Title: Fundamental local equivalences in quantum geometric Langlands

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Abstract: In quantum geometric Langlands, the Satake equivalence plays a less prominent role than in the classical theory. Gaitsgory--Lurie proposed a conjectural substitute, later termed the fundamental local equivalence, relating categories of arc-integrable Kac--Moody representations and Whittaker D-modules on the affine Grassmannian. With a few exceptions, we verified this conjecture non-factorizably, as well as its extension to the affine flag variety. This is a report on joint work with Justin Campbell and Sam Raskin.

Fundamental local equivalences in quantum geometric Langlands

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Overview

- Gaitsgory and Lurie conjectured deformations of the geometric Satake equivalence termed *fundamental local equivalences*
- Under a mild restriction on the deformation parameter, we proved these conjectures non-factorizably, in joint work with J. Campbell and S. Raskin
- Has arithmetic and representation-theoretic applications which are work in progress.

Motivation

- Everything over k alg. closed of characteristic zero
- X smooth projective curve
- G reductive group
- Bun_G moduli stack of G-bundles on X
- Given $x \in X(k)$, have completed local ring and its fraction field

$$\mathcal{O} \quad \text{and} \quad \mathcal{K}$$

and corresponding arc and loop groups

$$G(\mathcal{O}) \quad \text{and} \quad G(\mathcal{K}).$$

Motivation

- Hecke correspondences give action

$$D(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})) \otimes D(Bun_G) \to D(Bun_G).$$

- Appearance of Langlands dual group $\check{G}$:

Theorem

(Lusztig–Drinfeld–Ginzburg–Mirkovic–Vilonen) There is a canonical (up to signs symmetric) monoidal equivalence of abelian categories

$$D(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}))^{\heartsuit} \simeq (\mathrm{Rep}\ \check{G})^{\heartsuit}.$$

- Count parameters - double cosets on the LHS are indexed by dominant coweights for G, i.e. dominant weights for $\check{G}$.

Motivation

- Quantum Langlands concerns the deformation

$$D\text{-modules on } Bun_G \rightsquigarrow \text{twisted } D\text{-modules on } Bun_G.$$

- Parameters given by

$$\mathrm{Sym}^2(\mathfrak{g}^*)^G \to \mathrm{Pic}(Bun_G) \underset{\mathbb{Z}}{\otimes} k$$

$$\kappa \rightsquigarrow D_\kappa(Bun_G)$$

- Basic question - find helpful deformation of Satake isomorphism
- Basic problem - twisted spherical Hecke category

$$D_\kappa(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}))$$

is often 'too small', e.g. is supported on trivial coset for κ generic.

Formulation of the conjectures

- Idea of Gaitsgory–Lurie: fewer Hecke operators, but same number of Whittaker coefficients
- Fix $N \subset B \subset G$ the unipotent radical of a Borel, and a nondegenerate character of conductor zero

$$\psi : N(\mathcal{K}) \to \mathbb{G}_a.$$

Theorem

(Frenkel–Gaitsgory–Vilonen) There is a canonical equivalence of abelian categories

$$D(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}))^{\heartsuit} \simeq D(N(\mathcal{K}), \psi\backslash G(\mathcal{K})/G(\mathcal{O}))^{\heartsuit}$$

- Parameter count - relevant orbits both indexed by dominant coweights
- One side of quantum deformation:

$$D(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})) \rightsquigarrow D_{\kappa}(N(\mathcal{K}), \psi\backslash G(\mathcal{K})/G(\mathcal{O})).$$

Formulation of the conjectures

- Need matching deformation of Rep $\check{G}$
- Basic idea - pass to representations of quantum group, or equivalently Kazhdan–Lusztig category for $\check{G}$
- Duality for levels via dual inner products on Cartan subalgebras

$$\mathrm{Sym}^2(\mathfrak{g}^*)^G \setminus 0 \simeq \mathrm{Sym}^2(\check{\mathfrak{g}}^*)^{\check{G}} \setminus 0.$$

$$\kappa \leftrightarrow \check{\kappa}$$

- The level $\check{\kappa}$ yields a Kac–Moody extension

$$0 \to k \to \widehat{\check{\mathfrak{g}}}_{\check{\kappa}} \to \check{\mathfrak{g}}(\mathcal{K}) \to 0.$$

- Associated Kazhdan–Lusztig category of $\check{G}(\mathcal{O})$-integrable modules

$$\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{-mod}^{\check{G}(\mathcal{O})}.$$

Formulation of the conjectures

Conjecture

(Gaitsgory–Lurie, 2006) For any nonzero κ, there is an equivalence of triangulated categories

$$D_\kappa(N(\mathcal{K}), \psi \backslash G(\mathcal{K})/G(\mathcal{O})) \simeq \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{-mod}^{\check{G}(\mathcal{O})}.$$

- Count parameters - both sides indexed by dominant coweights for G.

Remark

In fact, they conjectured more, namely an equivalence of factorization categories (informally, compatibilities with moving and colliding multiple points of the curve).

Formulation of the conjectures

- Gaitsgory also conjectured a tamely ramified variant, concerning the Iwahori subgroups I and $\check{I}$.

Conjecture

(Gaitsgory, 2006) For any nonzero κ, there is an equivalence of triangulated categories

$$D_\kappa(N(\mathcal{K}), \psi\backslash G(\mathcal{K})/I) \simeq \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{-mod}^{\check{I}}.$$

- Analog for $\kappa = 0$:

Theorem

(Arkhipov–Bezrukavnikov) There is an equivalence of triangulated categories

$$D(N(\mathcal{K}), \psi\backslash G(\mathcal{K})/I) \simeq \mathrm{QCoh}^{\check{G}}(T^*(\check{G}/\check{B})).$$

Results

Theorem

(Campbell-D.-Raskin) If κ satisfies a mild technical hypothesis, then both conjectures are true (the former non-factorizably).

- Hypothesis - after restriction to each simple factor of $\mathfrak{g}$, κ is either irrational or rational with denominator coprime to the bad primes of the root system.
- Hypothesis is vacuous in type A, in general should be removable by a variant of the argument.
- Gaitsgory and collaborators have a rich program which is expected to yield the conjectures with factorization

Methods

- In finite type, well known equivalence (Milicic–Soergel, Bezrukavnikov, etc.) between blocks of $\mathcal{O}$ and partial Whittaker sheaves

$$\mathfrak{g}\text{-mod}_\lambda^B \simeq D(B\backslash G/N, \chi_\lambda).$$

- (Aside) Does not arise from usual localization on G/B, but instead

Theorem

(Campbell-D.) Localization yields a fully faithful embedding

$$\mathfrak{g}\text{-mod}_\lambda \hookrightarrow D(G/N, \chi_\lambda).$$

Methods

- Similarly, we relate

$$D_\kappa(N(\mathcal{K}),\psi\backslash G(\mathcal{K})/I) \rightsquigarrow \text{category } \mathcal{O} \text{ for } \widehat{\mathfrak{g}}_\kappa$$

  using categorical representation theory of loop groups.
- Match the combinatorial descriptions provided by Soergel–Fiebig for blocks of $\mathcal{O}$ for $\widehat{\mathfrak{g}}$ at level κ and $\widehat{\check{\mathfrak{g}}}$ at level $\check{\kappa}$.

Next steps

- Pass from k to $\mathbb{F}_q$, so

$$\mathcal{O} \cong \mathbb{F}_{q_x}[[t]] \quad \text{and} \quad \mathcal{K} \cong \mathbb{F}_{q_x}((t)).$$

- An unramified principal series representation π of $G(\mathcal{K})$ has one dimensional spaces of $G(\mathcal{O})$ invariants and $(N(\mathcal{K}), \psi)$ coinvariants, hence yields a spherical Whittaker function

$$f_\pi \in \mathrm{Fun}(N(\mathcal{K}), \psi \backslash G(\mathcal{K})/G(\mathcal{O})).$$

- Frenkel–Gaitsgory–Vilonen equivalence

$$Perv(N(\mathcal{K}), \psi \backslash G(\mathcal{K})/G(\mathcal{O})) \simeq \mathrm{Rep}\, \check{G}$$

recovers after trace of Frobenius the *Casselman–Shalika* formula for f_π.

Next steps

- Fix a metaplectic cover $\widetilde{G(\mathcal{K})}$ of $G(\mathcal{K})$
- An unramified principal series representation π of $\widetilde{G(\mathcal{K})}$ has a line of $G(\mathcal{O})$ invariants, but in general a greater than one dimensional space of $(N(\mathcal{K}), \psi)$ coinvariants, hence yields a vector space of spherical Whittaker functions

$$V_\pi \subset \mathrm{Fun}(N(\mathcal{K}), \psi \backslash \widetilde{G(\mathcal{K})} / G(\mathcal{O}))$$

(Kubota, Matsumoto, Kazhdan–Patterson, Brubaker, Bump, Chinta, Friedberg, Gunnells, McNamara, Lysenko, etc.)

- A variant of our proof of the tamely ramified conjecture should yield a description of

$$Perv(N(\mathcal{K}), \psi \backslash \widetilde{G(\mathcal{K})} / G(\mathcal{O}))$$

in terms of the Kazhdan–Lusztig category, and after trace of Frobenius a metaplectic Cassleman–Shalika formula for a canonical basis for V_π.

Next steps

- Return to k alg closed of characteristic 0
- To $\mathfrak{g}$ and κ one may attach an (affine) W-algebra

$$\mathcal{W}_\kappa.$$

- Problem (Frenkel–Kac–Wakimoto, Arakawa, etc.) - determine behavior of the obtained functor

$$C^{\frac{\infty}{2}+*}(\mathfrak{n}(\mathcal{K}), - \otimes k_\psi) : \widehat{\mathfrak{g}}_\kappa\text{-mod}^I \to \mathcal{W}_\kappa\text{-mod}\,.$$

Next steps

- This functor fits into a family of functors

$$D_\kappa(N(\mathcal{K}), \psi \backslash G(\mathcal{K})/I) \otimes \widehat{\mathfrak{g}}_\kappa\text{-mod}^I \to \mathcal{W}_\kappa\text{-mod}\,.$$

- Similar arguments to those above yield a description of

$$D_\kappa(N(\mathcal{K}), \psi \backslash G(\mathcal{K})/I) \otimes \widehat{\mathfrak{g}}_\kappa\text{-mod}^I \to \mathcal{W}_\kappa\text{-mod}\,.$$

  in terms of 'two sided antispherical quotients' of the affine Hecke category (at least for κ negative).
- Should imply a conjecture of Gaitsgory on the compatibility of the above pairing with Feigin–Frenkel duality and the FLE

The end

Thanks for listening!