Title: Geometric class field theory and Cartier duality

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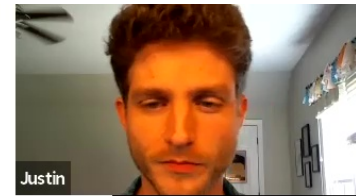
Abstract: I will explain a generalized Albanese property for smooth curves, which implies Deligne's geometric class field theory with arbitrary ramification. The proof essentially reduces to some well-known Cartier duality statements. This is joint work with Andreas Hayash.



Geometric class field theory and Cartier duality

Justin Campbell (Caltech)
Joint with Andreas Hayash (UMass)

Geometric Representation Theory, June 2020

The Jacobian

$\overline{X}$ = smooth projective closure of X
I = set of closed points of $\overline{X}$ not contained in X

The *Jacobian* of X is the commutative group stack $\mathrm{Pic}(X, \partial X)$ parameterizing line bundles on $\overline{X}$ equipped with a trivialization over the formal completion of each $x \in I$.

The Abel-Jacobi map

The *Abel-Jacobi map* is given by the formula

$$\begin{aligned} X &\longrightarrow \mathrm{Pic}(X, \partial X) \\ x &\mapsto (\mathcal{O}_{\overline{X}}(x), 1), \end{aligned}$$

where 1 denotes the section of $\mathcal{O}_{\overline{X}}(x)$ determined by the canonical morphism

$$\mathcal{O}_{\overline{X}} \longrightarrow \mathcal{O}_{\overline{X}}(x).$$

Albanese property

It is known that for many commutative group stacks G, restriction along the Abel-Jacobi map induces an isomorphism

$$\mathrm{Hom}(\mathrm{Pic}(X, \partial X), G) \xrightarrow{\sim} \mathrm{Map}(X, G).$$

This is called the *Albanese property* of the Abel-Jacobi map.

Geometric class field theory

For example, take $G = BA$ to be the classifying stack in the étale topology of a finite abelian group A. Then $\mathrm{Map}(X, BA)$ consists of étale A-local systems on X, and hence can be identified with

$$\mathrm{Hom}_{\mathrm{cts}}(\pi_1(X), A) = \mathrm{Hom}_{\mathrm{cts}}(\pi_1(X)^{\mathrm{ab}}, A).$$

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On the other hand $\mathrm{Hom}(\mathrm{Pic}(X, \partial X), BA)$ consists of *multiplicative* A-local systems on $\mathrm{Pic}(X, \partial X)$. A multiplicative structure on an A-local system $\mathcal{E}$ consists of isomorphisms

$$e^*\mathcal{E} \tilde{\to} k \qquad \text{and} \qquad m^*\mathcal{E} \tilde{\to} \mathcal{E} \boxtimes \mathcal{E}$$

satisfying natural associativity and unitality conditions, where e and m are the unit and the group operation on $\mathrm{Pic}(X, \partial X)$, respectively.

Geometric class field theory

Assume further that $k = \mathbb{F}_q$ is finite and write $J := \mathrm{Pic}(X, \partial X)$. The "trace of Frobenius" construction defines a map

$$\mathrm{Hom}(J, BA) \longrightarrow \mathrm{Hom}_{\mathrm{cts}}(J(\mathbb{F}_q), A),$$

which is in fact an isomorphism. Allowing A to vary, we formally obtain an isomorphism of profinite abelian groups

$$\widehat{J(\mathbb{F}_q)} \xrightarrow{\sim} \pi_1(X)^{\mathrm{ab}}.$$

The group $J(\mathbb{F}_q)$ is a suitable quotient of the idèle class group of the function field of X, and this isomorphism is precisely the Artin reciprocity map.

Internal Hom

For any prestacks Y and Z, we will write $\underline{\mathrm{Map}}(Y, Z)$ for the mapping space defined by the formula

$$\mathrm{Map}(S, \underline{\mathrm{Map}}(Y, Z)) := \mathrm{Map}(Y \times S, Z).$$

Similarly, given commutative group stacks G and H, their internal Hom is defined by

$$\mathrm{Map}(S, \underline{\mathrm{Hom}}(G, H)) := \mathrm{Hom}_S(G \times S, H \times S).$$

Cartier duality

The *Cartier* 1-*dual* of a commutative group stack G is defined by

$$G^\vee := \underline{\mathrm{Hom}}(G, B\mathbb{G}_m).$$

We say that G is 1-*reflexive* if the natural homomorphism $G \to G^{\vee\vee}$ is an isomorphism.

Albanese property in families

A naïve guess for a "parameterized" version of the Albanese property would be that the Abel-Jacobi map induces an isomorphism of commutative group stacks

$$\underline{\mathrm{Hom}}(\mathrm{Pic}(X, \partial X), G) \xrightarrow{\sim} \underline{\mathrm{Map}}(X, G)$$

for any 1-reflexive G. In fact, this is true for X proper, but not in general.

For example, take $X = \mathbb{A}^1$. In 1962 Bass exhibited a large class of affine schemes S for which there exist line bundles on $\mathbb{A}^1 \times S$ not pulled back from S. However, one can show in this case that any S-family of multiplicative line bundles on $\mathrm{Pic}(X, \partial X)$ is pulled back from S, for any affine scheme S.

Albanese property in families

A map $X \times S \to G$ will be called $B\mathbb{G}_m$-*extendable* if for any homomorphism $G \times S \to B\mathbb{G}_m \times S$ over S, the composition

$$X \times S \longrightarrow G \times S \longrightarrow B\mathbb{G}_m \times S$$

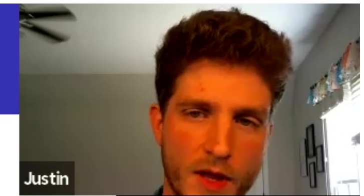
extends to $\overline{X} \times S$ locally on S.

We will write

$$\underline{\mathrm{Map}}(X, G)^{\mathrm{ext}} \subset \underline{\mathrm{Map}}(X, G)$$

for the substack consisting of $B\mathbb{G}_m$-extendable maps. Note that any line bundle on X extends to $\overline{X}$, so this inclusion is an isomorphism on k-points.

Albanese property in families

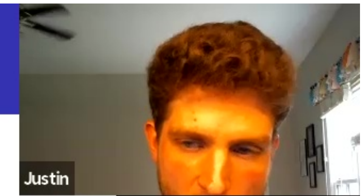


For example, consider the case $G = B\mathbb{G}_m$. Then the theorem says that restriction along the Abel-Jacobi map induces an isomorphism

$$\mathrm{Pic}(X, \partial X)^\vee \xrightarrow{\sim} \mathrm{Pic}(X)^{\mathrm{ext}},$$

where $\mathrm{Pic}(X)^{\mathrm{ext}}$ denotes the stack of extendable line bundles on X.

The proof



The proof involves two major reduction steps. The first is to reduce to the case $G = B\mathbb{G}_m$.

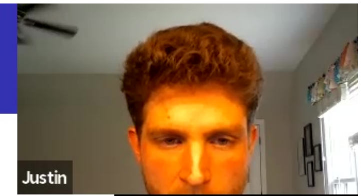
Namely, any commutative group stack G sits in a cofiber sequence

$$\bigoplus_j \mathbb{Z}[S_j] \longrightarrow \bigoplus_i \mathbb{Z}[S_i] \longrightarrow G,$$

where for any affine scheme S we write $\mathbb{Z}[S]$ for the free sheaf of abelian groups on S. Dualizing, we obtain a fiber sequence

$$G^\vee \longrightarrow \prod_i \underline{\mathrm{Map}}(S_i, B\mathbb{G}_m) \longrightarrow \prod_j \underline{\mathrm{Map}}(S_j, B\mathbb{G}_m).$$

The proof



Thus any 1-reflexive G can be built up from commutative group stacks of the form $\underline{\mathrm{Map}}(S, B\mathbb{G}_m)$ by taking products and fibers, which allows us to make the desired reduction.

We then prove and combine the following two more basic results:

- (unramified Albanese property) the special case of the theorem when $X = \overline{X}$ is projective,
- (local Albanese property) a version of the theorem with X replaced by a formal punctured disk.

The proof

For any closed point x in X, we write $D_x := \widehat{X}_x$ for the formal completion of X at x, and

$$\mathfrak{L}_x^+ \mathbb{G}_m := \mathrm{Map}(D_x, \mathbb{G}_m)$$

for the corresponding group of arcs in $\mathbb{G}_m$.

The second reduction is performed by exploiting the exact triangle

$$\prod_{x \in I} \mathfrak{L}_x^+ \mathbb{G}_m \longrightarrow \mathrm{Pic}(X, \partial X) \longrightarrow \mathrm{Pic}(\overline{X}).$$

The unramified case

We then prove and apply the following two lemmas.

Lemma

The commutative group stack $\mathrm{Rat}^{\vee}$ is trivial, whence the canonical map

$$\mathrm{Pic}^{\vee} \longrightarrow \mathrm{Gr}^{\vee}$$

is an isomorphism.

Note that the Abel-Jacobi map $X \to \mathrm{Pic}$ canonically lifts to $X \to \mathrm{Gr}$.

Lemma

Restriction along $X \to \mathrm{Gr}$ induces an isomorphism

$$\mathrm{Gr}^{\vee} = \underline{\mathrm{Hom}}(\mathrm{Gr}, B\mathbb{G}_m) \xrightarrow{\sim} \underline{\mathrm{Map}}(X, B\mathbb{G}_m) = \mathrm{Pic}\,.$$

The local case

Fix a closed point x in X. The role of the local Jacobian is played by the loop group $\mathfrak{L}\mathbb{G}_m$, defined by

$$\mathrm{Map}(\operatorname{Spec} A, \mathfrak{L}\mathbb{G}_m) := (A \widehat{\otimes} K_x)^{\times}.$$

Here K_x denotes the fraction field of the completed local ring of X at x.

Theorem (Contou-Carrère)

There is a canonical perfect pairing

$$\mathfrak{L}\mathbb{G}_m \times \mathfrak{L}\mathbb{G}_m \longrightarrow \mathbb{G}_m.$$

The local case

In keeping with the Albanese theme, we should mention that Contou-Carrère's pairing can be constructed using a kind of local Abel-Jacobi map. However, if we write $\mathring{D}_x := \operatorname{Spec} K_x$, then the expected map

$$\mathring{D}_x \longrightarrow \mathfrak{L}\mathbb{G}_m$$

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does not exist!

This is for the same reason that there is no nonconstant map $\mathring{D}_x \to D_x$, and the difficulty can be overcome in the same way: by replacing the formal component of $\mathfrak{L}\mathbb{G}_m$ by an affine scheme.

The local case

The resulting local Abel-Jacobi map induces a homomorphism

$$\underline{\mathrm{Hom}}(\mathfrak{L}\mathbb{G}_m, B\mathbb{G}_m) \longrightarrow \mathrm{Pic}(\mathring{D}_x)^{\mathrm{ext}} = B\mathfrak{L}\mathbb{G}_m,$$

and Contou-Carrère's result shows that this is an isomorphism on H^{-1} (i.e. on automorphisms of the unit object).

To see that this is an isomorphism on H^0, the main point is to show that any family of multiplicative line bundles on $\mathfrak{L}^+\mathbb{G}_m$ is trivial locally on the base. We deduce this from the following general vanishing result.

Lemma

For any commutative affine group scheme G, we have $\underline{\mathrm{Ext}}^1(G, \mathbb{G}_m) = 1$.