Title: Categorification of the Hecke algebra at roots of unity.

Speakers: Ben Elias

Collection: Geometric Representation Theory

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Abstract: Categorical representation theory is filled with graded additive categories (defined by generators and relations) whose Grothendieck groups are algebras over \mathbb{Z}[q,q^{-1}]. For example, Khovanov-Lauda-Rouquier (KLR) algebras categorify the quantum group, and the diagrammatic Hecke categories categorify Hecke algebras. Khovanov introduced Hopfological algebra in 2006 as a method to potentially categorify the specialization of these \mathbb{Z}[q,q^{-1}]-algebras at q = \zeta_n a root of unity. The schtick is this: one equips the category (e.g. the KLR algebra) with a derivation d of degree 2, which satisfies d^p = 0 after specialization to characteristic p, making this specialization into a p-dg algebra. The p-dg Grothendieck group of a p-dg algebra is automatically a module over \mathbb{Z}[\zeta_{2p}]... but it is NOT automatically the specialization of the ordinary Grothendieck group at a root of unity!

Upgrading the categorification to a p-dg algebra was done for quantum groups by Qi-Khovanov and Qi-Elias. Recently, Qi-Elias accomplished the task for the diagrammatic Hecke algebra in type A, and ruled out the possibility for most other types. Now the question is: what IS the p-dg Grothendieck group? Do you get the quantum group/hecke algebra at a root of unity, or not?

This is a really hard question, and currently the only techniques for establishing such a result involve explicit knowledge of all the important idempotents in the category. These techniques sufficed for quantum \mathfrak{sl}_n with n \le 3, but new techniques are required to make further progress.

After reviewing the theory of p-dg algebras and their Grothendieck groups, we will present some new techniques and conjectures, which we hope will blow your mind.

Everything is joint with You Qi.

Categorification of the Hecke Algebra at a root of unity

Ben Elias (U. Oregon)

GRT 2020

jt w/ You Qi

June 2020

The "original goal" of Crane-Frenkel's categorification program was to categorify quantum gps at roots of unity.

A practical approach to catfn at a rou was suggested by Khovanov in 2006.

1) History: catfn at rou so far

2) Conjectures; moving forward.

Common Household categories:

KLR Category $\mathcal{U}^+$ $\quad [\ \ ]_{\oplus} \quad$ $U_q^+(\mathfrak{g})$

KL category $\mathcal{U}$ $\rightsquigarrow$ $U_q(\mathfrak{g})$

diag. Hecke category $\mathcal{H}$ $\quad$ $\mathbb{H}(W)$

These are $\otimes, \oplus, (1)$ categories. Think: Projective modules.

$\rightsquigarrow$ $[\ \ ]_{\oplus}$ is a $\mathbb{Z}[q,q^{-1}]$-algebra.

Presented by gens+rels, so categories have a $\mathbb{Z}$-form.

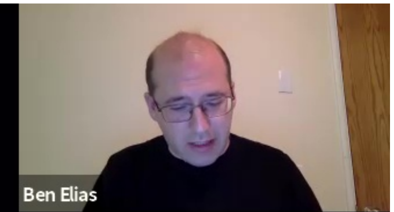
They have been studied extensively for last decade.

Describe Ext algebras b/w perverse sheaves on quiver/Flag vtys.

SURPRISE: New structure. Secret "differential"

<u>Def</u>: A <u>gaea</u> (short for $\mathbb{G}_a$-equiv alg) is a graded alg/$\mathbb{Z}$ A equipped w/ a derivation d (of degree 2)

st
- $d(fg) = d(f)g + f\,d(g)$
- in $\otimes$ setting, $d(f \otimes 1) = d(f) \otimes 1$ and $d(1 \otimes f) = 1 \otimes d(f)$
- divided powers are defined integrally, $d^{(k)} = \frac{d^k}{k!}$

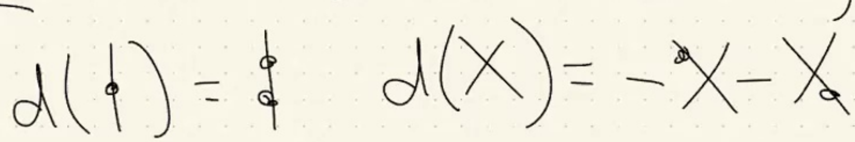


Ex 1: $\mathbb{Z}[x]$ $\quad d(x)=x^2 \Rightarrow d(x^k)=kx^{k+1}$

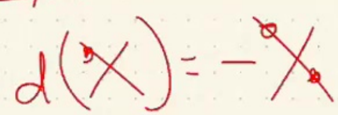
$$d^{(\ell)}(x^k)=\binom{k+\ell}{\ell}x^{k+\ell}$$

$$1 \qquad x \xrightarrow{1} x^2 \xrightarrow{2} x^3 \xrightarrow{3} x^4 \longrightarrow$$

Ex 2: NilHecke alg (i.e. $\mathrm{End}_{\mathcal{U}^+}(E^n)$)



Exercise:



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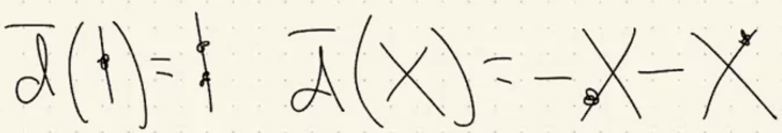
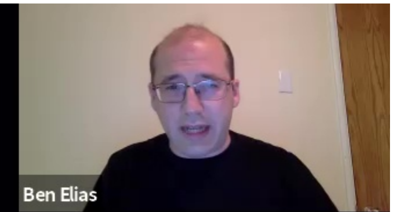


flip upside down (duality)

A gaea is like a dg-algebra but in a weird world with unusual homological algebra (like ours!)

The Ext algebras described by $\mathcal{U}, \mathcal{P}$ are formal, even, no dg structure..., but they are gaeas! WHY?

Connection to Khovanov's programme:



Why is $[\mathcal{A}]_\oplus$ a $\mathbb{Z}$-module? (By defn, silly!)

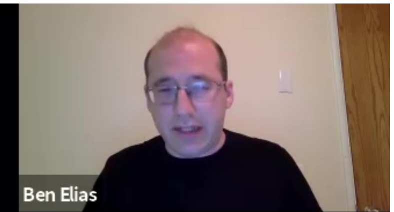
If we're over a field $\Bbbk$, then $\mathrm{Vect}(\Bbbk) \circlearrowright \mathcal{A}$

$$k^n \otimes M = M^{\oplus n}.$$

$$\text{so } \underset{\mathbb{Z}}{\overset{\shortparallel}{[\mathrm{Vect}]}} \circlearrowright [\mathcal{A}]$$

Similarly, $K^b(\mathrm{Vect}) \circlearrowright K^b(\mathcal{A})$ so $[K^b(\mathrm{Vect})]_\Delta \circlearrowright [\mathcal{A}]_\Delta$

$$[K^b(\mathrm{Vect})]_\Delta = \mathbb{Z}[t,t^{-1}]/t=-1$$



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and in graded setting,

$$\mathrm{Vect}_{\mathbb{Z}} \circlearrowright \mathcal{A} \qquad [\mathrm{Vect}_{\mathbb{Z}}] = \mathbb{Z}[q,q^{-1}]$$

What base category has groth gp $\mathbb{Z}[\zeta]$ for ζ root of unity?

Khovanov: H a f.d. graded Hopf alg.

H-gmod – f.g. graded modules

H-$\underline{\text{gmod}}$ – stable module cat. $:= H\text{-gmod} / H\text{-gproj}$

Then H-$\underline{\text{gmod}}$ is Δ, $\otimes$. If H graded local then

$$[H\text{-}\underline{\text{gmod}}]_\Delta \cong \mathbb{Z}[q,q^{-1}] / \text{gdim } H$$

If A is an H-module alg. then $\exists\ D_H(A\text{-gmod})$ and

$$[H\text{-}\underline{\text{gmod}}]_\Delta \circlearrowright [D_H(A)]_\Delta$$

Ex 1: $H = k[d]/d^2$ w/ $\Delta(d) = d \otimes 1 + 1 \otimes d$.

To get $\Delta(d)^2 = 0$ need to work in SuperVect.

Then $H\text{-}\underline{\text{gmod}} = K^b(\mathrm{Vect}_{\mathbb{Z}}(k))$.

An H-mod alg is just a dg-algebra.

ordinary homological algebra

$[H] = 1 + t$

Ex 2: $H = k[d]/d^p$ w/ $\Delta(d) = d \otimes 1 + 1 \otimes d$.

to get $\Delta(d)^p = 0$ need char $k = p$.

H-mod-alg called a p-dg algebra.

$[H] = 1 + q^2 + \dots + q^{2(p-1)}$

p-homological alg

see Qi.

"Ex 3:" $H = \dot{U}^{+}(sl_2)_{\mathbb{Z}}$. H-mod-alg = gaea.

but doesn't fit the framework yet.

15 minutes

However, gaea $\xrightarrow{\text{char } k = p}$ p-dg alg since $d^p = p!\, d^{(p)}$

PUNCHLINE:
(in Ex 2)

$[H\text{-gmod}]_{\partial} = \mathbb{Z}[\zeta]$ / ζ is primitive p^{th} rou.

Want to know more? QUACKS, August 2020.

So, we take U^+, equip w/ d, work over char p.

Is $[\ ]_\Delta \cong U_\zeta(\mathfrak{g})$? Sometimes! Not for free!

In fact, NH_n has a 1-parameter family of derivations d_a

but only two (d and $\bar{d}$) give correct groth gp.

What's the catch?

Ex: $EE \cong E^{(2)} \oplus E^{(2)}$ (ignoring grading)

In additive setting, prove by constructing idempotent decomp of identity.

$$|| = \times + (-\times)$$
$$1 = e_1 + e_2$$

So $\mathrm{Hom}(EE,-) \cong \mathrm{Hom}(EE,-)e_1 \oplus \mathrm{Hom}(EE,-)e_2$

a left/top U^+-mod

both isom to $\mathrm{Hom}(E^{(2)},-)$

Now add d. This is NOT a direct sum of (U^+,d)-modules.

$d(e_1) = -\times = +x_1 e_2 \in U^+ \cdot e_2$ || $d(fe_1) = \underline{d(f)e_1} + x_1 f e_2$

But it is filtered! $d(e_2) = \cancel{X} = -x_1 e_2 \in \mathcal{U}^+ \cdot e_2$.

$$\begin{tikzcd} \mathcal{U}^+ \cdot \mathrm{id}_{EE} \cong \mathcal{U}^+ \cdot e_1 \arrow[loop below, "d"] \arrow[rr, bend left, "d"] & \oplus & \mathcal{U}^+ \cdot e_2 \arrow[loop below, "d"] \end{tikzcd}$$

UPSHOT: Still get $[EE] = [\mathrm{Im}\, e_1] + [\mathrm{Im}\, e_2]$ in $[D_p(\mathcal{U}^+)]_\Delta$

Think: It looks like a "complex" of objects, which is sum of its pieces in Groth gp.

WARNING! Not every idemp. decomp. will work.

$$\mathbb{1} = X + (-X) \qquad \begin{tikzcd} \bullet \arrow[loop below, "d"] \arrow[r, bend left, "d"] & \bullet \arrow[loop below, "d"] \arrow[l, bend left, "d"] \end{tikzcd}$$

Not filtered!

WARNING CONT: Also need to get right d structure on associated graded or $\mathrm{Hom}(EE,-)_{e_1} \cong \mathrm{Hom}(E^{(2)},-)$

won't respect d!

Def: An Fc-filtration is ... an idemp decomp where it all works out.

Moral: To categorify defining relns of $U_q(\mathfrak{g})$ or $H(W)$ need key direct sum decomps to lift to Fc filtrations.

A severe restriction both on d and on idempotents!!..

Which is good!!..

What's been done:

		Defining FC	$[\ ]_\Delta = ?$	Explicit idempotents
$U^+(\mathfrak{sl}_2)$	Khovanov-Qi	✓	✓	✓
U^+(simply laced)	— // —	✓	✗	✗
$U(\mathfrak{sl}_2)$	Elias-Qi	✓	✓	✓
$\dot{U}(\mathfrak{sl}_2)$	— // —	✓	✓	✓
$\dot{U}^+(\mathfrak{sl}_3)$	Stephens	✓	✓	✓
Some Webster $\otimes$ alg's	Khovanov-Sussan-Qi	✓	✓ New tool	✗
$\mathcal{H}$(simply laced)	Elias-Qi	✓	✗ New tool needed!	✗

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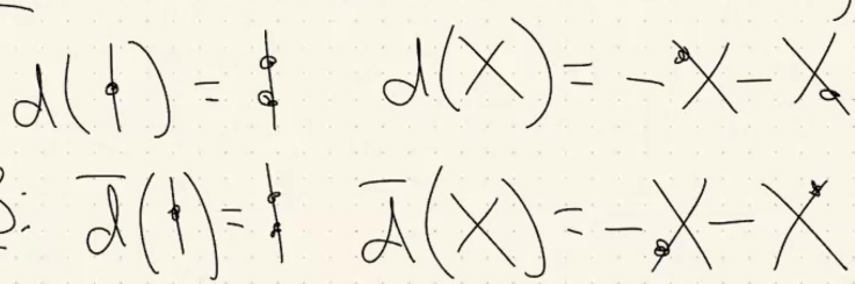

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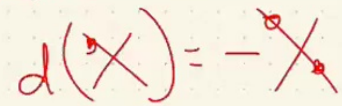

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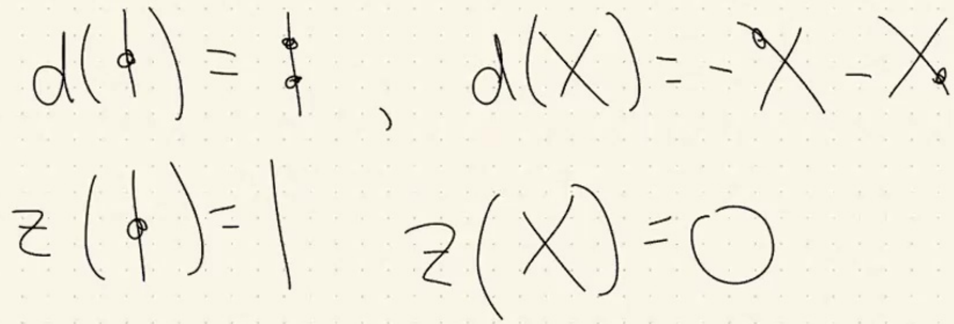
In our search for a proof that $[D_p(\mathcal{H})]_\Delta \cong \mathbb{H}(S_n)$, we found some surprising new structure.

Thm: The categories $\mathcal{H}, \mathcal{U}^+, \mathcal{U}$, etc have actions of $\mathfrak{sl}_2$ by derivations!

I.e. there is a degree -2 deriv. z s.t. both

$$(d, \text{degree}, -z) \text{ and } (\bar{d}, \text{degree}, -z) \text{ are } \mathfrak{sl}_2\text{-triples.}$$



"Base ring" ex: $R = \mathbb{Z}[x_1, \dots, x_n]$

$$d(x_i) = x_i^2 \rightsquigarrow d = \sum_i x_i^2 \frac{\partial}{\partial x_i}$$

$$z(x_i) = 1 \rightsquigarrow z = \sum_i \frac{\partial}{\partial x_i}$$

$$\deg(x_i) = 2x_i \rightsquigarrow \deg = 2\sum_i x_i \frac{\partial}{\partial x_i}$$

Witt algebra

All hom spaces in these categories are simultaneously (R,R)-bimodules and $\mathfrak{sl}_2$-modules, compatibly.

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Witt algebra

Can't separate – derivations are S_n-equiv.

All hom spaces in these categories are simultaneously (R,R)-bimodules and $\mathfrak{sl}_2$-modules, compatibly.

What do (R, ∂_2)-modules look like?

Ex: $R = \mathbb{Z}[x] \circlearrowright M = R$

$$\begin{tikzcd} 1 & x \arrow[l, "1"'] \arrow[r, bend left, "1"] & x^2 \arrow[l, "2"'] \arrow[r, bend left, "2"] & x^3 \arrow[l, "3"'] \arrow[r, bend left] & \cdots \arrow[l] \end{tikzcd}$$

f.d. submod

It's the coverma $\nabla(0)$!

Ex: $f = ax \in R$. $M = R\langle f \rangle$ i.e. free rank 1 as R-mod w/ gen. v, and

$$d(v) = f \cdot v$$
$$z(v) = 0$$
$$\deg(v) = a$$

It's $\nabla(a)$

Ex: $R = \mathbb{Z}[x_1, \dots, x_n]$, $f = \sum a_i x_i$, $M = R\langle f \rangle$.

Then as $\mathfrak{sl}_2$-mod, $M \cong \nabla(a_1) \otimes \nabla(a_2) \otimes \dots \otimes \nabla(a_n)$

M has f.d. subrepn iff $a_i \leq 0 \ \forall i$, subrepn is

$$L_{-a_1} \otimes \dots \otimes L_{-a_n}.$$

Why do I care? Bear with me ...

Previously known: Hom spaces in $\mathcal{U}^+$ or $\mathcal{H}$ are free

as left R-mod w/ diagrammatic bases

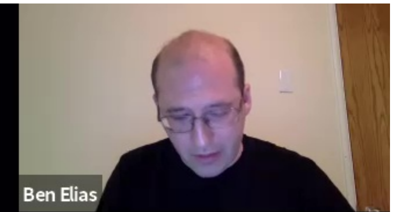
Thm: Z preserves this R-mod decomposition

d acts in a filtered way wrt known p.o.

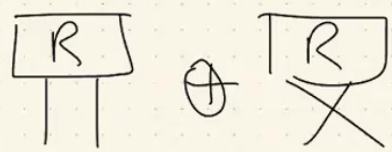
Comb formula for shifts $R\langle f \rangle$ on subquots

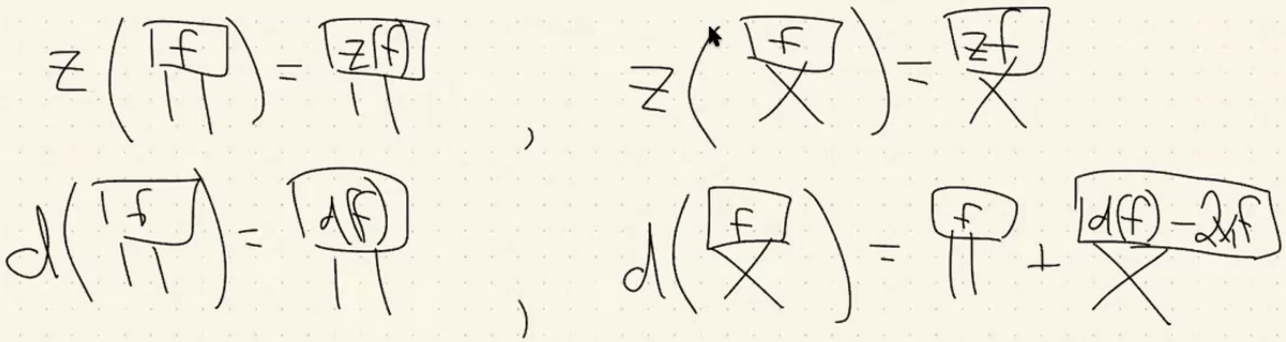
Rmk 1: Still not easy to determine f.d. subreps

Rmk 2: Hom spaces live in category "$\mathcal{O}$" for $(R, \mathfrak{sl}_2)$



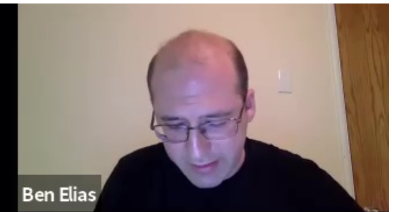
Ex: NH_2





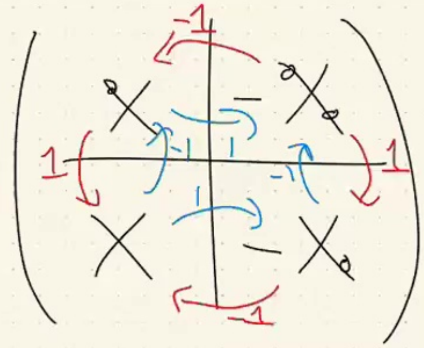
get $0 \to R\langle 0 \rangle \to NH_2 \to R\langle -2x_1 \rangle \to 0$

which is (R, z)-split.



Ex: NH_2

$$NH_2 \cong Mat_2(R^{S_2})$$

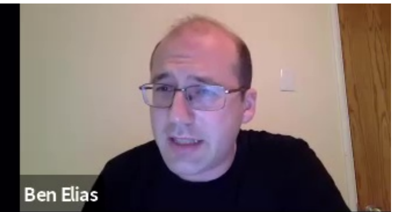


$$d(X) = 0 \qquad z(X) = 0$$

MAX'L F.D. SUBREPN OF NH_2 IS

$$Mat_2(\mathbb{k}) \cong NH_2 / J(NH_2),$$

IT'S SEMISIMPLE!!!



MAX'L F.D. SUBREPN OF NH_2 IS

$$Mat_2(\Bbbk) \cong NH_2 / J(NH_2),$$

IT'S SEMISIMPLE!!!

But it's a SUB not a quotient!

(R-modules)

$$0 \rightleftarrows Mat_2(R_+^{S_2}) \rightleftarrows Mat_2(R^{S_2}) \rightleftarrows Mat_2(\Bbbk) \rightleftarrows 0$$

($\mathfrak{sl}_2$ modules)

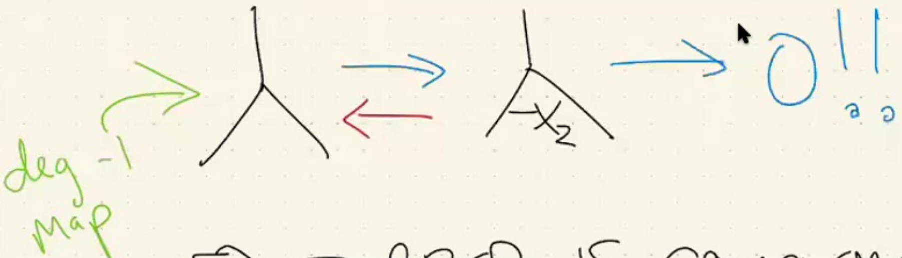
Ex: In $\mathcal{H}$, $B_sB_s \cong B_s(1) \oplus B_s(-1)$.

$$\mathrm{Hom}^\circ(B_sB_s, B_s) \times \mathrm{Hom}^\circ(B_s, B_sB_s) \xrightarrow[\text{compose}]{} \mathrm{End}(B_s) \to \mathrm{End}(B_s)/\mathfrak{m} = \mathbb{k}$$

is local intersection form.

- Dual bases yield ortho inclusion + proj maps.
- Kernel = $\mathrm{Hom} \cap J(\mathcal{H})$
- Hom/Kernel is "multiplicity space"
- Duality makes it a form on $\mathrm{Hom}(B_sB_s, B_s)$.

Hom(B_sB_s, B_s) has grank $q^{-1} + 2q + q^3$ over R.



f.d subrepn.

- FD SUBREP IS COMPLEMENT TO J!!
- A "CANONICAL" SECTION OF MULT. SPACE, a sub not a quotient.

Hodge-Riemann bilinear relations: $(\curlywedge, d(\curlywedge)) = (-x_2) = +\mathrm{id}_{B_s}$ positive sign.

Let's not get too excited.

Ex 2: $X := B_s B_t B_s \cong B_{sts} \oplus B_s$

$$V = \mathrm{Hom}(X, B_s) \oplus \mathrm{Hom}(X, B_{sts})$$

f.d.(V) is spanned by projection to B_s, no map to B_{sts}.

But now look at $F_1 = d\rho \circ \mathrm{f.d.}(V)$, and $V_1 = V/F_1$.

f.d.(V_1) spanned by (image of) proj. to B_{sts}.

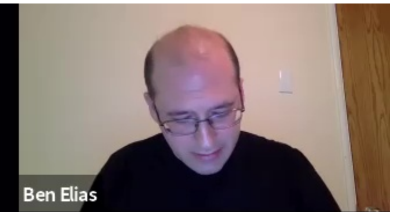
In conclusion:

- I have a concrete conjecture which encapsulates this behavior but technical to state.

"f.d. $\mathfrak{sl}_2$ pieces are sufficient to construct enough idemps."

- Conjecture $\Longrightarrow [D_p(\mathcal{H})] \cong \mathcal{H}_q(S_n)$

Conj $\Uparrow$ enough to Fc-filter anything into ∂-indecomposables.

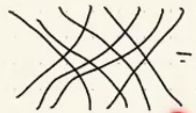
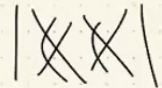


In conclusion:

- Also conjecture a version of HR.

E-Williamson already proved some mult. spaces have HR wrt "middle mult." Relationship unclear.

Ex 3: Kashiwara-Saito singularity in S_8

X = some object  = w   y = 

In char 0, $X \cong B_w \oplus B_y$. In char 2, X indecomposable.

Only id_X lives in f.d. subrep. It is $\mathfrak{sl}$-indecomposable.

$\mathfrak{sl}_2$ DETECTS p-TORSION?

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