study their cohomology rings. We give a partial characterization, in terms of the Springer correspondence, of the irreducible representations which appear in the action of the Weyl group on the cohomology ring of a regular semisimple Hessenberg variety. We also prove a support theorem for the universal family of regular Hessenberg varieties, and we deduce that its fibers, though not necessarily smooth, always have the "Kahler package". This is joint work with Peter Crooks.

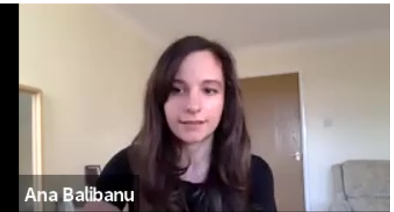
Perverse sheaves & the cohomology of regular Hessenberg varieties

jt. w/ P. Crooks

Geometric Representation Theory

Perimeter Institute / MPI Bonn

June 2020



G semisimple, simply-connected alg group $/\mathbb{C}$

$T \subset B \subset G$ max torus $\subset$ Borel

$\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g} = \mathrm{Lie}\, G$

Def: $H \subset \mathfrak{g}$ is a Hessenberg subspace if
- $[\mathfrak{b}, H] \subset H$
- $\mathfrak{b} \subseteq H$

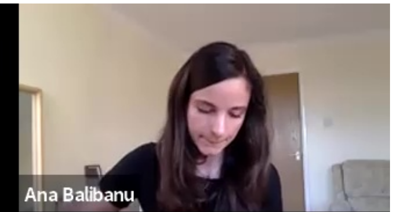
Ex: - all std parabolics

- $\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \subset \mathfrak{sl}_3$

$\rightsquigarrow$ proper family $\mu : G \times_B H \longrightarrow \mathfrak{g}$

$[g : x] \longmapsto g \cdot x$

Poisson variety w/ moment map

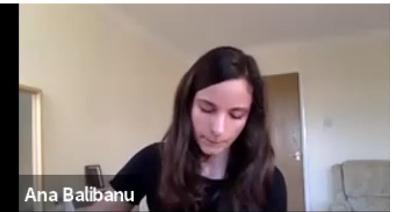


Fix H $\rightsquigarrow$
$$\mu : G \times_B H \longrightarrow \mathfrak{g}$$
$$[g:x] \longmapsto g\cdot x$$

Def: The <u>Hessenberg variety</u> associated to $x \in \mathfrak{g}$ is

$$\mathrm{Hess}(x) = \mu^{-1}(x)$$
$$= \{ gB \in G/B \mid g^{-1}\cdot x \in H \}$$

Rks:
- $x \in \mathfrak{g}^{rs} \Rightarrow \mathrm{Hess}(x)$ is smooth — $\mu|_{rs}$ smooth
- $x \in \mathfrak{g}^{r} \Rightarrow \dim \mathrm{Hess}(x) = \dim H/\mathfrak{b}$ — $\mu|_{r}$ flat



Ex 1: $H = \mathfrak{b} \rightsquigarrow \tilde{\mathfrak{g}} = G \times_B \mathfrak{b} \longrightarrow \mathfrak{g}$ Grothendieck - Springer

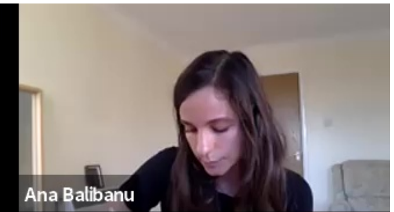
$$\mathrm{Hess}(x) = \mathcal{B}_x = \{ gB \in G/B \mid x \in g \cdot \mathfrak{b} \}$$

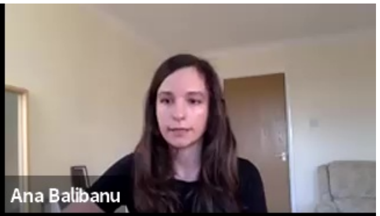
Rk: $\forall$ Hessenberg H, $\mathfrak{b} \subset H \Rightarrow$

$$\begin{tikzcd} G \times_B \mathfrak{b} \arrow[r, hook] \arrow[dr] & G \times_B H \arrow[d] \\ & \mathfrak{g} \end{tikzcd}$$

$\Rightarrow \mathcal{B}_x \hookrightarrow \mathrm{Hess}(x)$

$x \in \mathfrak{g}^r \Rightarrow \mathcal{B}_x = H \cdot$





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$$\Rightarrow \mathcal{B}_x \hookrightarrow \mathrm{Hess}(x)$$

$$x \in \mathfrak{g}^r \Rightarrow \mathcal{B}_x = \mathrm{Hess}(x)^{G^x}$$

Ex 2: $H = \mathfrak{b} \oplus \left(\sum_{\alpha \in \Delta} \mathfrak{g}_{-\alpha}\right)$, $\Delta = \{\text{simple roots}\}$

$\Rightarrow \forall x \in \mathfrak{g}^r$, • $\dim \mathrm{Hess}(x) = \mathrm{rk}\, \mathfrak{g}$

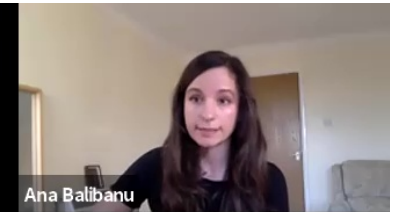
• $G^x \curvearrowright \mathrm{Hess}(x)$ w/ open dense orbit

• $s \in \mathfrak{g}^{rs} \Rightarrow G^s = T$ max torus

Hess(s) smooth proj toric variety

fan of Weyl chambers

• $e \in \mathfrak{g}^r$, nilpotent $\Rightarrow \mathrm{Hess}(e) =$ Peterson variety



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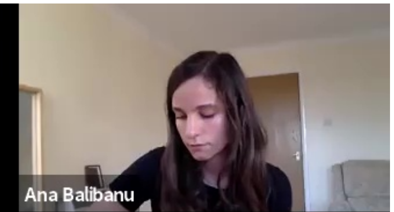
singular, not nor



Back to general H — $\underline{\mu : G \times_B H \longrightarrow \mathfrak{g}, \quad \mathrm{Hess}(x) = \mu^{-1}(x)}$

Goal: Understand the geometry of $\mathrm{Hess}(x)$.

$\underline{s \in \mathfrak{g}^{rs}} \rightsquigarrow T = G^s, \quad W = N_G(T)/T$

$T \circlearrowright \mathrm{Hess}(s)$ is a $\underline{\text{GKM variety}}$ $\Big\langle$ $\#\{\text{fixed pts}\} < \infty$, $\#\{1\text{-dim'l orbits}\} < \infty$

$$\Rightarrow \begin{array}{ccc} H_T^*(\mathrm{Hess}(s)) & \hookrightarrow & H_T^*(\mathrm{Hess}(s)^T) = H_T^*(\mathcal{B}_s) \\ \downarrow & & \cong \\ & & \bigoplus_{w \in W} H_T^*(pt) \\ & & \cong \\ H^*(\mathrm{Hess}(s)) & & \bigoplus_{w} \end{array}$$

(*)

$$\begin{tikzcd} H_T^*(\mathrm{Hess}(s)) \arrow[r] \arrow[d] & H_T^*(\mathrm{Hess}(s)^T) = H_T^*(B_s) \\ H^*(\mathrm{Hess}(s)) & \end{tikzcd}$$

$$H_T^*(B_s) \cong \bigoplus_{w \in W} \mathbb{C}[t]$$

- $W \curvearrowright \mathbb{C}[t]$
- permutes summands

Tymoczko dot-action: $W \curvearrowright H_T^*(B_s)$

$$u \cdot (f_w)_{w \in W} = (f_{uw} \circ u^{-1})_{w \in W}$$

$\xrightarrow{[\mathrm{Tym}]}$ $W \curvearrowright H^*(\mathrm{Hess}(s))$ through (*)

$W \curvearrowright H^*(\mathrm{Hess}(s))$

Question: Which W-irreps appear? (Known in type A)

Recall: $\mu : G \times_B H^{rs} \longrightarrow \mathfrak{g}^{rs}$ is a smooth fibration,

and $\mathrm{Hess}(s) = \mu^{-1}(s)$.

Observation: [Brosnan - Chow]

$$\begin{tikzcd} \pi_1(\mathfrak{g}^{rs}, s) \arrow[drr] & \curvearrowright & H^*(\mathrm{Hess}(s)) \\ & & W \arrow[u, phantom, "\circlearrowleft"] \end{tikzcd}$$

Observation: [Brosnan - Chow] $\pi_1(\mathfrak{g}^{rs}, s) \curvearrowright H^*(\mathrm{Hess}(s))$

$$\pi_1(\mathfrak{g}^{rs}, s) \twoheadrightarrow W \curvearrowright H^*(\mathrm{Hess}(s))$$

Decomposition thm

$$R\mu_* \underline{\mathbb{C}}_{G \times_B H} \cong \left[\bigoplus_{\psi \in \mathrm{Irr} W} \mathcal{IC}_{\mathfrak{g}^{rs}}(\mathcal{L}_\psi^{\oplus m})[b] \right] \oplus \left[\bigoplus \mathcal{IC}_S(\mathcal{L}^{\oplus m})[b] \right]$$

smaller strata

Question redux: Which irred local systems $\mathcal{L}_\psi$ appear in this decomposition?

Observation: [Brosnan - Chow] $\pi_1(\mathfrak{g}^{rs}, s) \curvearrowright H^*(\mathrm{Hess}(s))$ $\circlearrowleft$ W

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Springer correspondence: $\mathrm{Irr}\, W \hookrightarrow \left\{ (\mathcal{O}, \mathcal{M}) \;\middle|\; \begin{array}{l} \mathcal{O} \subset \mathfrak{g} \text{ nilp orbit} \\ \mathcal{M} \text{ loc sys on } \mathcal{O} \end{array} \right\}$

$$\chi_\psi \longmapsto (\mathcal{O}_\psi, \mathcal{M}_\psi)$$

Let $H^\perp$ = annihilator of H under Killing form

$\mathfrak{b} \subset H \Rightarrow H^\perp = [\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$

Thm: [B.-Crooks] If $\psi \in \mathrm{Irr}\, W$ appears in $W \curvearrowright H^*(\mathrm{Hess}(s))$,

then $\mathcal{O}_\psi \cap H^\perp \neq \emptyset$.

$\mu : G \times_B H^\Gamma \longrightarrow \mathfrak{g}^\Gamma$ proper flat family

$x \in \mathfrak{g}^\Gamma \rightsquigarrow x = x_s + x_n$ Jordan decomp

wlog $x_s \in \mathfrak{t}$

$\rightsquigarrow W^{x_s} = \mathrm{Stab}_W(x_s)$

$s \in \mathfrak{t}^\Gamma$ "nearby x"

$\rightsquigarrow$ local invariant cycle map

$$\lambda : H^*(\mathrm{Hess}(x)) \longrightarrow\!\!\!\!\rightarrow H^*(\mathrm{Hess}(s))^{\pi_1^{loc}}$$

$$H^*(\mathrm{Hess}(s))^{\pi_1^{loc}} = H^*(\mathrm{Hess}(s))^{W^{x_s}}$$

Prop: [Brosnan - Chow in type A, but holds generally]

$\lambda : H^*(\mathrm{Hess}(x)) \longrightarrow H^*(\mathrm{Hess}(s))^{W^{x_s}}$ is an isomorphism.

Thm: [B. - Crooks] $\forall x \in \mathfrak{g}^r$, $H^*(\mathrm{Hess}(x))$ has the "Kähler package":

1) Poincaré duality

2) Hard Lefschetz property

3) Hodge - Riemann relations