Title: Elliptic stable envelopes via loop spaces

Speakers: Michael McBreen

Collection: Geometric Representation Theory

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Abstract: Elliptic stable envelopes, introduced by Aganagic and Okounkov, are a key ingredient in the study of quantum integrable systems attached to a symplectic resolution. I will describe a relation between elliptic stable envelopes on a hypertoric variety and a certain 'loop space' of that variety. Joint with Artan Sheshmani and Shing-Tung Yau.

Wednesday, June 24, 2020 2:47 AM

Fun w/ hypertoric loops

jt w/ A. Sheshmani & S.T. Yau.

[Starting from a hypertoric variety X produce two non-finite type hypertorics PX & $\mathcal{L}X$, & use them to study enumerative geometry of X & (eventually) elliptic stable env. on X.].

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Conical Symplectic Resolutions: $X, \omega \circlearrowleft \mathbb{C}^\times \quad \omega \mapsto t\omega$

(CSR)

$$X, \omega \xrightarrow{\text{proper, birational}} \operatorname{Spec} \mathcal{O}(X)$$

Ex: $X = T^*G/B$, $\operatorname{Hilb}_n \mathbb{C}^2$.

Pick $\eta \in H^2(X, \mathbb{C})$. $\mathcal{O}(X) \overset{\text{quantize}}{\rightsquigarrow} \mathcal{U}_X^\eta$,

$\mathcal{U}_X^\eta$ shares many features with $\mathcal{U}\mathfrak{g}$ / central char

Given $A \curvearrowright X$ hamiltonian, $\sigma : \mathbb{C}^\times \to A$

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$$\mathrm{cat}\,\mathcal{O}_X^{\sigma,\eta} = \text{"}\mathcal{U}_X^\eta\text{-mod w/ }\sigma\text{-bounded } A \text{ wts."}$$

$$L \in \mathrm{cat}\,\mathcal{O}_X^{\sigma,\eta} \longrightarrow \mathrm{supp}(L) \subset \mathrm{Att}_\sigma \subset X$$

lagrangian contracted by σ.

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We say $X, X^!$ are symplectic duals if:

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Lagrangian contracted by σ.

We say $X, X^!$ are symplectic duals if:

$$\begin{tikzcd}
\tilde{X} \arrow[loop left] & A \arrow[l, bend right] \arrow[rrd, leftrightarrow] & & A^! \arrow[r, bend left] \arrow[lld, leftrightarrow] & \tilde{X}^! \\
& H_2(X, \mathbb{C}^\times) & & H_2(X^!, \mathbb{C}^\times) &
\end{tikzcd}$$

& given $\eta \in H^2(X, \mathbb{Z}) = \mathfrak{a}^!_{\mathbb{Z}}$, $\sigma \in \mathfrak{a}_{\mathbb{Z}} = H^2(X^!, \mathbb{Z})$

cat $\mathcal{O}_X^{\eta, \sigma}$ is Koszul dual to cat $\mathcal{O}_{X^!}^{-\sigma, -\eta}$

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X $\qquad$ $X^!$

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$$\begin{tikzcd}
X \arrow[d, phantom, "\wr"] & & X^! \arrow[d, phantom, "\wr"] \\
\text{eq. } A \arrow[drr, leftrightarrow] & & A^! \arrow[dll, leftrightarrow] \\
\text{Kähler } H_2(X, \mathbb{C}^\times) & & H_2(X^!, \mathbb{C}^\times)
\end{tikzcd}$$

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question: how does Koszul duality interact w/ enumerative geometry of X, $X^!$?

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question: how does Koszul duality interact w/ enumerative geometry of X & $X^!$?

Eg: let $Z_X = \sum_{\beta \in H_2(X,\mathbb{Z})} \# \left\{ \mathbb{P}^1 \xrightarrow{f} X \;\middle|\; f_*[\mathbb{P}^1] = \beta \right\} z^\beta$

Describe Z_X in terms of $X^!$.

[AO, BDHG, KMP, SZ, ...]

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· Hypertoric varieties / toric Hyperkähler varieties

$G \curvearrowright T^*V \qquad X := T^*V /\!/\!/_\theta G$ is a C.S.R.

torus $\quad \theta : G \to \mathbb{C}^\times$ generic.

- geometrize matroids.
- abelianize quiver varieties.
- subclass associated to graphs is particularly nice:

$$\Gamma \longrightarrow X(\Gamma) = T^*\mathbb{C}^E /\!/\!/ (\mathbb{C}^\times)^V.$$

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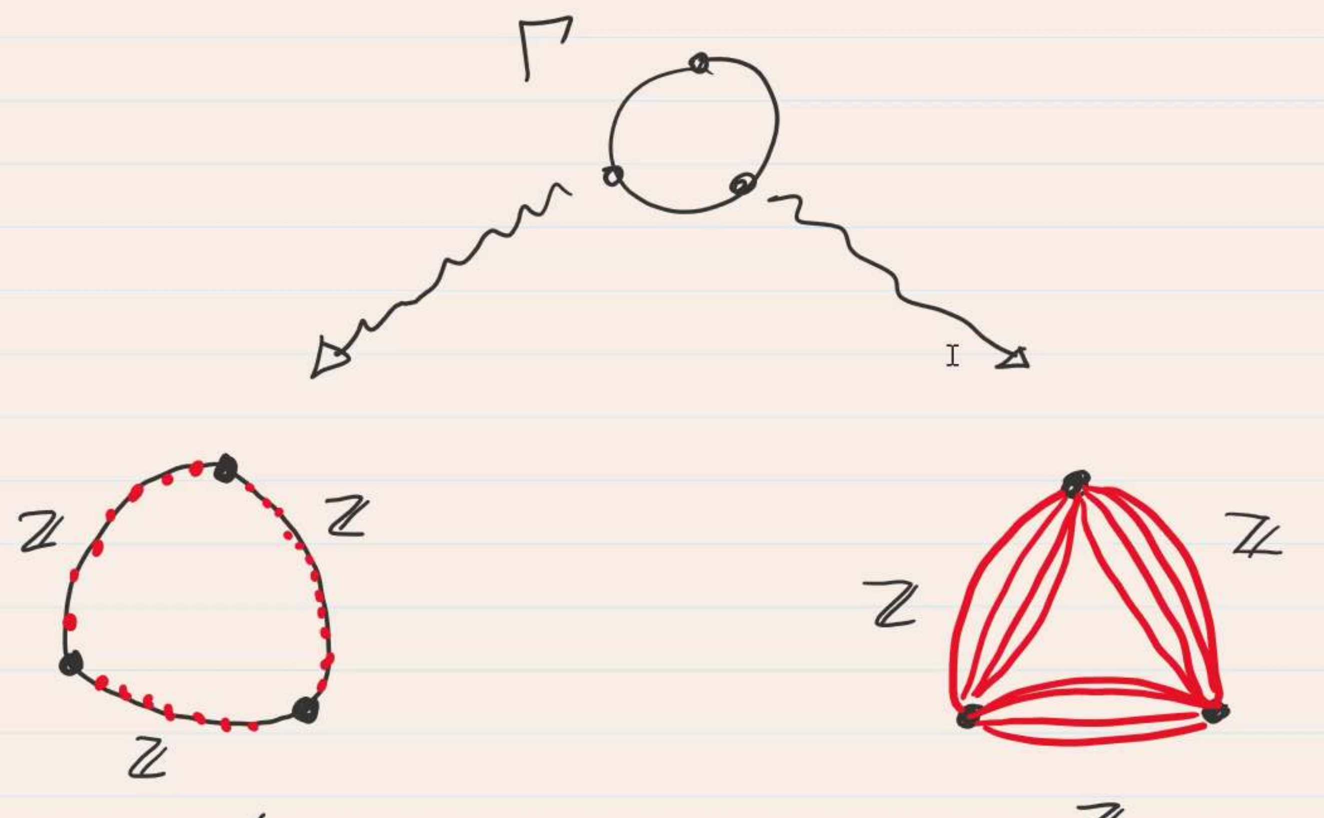
Non-finitary hypertorics: warm-up with graphs.



'periodization'

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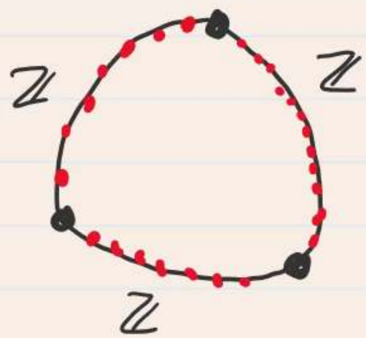
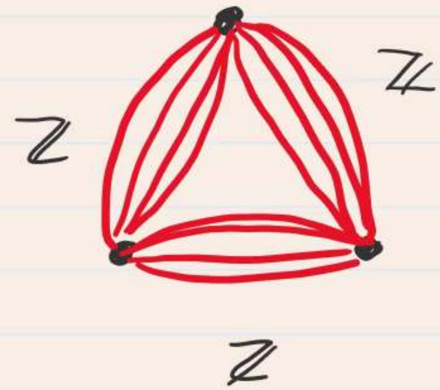


'periodization' of Γ

$P\Gamma$



'loopification of Γ'

$\widetilde{\mathcal{L}}\Gamma$

Non-finitary hypertorics

$A_G V = T^* V /\!/\!/ G$

Non-finitary hypertorics

$$\begin{tikzcd}[column sep=small]
 & A \curvearrowright X = T^*V/\!/\!/G \arrow[dl] \arrow[dr] & \\
PX & & \widetilde{\mathcal{L}}X = T^*V \otimes \mathbb{C}[t,t^{-1}]/\!/\!/G \\
A \times \mathfrak{t}_{\mathbb{Z}} \arrow[u] & & A \times \underbrace{\mathbb{C}^\times_q}_{\text{loop rotation}} \times H_2(X,\mathbb{Z}). \arrow[u]
\end{tikzcd}$$

$$PX^A = X^A \times \mathfrak{t}_{\mathbb{Z}} \qquad \widetilde{\mathcal{L}}X^{A \times \mathbb{C}^\times_q} = X^A \times H_2(X,\mathbb{Z})$$

loop rotation

$PX^A = X^A \times t_{\mathbb{Z}}$

$\tilde{\mathcal{L}}X^{A \times \mathbb{C}^\times_q} \stackrel{!}{=} X^A \times H_2(X,\mathbb{Z})$

PX 'periodic version of X'

X PX

$\tilde{\mathcal{L}}X$ 'universal cover of the loop space'

$T^*\mathbb{P}^1 \rightsquigarrow T^*\mathbb{P}^\infty$

Action of $H_2(X,\mathbb{Z})$ $\leftrightarrow$ Deck transform
by $\pi_1(\mathcal{L}X) = \pi_2$

$\underline{Cat \circlearrowright} \, \tilde{\mathcal{L}}X$

$PX^A = X^A \times t_{\mathbb{Z}}$

$\widetilde{\mathcal{L}X}^{A\times\mathbb{C}^\times_q} = X^A \times H_2(X,\mathbb{Z})$

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Action of $H_2(X,\mathbb{Z})$ $\leftrightarrow$ Deck transformations by $\pi_1(\mathcal{L}X) = \pi_2(X)$

$\underline{Cat} \circlearrowright \widetilde{\mathcal{L}X}$

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$\mathrm{Cat} \circlearrowright \widetilde{\mathcal{L}X}$

Let $\sigma^+ : \mathbb{C}^\times \longrightarrow T \times \mathbb{C}^\times_q$ pos $\mathbb{C}^\times_q$ comp.

$\sigma^- : \mathbb{C}^\times \longrightarrow T \times \mathbb{C}^\times_q$ neg $\mathbb{C}^\times_q$ comp.

Then $\mathrm{Att}_{\sigma^+} \sim$ holomorphic loops into X

$\mathrm{Att}_{\sigma^-} \sim$ anti-hol. loops into X.

$\mathrm{Att}_{\sigma^+} \cap \beta \cdot \mathrm{Att}_{\sigma^-} \sim \{ \mathbb{P}^1 \dashrightarrow X$ of class $\beta \in H_2(X,\mathbb{Z})$. $\}$

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Simples in cat $\mathcal{O}^{\sigma^\pm}_{\tilde{\mathcal{L}}X}$ indexed by $\tilde{\mathcal{L}}X^{A \times \mathbb{C}^*_q}$

$= X^A \times H_2(X, \mathbb{Z})$

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[M-Sheshmani-Yau]

Let $L^+_{0,p}$ $L^-_{\beta,p}$ be simples in $\mathrm{cat}\,\mathcal{O}^{\sigma^+}$, $\mathrm{cat}\,\mathcal{O}^{\sigma^-}$.

$p \in X^A$ $\beta \in H_2(X,\mathbb{Z})$.

$$\boxed{\mathrm{Ext}^\bullet(L^+_{0,p}, L^-_{\beta,p}) = H^\bullet(\text{twisted quasimaps of class } \beta)}$$

Let $X, X^!$ be symplectic dual hypertorics.

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Then Att_{σ^+} ~ holomorphic loops into X

Att_{σ^-} ~ anti-hol. loops into X.

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Simples in cat $\mathcal{O}^{\sigma^\pm}_{\tilde{\mathcal{L}}X}$ indexed by: $\tilde{\mathcal{L}}X^{A\times\mathbb{C}^*_q}$

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$$\boxed{\mathrm{Ext}^\bullet(L^+_{0,\rho}, L^-_{\beta,\rho}) = H^\bullet(\text{twisted quasimaps of class } \beta)}$$

Let $X, X^!$ be symplectic dual hypertorics.

$$\boxed{(\tilde{\mathcal{L}}X)^! = \mathbb{P}(X^!)}$$

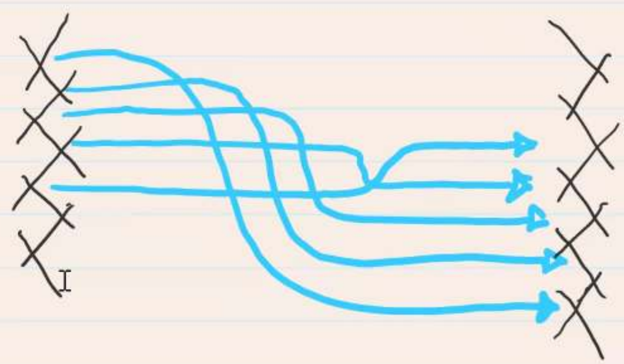
Recall: cocharacters of $A \curvearrowright X$

$\sigma^+, \sigma^- : \mathbb{C}^* \longrightarrow 1 \times \mathbb{C}_q$

$\rightsquigarrow$ birational models

$$P(X^!)_{\sigma^+} \quad \& \quad P(X^!)_{\sigma^-}$$

0_+0



$$\operatorname{cat} \mathcal{O}_{PX^!_{\sigma^+}} \xrightarrow[\text{'ringel duality'}]{} \operatorname{cat} \mathcal{O}_{PX^!_{\sigma^-}}$$

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Applying Koszul duality + Ringel duality

$$\boxed{\begin{array}{l}\text{generating function } Z_X \\ \text{for } H^\cdot(\text{twisted quasimaps})\end{array} = \begin{array}{l}\text{character of } \mathcal{U}_{PX^!} \\ \text{- tilting module}\end{array}}$$

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Now consider $\sigma^0 : \mathbb{C}^\times \longrightarrow T \times \mathbb{C}_q^\times$ trivial on $\mathbb{C}_q^\times$.

$\mathrm{Att}_{\sigma^0} \sim$ loops into σ-positive part of X.

In progress: recover the 'elliptic stable envelopes' of Aganagic & Okounkov from σ^0-bounded lagrangians on $\widetilde{\mathcal{L}}X \times PX$!

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[M-Sheshmani-Yan]

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Let $X, X^!$ be symplectic dual hypertorics.

$$\boxed{(\tilde{\mathcal{L}}X)^! = P(X^!)}$$

Recall: cocharacters of $A \curvearrowright X$

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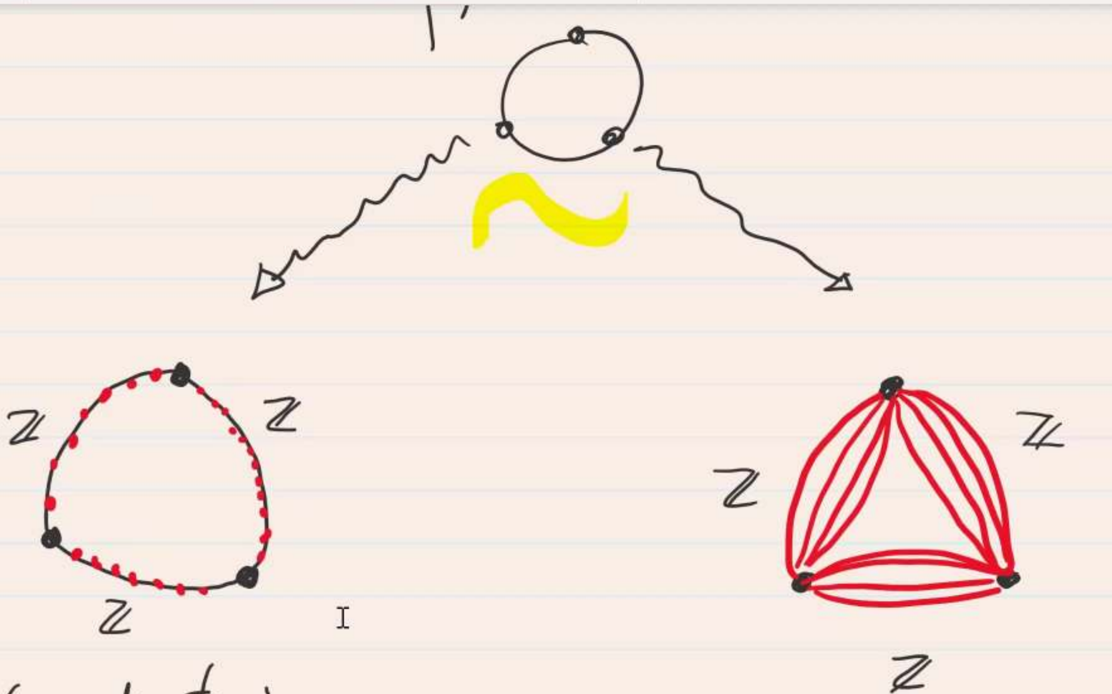

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