

Title: Conjectures on p-cells, tilting modules, and nilpotent orbits

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Abstract: For quantum groups at a root of unity, there is a web of theorems (due to Bezrukavnikov and Ostrik, and relying on work of Lusztig) connecting the following topics: (i) tilting modules; (ii) vector bundles on nilpotent orbits; and (iii) Kazhdan–Lusztig cells in the affine Weyl group. In this talk, I will review these results, and I will explain a (partly conjectural) analogous picture for reductive algebraic groups over fields of positive characteristic, inspired by a conjecture of Humphreys. This is joint work with W. Hardesty and S. Riche.

Conjectures on p -cells, tilting modules, and nilpotent orbits

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Notation

Consider two cases simultaneously:

Quantum case

$\mathbb{k} = \mathbb{C}$

G = quantum group over \mathbb{k} at ℓ -th root of unity

Assume: $\ell > h = \text{Coxeter number}$

Reductive group case

\mathbb{k} = algebraically closed field of char. $p > 0$

G = connected reductive group over \mathbb{k}

Assume: $p > h = \text{Coxeter number}$

Both cases:

\mathbf{X} = weight lattice, \mathbf{X}^+ = dominant weights for G

$T(\lambda)$, $\lambda \in \mathbf{X}^+$: indecomposable tilting module of highest weight λ

$\text{Tilt}(G)$ = the additive \otimes -category of tilting G -modules

A \otimes -**ideal** is a full additive subcategory $I \subset \text{Tilt}(G)$ such that:

- ▶ For any $X \in \text{Tilt}(G)$ and $Y \in I$, we have $X \otimes Y \in I$.
- ▶ I is closed under \oplus and direct summands

The **principal** \otimes -**ideal** generated by $T(\lambda)$ is the smallest \otimes -ideal containing $T(\lambda)$.

Warm-up: tensor ideals of tilting modules for SL_2

Problem

Classify principal \otimes -ideals in $\text{Tilt}(G)$ for $G = SL_2$ (either quantum or reductive case)

Dominant weights: $\mathbf{X}^+ = \mathbb{Z}_{\geq 0}$.

- ▶ $T(0) = \text{triv rep}$, $T(1) = \text{defining rep on } \mathbb{k}^2$
- ▶ All summands of $T(1)^{\otimes m}$ are tilting
- ▶ $T(m)$ occurs as summand of $T(1)^{\otimes m}$

Specific cases

0. $SL_2(\mathbb{C})$: every rep completely reducible & tilting. $T(m) = \text{irreducible rep of highest weight } m$.
 - ▶ Only 1 principal \otimes -ideal (the **unit ideal** = the whole category $\text{Tilt}(G)$)
 - ▶ Proof: $\forall m, T(m) \otimes T(m)^*$ contains the triv module $T(0)$ as a direct summand

1. (Quantum case) Quantum SL_2 at ℓ th root of 1: Exactly 2 principal \otimes -ideals:

- ▶ For $0 \leq m \leq \ell - 2$, $\langle T(m) \rangle_{\otimes} = \text{the unit ideal}$.
- ▶ $\forall m \geq \ell - 1$, $\langle T(m) \rangle_{\otimes} = \langle T(\ell - 1) \rangle_{\otimes}$

2. (Reductive group case) $SL_2(\mathbb{k})$, $\text{char } \mathbb{k} = p > 2$: Infinitely many principal \otimes -ideals

- ▶ For $m \in \mathbb{Z}_{\geq 0}$, let

$$\nu(m) = \min\{j \mid m < p^{j+1} - 1\}$$

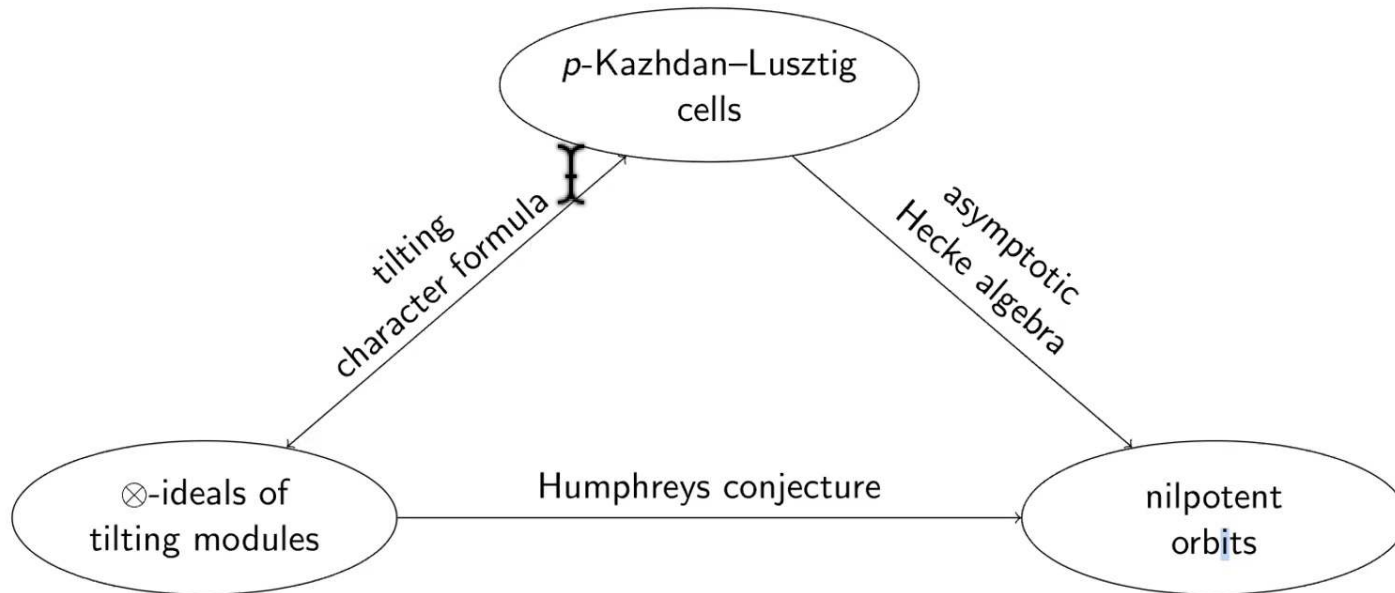
= largest power of p in

the p -adic expansion of $m + 1$

- ▶ $\langle T(m) \rangle_{\otimes} = \langle T(n) \rangle_{\otimes} \iff \nu(m) = \nu(n)$.



Topics in this talk



Based rings

A = a commutative ring (usually \mathbb{Z} or $\mathbb{Z}[v, v^{-1}]$)
 R = an A -algebra, free over A , with A -basis \mathcal{B}

Defn

A **based ideal** is an ideal $I \subset R$ spanned by $I \cap \mathcal{B}$.
 The **principal based ideal** gen by $x \in R$ is the smallest based ideal containing x .

Example 1

$K_0(\text{Tilt}(G))$ = split Grothendieck group of $\text{Tilt}(G)$.
 Basis: $\mathcal{B} = \{[T(\lambda)] \mid \lambda \in \mathbf{X}^+\}$.

$$\begin{array}{ccc} \text{principal based} & & \text{principal} \\ \text{ideals in } K_0(\text{Tilt}(G)) & \xleftrightarrow{\sim} & \otimes\text{-ideals in } \text{Tilt}(G) \end{array}$$

Example 2

\mathcal{H}_{ext} = extended affine Hecke algebra over $\mathbb{Z}[v, v^{-1}]$

- ▶ Basis: “standard basis” $\{T_w : w \in W_{\text{ext}}\}$
 The unit ideal is the unique based ideal.

Example 2 continued

- ▶ Basis: Kazhdan–Lusztig (canonical) basis:

$$\begin{array}{ccc} \text{(left, right, 2-sided)} & & \text{(left, right, 2-sided)} \\ \text{principal based ideals} & = & \text{KL-cells} \end{array}$$

- ▶ Basis: p -canonical basis:

$$\begin{array}{ccc} \text{(left, right, 2-sided)} & & \text{(left, right, 2-sided)} \\ \text{principal based ideals} & = & p\text{-cells} \end{array}$$

Example 3

$\mathcal{M}_{\text{asph}} = (\text{sgn}) \otimes_{\mathcal{H}_f} \mathcal{H}_{\text{ext}} =$ **antispherical module**

Basis: inherits KL- and p -canonical bases from \mathcal{H}_{ext}

$$\begin{array}{ccc} \text{principal} & & \text{antispherical} \\ \text{based submodules} & = & \text{KL- or } p\text{-cells} \end{array}$$

Tensor ideals and antispherical cells

Thm (Ostrik)

In the quantum case, there is a bijection

$$\left\{ \begin{array}{c} \text{principal} \\ \otimes\text{-ideals in Tilt}(G) \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{antispherical} \\ \text{KL-cells} \end{array} \right\}$$

Thm (A.–Hardesty–Riche)

In the reductive group case, there is a bijection

$$\left\{ \begin{array}{c} \text{principal} \\ \otimes\text{-ideals in Tilt}(G) \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{antispherical} \\ \rho\text{-cells} \end{array} \right\}$$

Proof sketch

Define an explicit map

$$\alpha : K_0(\text{Tilt}(G)) \rightarrow \mathcal{M}_{\text{asph}}|_{v=1}$$

(Roughly, α is the “principal block part of the character formula.”)

To show: α sends based ideals to based submodules. Use:

$$\text{in the } \left\{ \begin{array}{l} \text{quantum [Soergel]} \\ \text{reductive group [A.–Makisumi–Riche–Williamson]} \end{array} \right. \text{ case, the } \left\{ \begin{array}{l} \text{KL-basis} \\ \rho\text{-canonical basis} \end{array} \right. \text{ for } \mathcal{M}_{\text{asph}}$$

controls characters of tilting modules in the principal block of $\text{Rep}(G)$.

How many antispherical cells are there?

Quantum case

\mathbf{G} = reductive gp/ \mathbb{C} with same root datum as G

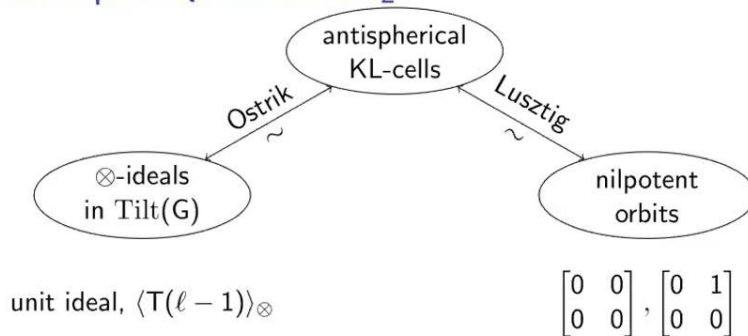
\mathcal{N} = nilpotent cone in $\text{Lie}(\mathbf{G})$

\mathbf{G} acts with finitely many orbits on \mathcal{N}

Thm (Lusztig)

$$\left\{ \begin{array}{l} \text{antispherical} \\ \text{KL-cells} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{l} \text{nilpotent} \\ \text{orbits} \end{array} \right\}$$

Example: Quantum SL_2



Reductive group case

\mathbf{G} = Frobenius twist of G (i.e. $G^{(1)}$)

\mathcal{N} = nilpotent cone in $\text{Lie}(\mathbf{G})$

\mathbf{G} acts with finitely many orbits on \mathcal{N}

- ▶ Infinitely many \otimes -ideals
- ▶ Infinitely many antispherical right p -cells

To do:

- ▶ Quantum case: fill in bottom arrow
- ▶ Reductive group case: everything

g-cohomology

Quantum case

\mathfrak{g} = small quantum group

\mathcal{N} = nilpotent cone in $\text{Lie}(\mathbf{G})$

Thm (Ginzburg–Kumar)

There is a \mathbf{G} -equivariant ring isomorphism

$$\text{Ext}_{\mathfrak{g}}^{\bullet}(\mathbb{C}, \mathbb{C}) = \mathbb{C}[\mathcal{N}].$$

Reductive group case

\mathfrak{g} = first Frobenius kernel

\mathcal{N} = nilpotent cone in $\text{Lie}(\mathbf{G})$

Thm (Andersen–Jantzen, Friedlander–Parshall)

There is a \mathbf{G} -equivariant ring isomorphism

$$\text{Ext}_{\mathfrak{g}}^{\bullet}(\mathbb{k}, \mathbb{k}) = \mathbb{k}[\mathcal{N}].$$

Defn

For any \mathbf{G} -module M , its **g-cohomology** is

$$H_{\mathfrak{g}}^{\bullet}(M) := \text{Ext}_{\mathfrak{g}}^{\bullet}(\mathbb{k}, M).$$

This is naturally a $\begin{cases} \mathbb{k}[\mathcal{N}]\text{-module, with a compatible } \mathbf{G}\text{-action} \\ \text{a } \mathbf{G}\text{-equivariant coherent sheaf on } \mathcal{N}. \end{cases}$



g-cohomology of tilting G-modules

Recall: for any G-module M , $H_g^\bullet(M) \in \text{Coh}^G(\mathcal{N})$.

Prop

1. $H_g^\bullet(T(\lambda)) = 0$ unless $\lambda = w \bullet_\ell 0$ for w min'l coset representative for $W_f \backslash W_{\text{ext}} / W_f$.
2. Every principal \otimes -ideal has a generator $T(\lambda)$ with $H_g^\bullet(T(\lambda)) \neq 0$.

For the remaining parts, assume $H_g^\bullet(T(\lambda)) \neq 0$:

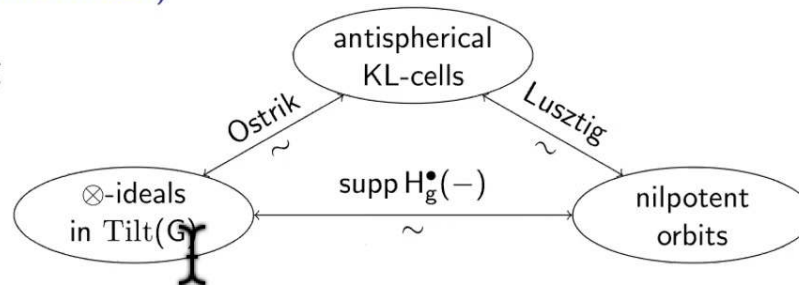
3. $\text{supp } H_g^\bullet(T(\lambda))$ is the closure of 1 nilpotent orbit
4. If $\langle T(\lambda) \rangle_\otimes = \langle T(\mu) \rangle_\otimes$, then $\text{supp } H_g^\bullet(T(\lambda)) = \text{supp } H_g^\bullet(T(\mu))$.

Get a map

$$\{\text{principal } \otimes\text{-ideals in } \text{Tilt}(G)\} \rightarrow \{\text{nilpotent orbits}\}$$

Conj (Humphreys), Thm (Bezrukavnikov)

In the quantum case, the following diagram commutes:

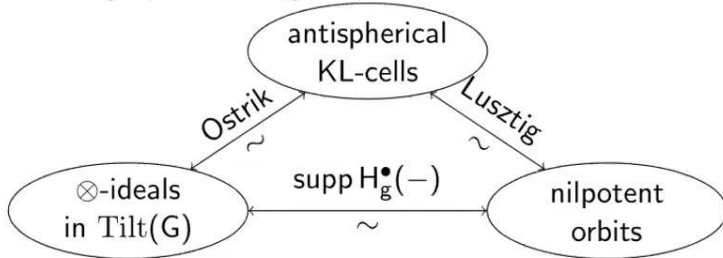


Humphreys conjecture

Quantum case

Thm (Bezrukavnikov)

For any quantum group:



Humphreys's formulation: formula for

$$\mathbf{X}^+ \xrightarrow{\text{supp } H_g^\bullet(T(-))} \{\text{nilpotent orbits}\}$$

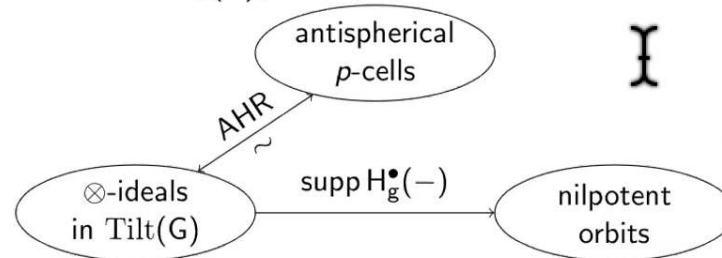
in terms of KL-cells, not \otimes -ideals.

3 nice bijections

Reductive group case

Thm (Hardesty for SL_n , A.–Hardesty–Riche)

For $G = SL_n(\mathbb{k})$, or if $p \gg 0$:



where the bottom arrow is given by the same formula

$$\mathbf{X}^+ \xrightarrow{\text{supp } H_g^\bullet(T(-))} \{\text{nilpotent orbits}\}$$

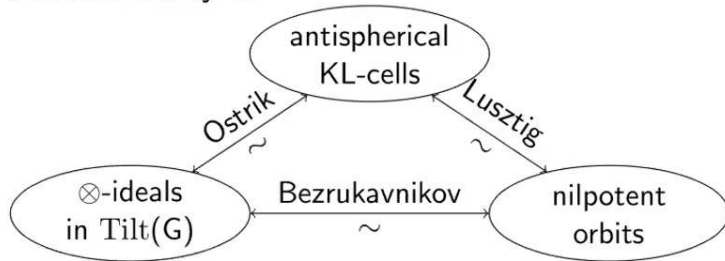
as in the quantum case (using KL-cells, **not** p -cells).

1 bijection, 1 surjection, 1 missing map

Goal for the rest of this talk: explain (conjecturally) how to improve this picture, get 3 bijections.

More on $\text{Coh}^G(\mathcal{N})$ in the quantum case

Further study of



Vector bundles on orbits

For $\mathcal{O} \subset \mathcal{N}$, $x \in \mathcal{O}$, recall

$$\text{Coh}^G(\mathcal{O}) \cong \text{Rep}(\mathbf{G}^x) \supset \text{Rep}(\mathbf{G}_{\text{red}}^x)$$

$\mathbf{G}_{\text{red}}^x$ = reductive quotient of \mathbf{G}^x .

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A vector bundle $\mathcal{F} \in \text{Coh}^G(\mathcal{O})$ is

- ▶ **irreducible** if corresp to irred \mathbf{G}^x - or $\mathbf{G}_{\text{red}}^x$ -rep
- ▶ **tilting** if corresp to tilting $\mathbf{G}_{\text{red}}^x$ -rep

Thm (Bezrukavnikov)

In the quantum case:

Assume $H_g^\bullet(T(\lambda)) \neq 0$ and $\text{supp } H_g^\bullet(T(\lambda)) = \overline{\mathcal{O}}$.

$H_g^\bullet(T(\lambda))|_{\mathcal{O}}$ = an **irreducible G -eqvt** vec bdle

The assignment $\lambda \mapsto H_g^\bullet(T(\lambda))|_{\mathcal{O}}$ realizes a bijection

$$W_f \backslash W_{\text{ext}} / W_f \xleftrightarrow{\sim} \left\{ \left(\begin{array}{cc} \text{nilp} & \text{irred} \\ \text{orbit} & \text{vec bdle} \end{array} \right) \right\}$$

called the **Lusztig–Vogan bijection**.

But...

In the reductive group case, the thm above is **false**, already for $G = \text{SL}_2$. Instead, examples suggest:

$H_g^\bullet(T(\lambda))|_{\mathcal{O}}$ = an **indecomp tilting G -eqvt** vec bdle

Tilting vector bundles on nilpotent orbits

Quantum case

Thm (Bezrukavnikov)

Assume $H_g^\bullet(T(\lambda)) \neq 0$ and $\text{supp } H_g^\bullet(T(\lambda)) = \overline{\mathcal{O}}$.

$H_g^\bullet(T(\lambda))|_{\mathcal{O}}$ = an **irreducible** G -eqvt vec bdle

The assignment $\lambda \mapsto H_g^\bullet(T(\lambda))|_{\mathcal{O}}$ realizes a bijection

$$W_f \backslash W_{\text{ext}} / W_f \xrightarrow{\sim} \left\{ \left(\begin{array}{cc} \text{nilp} & \text{irred} \\ \text{orbit} & \text{vec bdle} \end{array} \right) \right\}$$

called the Lusztig–Vogan bijection.

Reductive group case

Conj (A.–Hardesty–Riche)

Assume $H_g^\bullet(T(\lambda)) \neq 0$ and $\text{supp } H_g^\bullet(T(\lambda)) = \overline{\mathcal{O}}$.

$H_g^\bullet(T(\lambda))|_{\mathcal{O}}$ = an **indecomp tilting** G -eqvt vec bdle

The assignment $\lambda \mapsto H_g^\bullet(T(\lambda))|_{\mathcal{O}}$ realizes a bijection

$$W_f \backslash W_{\text{ext}} / W_f \xrightarrow{\sim} \left\{ \left(\begin{array}{cc} \text{nilp} & \text{indecomp tilting} \\ \text{orbit} & \text{vec bdle} \end{array} \right) \right\}$$

matching the Lusztig–Vogan bijection. **Moreover,**

$$\langle T(\lambda) \rangle_{\otimes} = \langle T(\mu) \rangle_{\otimes} \quad \text{in } \text{Tilt}(G)$$

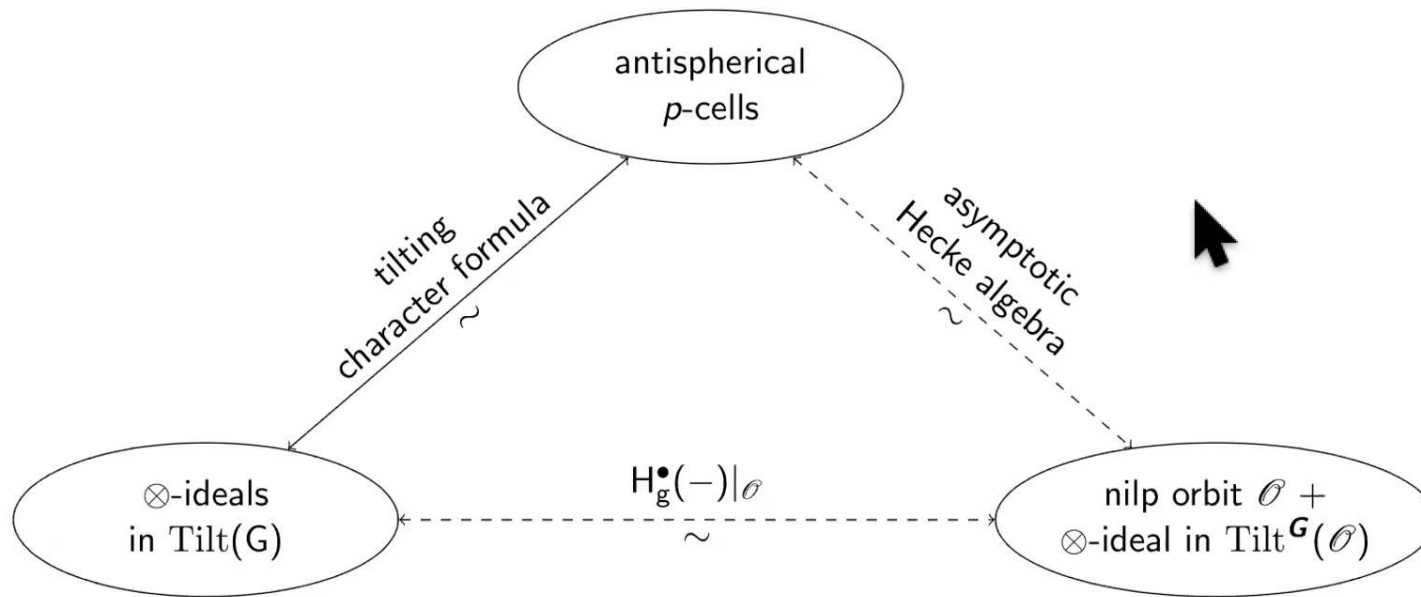
if and only if $\text{supp } H_g^\bullet(T(\lambda)) = \text{supp } H_g^\bullet(T(\mu))$ and

$$\langle H_g^\bullet(T(\lambda))|_{\mathcal{O}} \rangle_{\otimes} = \langle H_g^\bullet(T(\mu))|_{\mathcal{O}} \rangle_{\otimes} \quad \text{in } \text{Tilt}^G(\mathcal{O})$$

Main conjecture

Conj (A.–Hardesty–Riche)

There is a commutative diagram of three bijections



True for SL_2, SL_3, SO_5 .

Tensor categories attached to antispherical KL-cells

Quantum case

\mathfrak{c} = an antispherical KL-cell

Lusztig defined a category

$$\mathcal{A}_{\mathfrak{c}}$$

as a subquotient of semisimple obj in $\text{Perv}_I(\text{Fl})$.

- ▶ monoidal, under “truncated convolution” \odot
- ▶ categorifies (part of) **asymptotic Hecke alg** (modify struc consts for \mathcal{H}_{ext} in KL-basis).

Thm (Bezrukavnikov, Finkelberg, Ostrik)

Suppose \mathfrak{c} corresp to $\mathcal{O} \ni x$.

$$(\mathcal{A}_{\mathfrak{c}}, \odot) \cong (\text{Rep}(\mathbf{G}_{\text{red}}^x), \otimes).$$

Reductive group case

\mathfrak{c} = an antispherical KL-cell (**not** a p -cell)

Hope: still possible to define

$${}^p\mathcal{A}_{\mathfrak{c}}$$

as a subquotient of parity sheaves on Fl , such that:

- ▶ “truncated convolution” \odot still makes sense
- ▶ still categorifies the same part of the asymptotic Hecke algebra

Hope

Suppose \mathfrak{c} corresp to $\mathcal{O} \ni x$.

$$({}^p\mathcal{A}_{\mathfrak{c}}, \odot) \cong (\text{Tilt}(\mathbf{G}_{\text{red}}^x), \otimes).$$

Combinatorics for GL_n

Quantum case

Lusztig, Ostrik, Bezrukavnikov \implies

$$\left\{ \begin{array}{l} \otimes\text{-ideals} \\ \text{in Tilt}(GL_n) \end{array} \right\} \begin{array}{l} \xrightarrow{\sim} \\ \xrightarrow{\sim} \end{array} \left\{ \begin{array}{l} \text{nilpotent} \\ \text{orbits} \end{array} \right\} \\ \xrightarrow{\sim} \{ \text{partitions } \pi \vdash n \}$$

For $\pi \vdash n$, let $\mathcal{O}_\pi =$ corresp orbit

If $\pi = [p_1^{a_1}, \dots, p_k^{a_k}]$, let $\text{mult}(\pi) = (a_1, \dots, a_k)$.

$$x \in \mathcal{O}_\pi \implies (GL_n)_{\text{red}}^x \cong GL_{\text{mult}(\pi)}.$$

Reductive group case

Conjecture \implies

$$\left\{ \begin{array}{l} \otimes\text{-ideals} \\ \text{in Tilt}(GL_n) \end{array} \right\} \begin{array}{l} \xrightarrow{\sim} \\ \xrightarrow{\sim} \end{array} \left\{ \begin{array}{l} \text{nilpotent } \otimes\text{-ideal in} \\ \text{orbit } \text{ , Tilt}(\text{Coh}^G(\mathcal{O})) \end{array} \right\} \\ \xrightarrow{\sim} \{ (\pi, \otimes\text{-ideal in Tilt}(GL_{\text{mult}(\pi)})) \}$$

Iterate:

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$$\left\{ \begin{array}{l} \otimes\text{-ideals} \\ \text{in Tilt}(GL_n) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{sequences of multipartitons} \\ (\pi_1, \pi_2, \dots,) \end{array} \right\}$$

where:

- ▶ $\pi_1 \vdash n$
- ▶ For $k > 1$, $\pi_k \vdash \text{mult}(\pi_{k-1})$
- ▶ For $k \gg 1$, π_k is trivial (all 1's)