Title: Conjectures on p-cells, tilting modules, and nilpotent orbits

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Collection: Geometric Representation Theory

Date: June 26, 2020 - 12:00 PM

URL: http://pirsa.org/20060036

Abstract: For quantum groups at a root of unity, there is a web of theorems (due to Bezrukavnikov and Ostrik, and relying on work of Lusztig) connecting the following topics: (i) tilting modules; (ii) vector bundles on nilpotent orbits; and (iii) Kazhdanâ€“Lusztig cells in the affine Weyl group. In this talk, I will review these results, and I will explain a (partly conjectural) analogous picture for reductive algebraic groups over fields of positive characteristic, inspired by a conjecture of Humphreys. This is joint work with W. Hardesty and S. Riche.

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Conjectures on p-cells, tilting modules, and nilpotent orbits

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June 26, 2020

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Notation

Consider two cases simultaneously:

Quantum case

$\Bbbk = \mathbb{C}$

G = quantum group over $\Bbbk$ at ℓ-th root of unity

Assume: $\ell > h$ = Coxeter number

Reductive group case case

$\Bbbk$ = algebraically closed field of char. $p > 0$

G = connected reductive group over $\Bbbk$

Assume: $p > h$ = Coxeter number

Both cases:

$\mathbf{X}$ = weight lattice, $\mathbf{X}^+$ = dominant weights for G

$\mathrm{T}(\lambda)$, $\lambda \in \mathbf{X}^+$: indecomposable tilting module of highest weight λ

$\mathrm{Tilt}(G)$ = the additive $\otimes$-category of tilting G-modules

A $\otimes$-**ideal** is a full additive subcategory $I \subset \mathrm{Tilt}(\mathrm{G})$ such that:

- For any $X \in \mathrm{Tilt}(\mathrm{G})$ and $Y \in I$, we have $X \otimes Y \in I$.
- I is closed under $\oplus$ and direct summands

The **principal** $\otimes$-**ideal** generated by $\mathrm{T}(\lambda)$ is the smallest $\otimes$-ideal containing $\mathrm{T}(\lambda)$.

Warm-up: tensor ideals of tilting modules for SL_2

Problem

Classify principal $\otimes$-ideals in $\mathrm{Tilt}(\mathrm{G})$ for $\mathrm{G} = \mathrm{SL}_2$ (either quantum or reductive case)

Dominant weights: $\mathbf{X}^+ = \mathbb{Z}_{\geq 0}$.

- $\mathrm{T}(0) =$ triv rep, $\mathrm{T}(1) =$ defining rep on $\Bbbk^2$
- All summands of $\mathrm{T}(1)^{\otimes m}$ are tilting
- $\mathrm{T}(m)$ occurs as summand of $\mathrm{T}(1)^{\otimes m}$

Specific cases

0. $\mathrm{SL}_2(\mathbb{C})$: every rep completely reducible & tilting. $\mathrm{T}(m) =$ irreducible rep of highest weight m.
   - Only 1 principal $\otimes$-ideal (the **unit ideal** = the whole category $\mathrm{Tilt}(\mathrm{G})$)
   - Proof: $\forall m$, $\mathrm{T}(m) \otimes \mathrm{T}(m)^*$ contains the triv module $\mathrm{T}(0)$ as a direct summand

1. (Quantum case) Quantum SL_2 at ℓth root of 1: Exactly 2 principal $\otimes$-ideals:
   - For $0 \leq m \leq \ell - 2$, $\langle \mathrm{T}(m) \rangle_\otimes =$ the unit ideal.
   - $\forall m \geq \ell - 1$, $\langle \mathrm{T}(m) \rangle_\otimes = \langle \mathrm{T}(\ell - 1) \rangle_\otimes$

2. (Reductive group case) $\mathrm{SL}_2(\Bbbk)$, $\mathrm{char}\Bbbk = p > 2$: Infinitely many principal $\otimes$-ideals
   - For $m \in \mathbb{Z}_{\geq 0}$, let

$$\begin{aligned}\nu(m) &= \min\{j \mid m < p^{j+1} - 1\} \\ &= \text{largest power of } p \text{ in} \\ &\quad \text{the } p\text{-adic expansion of } m+1\end{aligned}$$

   - $\langle \mathrm{T}(m) \rangle_\otimes = \langle \mathrm{T}(n) \rangle_\otimes \iff \nu(m) = \nu(n)$.

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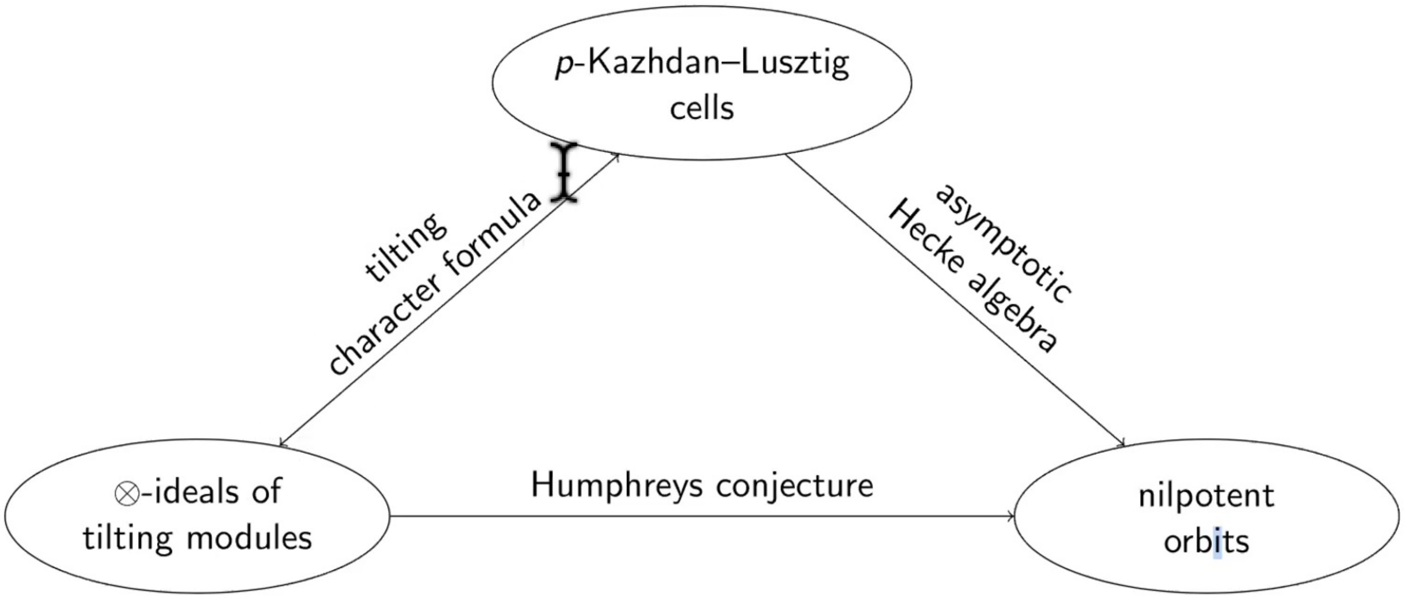
Topics in this talk



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Based rings

A = a commutative ring (usually $\mathbb{Z}$ or $\mathbb{Z}[v, v^{-1}]$)
R = an A-algebra, free over A, with A-basis $\mathcal{B}$

Defn

A **based ideal** is an ideal $I \subset R$ spanned by $I \cap \mathcal{B}$.
The **principal based ideal** gen by $x \in R$ is the smallest based ideal containing x.

Example 1

$K_0(\mathrm{Tilt}(G))$ = split Grothendieck group of $\mathrm{Tilt}(G)$.
Basis: $\mathcal{B} = \{[T(\lambda)] \mid \lambda \in \mathbf{X}^+\}$.

principal based ideals in $K_0(\mathrm{Tilt}(G))$ $\overset{\sim}{\longleftrightarrow}$ principal $\otimes$-ideals in $\mathrm{Tilt}(G)$

Example 2

$\mathcal{H}_{\mathrm{ext}}$ = extended affine Hecke algebra over $\mathbb{Z}[v, v^{-1}]$

- Basis: "standard basis" $\{T_w : w \in W_{\mathrm{ext}}\}$
  The unit ideal is the unique based ideal.

Example 2 continued

- Basis: Kazhdan–Lusztig (canonical) basis:

  (left, right, 2-sided) principal based ideals = (left, right, 2-sided) KL-cells

- Basis: p-canonical basis:

  (left, right, 2-sided) principal based ideals = (left, right, 2-sided) p-cells

Example 3

$\mathcal{M}_{\mathrm{asph}} = (\mathrm{sgn}) \otimes_{\mathcal{H}_f} \mathcal{H}_{\mathrm{ext}}$ = **antispherical module**
Basis: inherits KL- and p-canonical bases from $\mathcal{H}_{\mathrm{ext}}$

principal based submodules = antispherical KL- or p-cells

Tensor ideals and antispherical cells

Thm (Ostrik)
In the quantum case, there is a bijection

$$\left\{\begin{matrix}\text{principal}\\ \otimes\text{-ideals in } \mathrm{Tilt}(\mathsf{G})\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{antispherical}\\ \text{KL-cells}\end{matrix}\right\}$$

Thm (A.–Hardesty–Riche)
In the reductive group case, there is a bijection

$$\left\{\begin{matrix}\text{principal}\\ \otimes\text{-ideals in } \mathrm{Tilt}(\mathsf{G})\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{antispherical}\\ p\text{-cells}\end{matrix}\right\}$$

Proof sketch
Define an explicit map

$$\alpha : K_0(\mathrm{Tilt}(\mathsf{G})) \to \mathcal{M}_{\mathrm{asph}}|_{v=1}$$

(Roughly, α is the "principal block part of the character formula.")

To show: α sends based ideals to based submodules. Use:

$$\text{in the } \left\{\begin{matrix}\text{quantum [Soergel]}\\ \text{reductive group [A.–Makisumi–Riche–Williamson]}\end{matrix}\right. \text{ case, the } \left\{\begin{matrix}\text{KL-basis}\\ p\text{-canonical basis}\end{matrix}\right. \text{ for } \mathcal{M}_{\mathrm{asph}}$$

controls characters of tilting modules in the principal block of $\mathrm{Rep}(\mathsf{G})$.

How many antispherical cells are there?

Quantum case

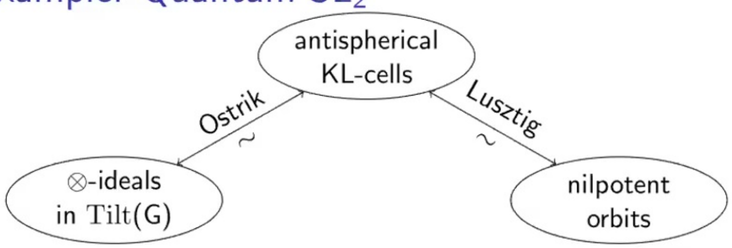
$\boldsymbol{G}$ = reductive gp/$\mathbb{C}$ with same root datum as G
$\mathcal{N}$ = nilpotent cone in $\mathrm{Lie}(\boldsymbol{G})$

$\boldsymbol{G}$ acts with finitely many orbits on $\mathcal{N}$

Thm (Lusztig)

$$\left\{\begin{matrix}\text{antispherical}\\ \text{KL-cells}\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{nilpotent}\\ \text{orbits}\end{matrix}\right\}$$

Example: Quantum SL_2



unit ideal, $\langle \mathrm{T}(\ell - 1)\rangle_{\otimes}$

$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$

Reductive group case

$\boldsymbol{G}$ = Frobenius twist of G (i.e. $\mathrm{G}^{(1)}$)
$\mathcal{N}$ = nilpotent cone in $\mathrm{Lie}(\boldsymbol{G})$

$\boldsymbol{G}$ acts with finitely many orbits on $\mathcal{N}$

- Infinitely many $\otimes$-ideals
- Infinitely many antispherical right p-cells

To do:

- Quantum case: fill in bottom arrow
- Reductive group case: everything

g-cohomology

Quantum case

g = small quantum group

$\mathcal{N}$ = nilpotent cone in $\mathrm{Lie}(\boldsymbol{G})$

Thm (Ginzburg–Kumar)

There is a $\boldsymbol{G}$-equivariant ring isomorphism

$$\mathrm{Ext}^{\bullet}_{\mathrm{g}}(\mathbb{C}, \mathbb{C}) = \mathbb{C}[\mathcal{N}].$$

Reductive group case

g = first Frobenius kernel

$\mathcal{N}$ = nilpotent cone in $\mathrm{Lie}(\boldsymbol{G})$

Thm (Andersen–Jantzen, Friedlander–Parshall)

There is a $\boldsymbol{G}$-equivariant ring isomorphism

$$\mathrm{Ext}^{\bullet}_{\mathrm{g}}(\Bbbk, \Bbbk) = \Bbbk[\mathcal{N}].$$

Defn

For any G-module M, its g-**cohomology** is

$$\mathrm{H}^{\bullet}_{\mathrm{g}}(M) := \mathrm{Ext}^{\bullet}_{\mathrm{g}}(\Bbbk, M).$$

This is naturally a $\begin{cases} \Bbbk[\mathcal{N}]\text{-module, with a compatible } \boldsymbol{G}\text{-action} \\ \text{a } \boldsymbol{G}\text{-equivariant coherent sheaf on } \mathcal{N}. \end{cases}$

$\mathfrak{g}$-cohomology of tilting G-modules

Recall: for any G-module M, $H^\bullet_{\mathfrak{g}}(M) \in \mathrm{Coh}^{G}(\mathcal{N})$.

Prop

1. $H^\bullet_{\mathfrak{g}}(T(\lambda)) = 0$ unless $\lambda = w \bullet_\ell 0$ for w min'l coset representative for $W_{\mathrm{f}} \backslash W_{\mathrm{ext}} / W_{\mathrm{f}}$.
2. Every principal $\otimes$-ideal has a generator $T(\lambda)$ with $H^\bullet_{\mathfrak{g}}(T(\lambda)) \neq 0$.

For the remaining parts, assume $H^\bullet_{\mathfrak{g}}(T(\lambda)) \neq 0$:

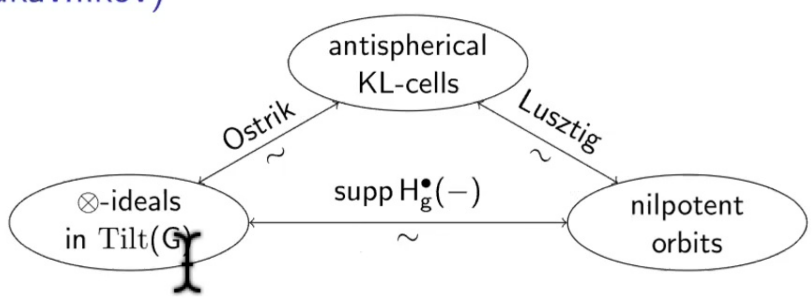
3. $\mathrm{supp}\, H^\bullet_{\mathfrak{g}}(T(\lambda))$ is the closure of 1 nilpotent orbit
4. If $\langle T(\lambda)\rangle_\otimes = \langle T(\mu)\rangle_\otimes$, then $\mathrm{supp}\, H^\bullet_{\mathfrak{g}}(T(\lambda)) = \mathrm{supp}\, H^\bullet_{\mathfrak{g}}(T(\mu))$.

Get a map

$$\{\text{principal } \otimes\text{-ideals in } \mathrm{Tilt}(G)\} \to \{\text{nilpotent orbits}\}$$

Conj (Humphreys), Thm (Bezrukavnikov)

In the quantum case, the following diagram commutes:

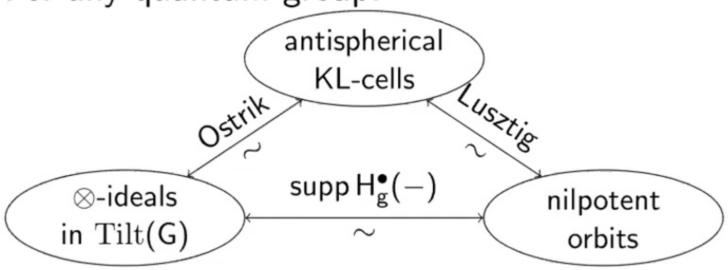


Humphreys conjecture

Quantum case

Thm (Bezrukavnikov)

For any quantum group:



Humphreys's formulation: formula for

$$\mathbf{X}^+ \xrightarrow{\operatorname{supp} H^\bullet_{\mathfrak{g}}(T(-))} \{\text{nilpotent orbits}\}$$

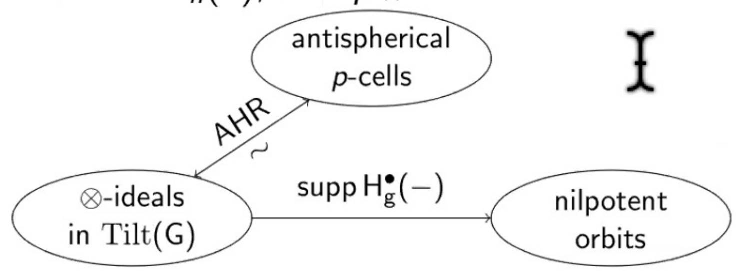
in terms of KL-cells, not ⊗-ideals.

3 nice bijections

Reductive group case

Thm (Hardesty for SL_n, A.–Hardesty–Riche)

For $G = SL_n(\Bbbk)$, or if $p \gg 0$:



where the bottom arrow is given by the same formula

$$\mathbf{X}^+ \xrightarrow{\operatorname{supp} H^\bullet_{\mathfrak{g}}(T(-))} \{\text{nilpotent orbits}\}$$

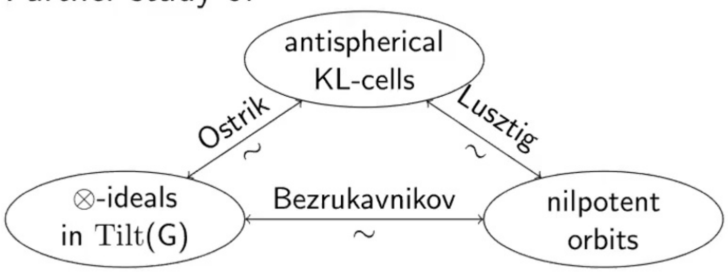
as in the quantum case (using KL-cells, **not** p-cells).

1 bijection, 1 surjection, 1 missing map

Goal for the rest of this talk: explain (conjecturally) how to improve this picture, get 3 bijections.

More on $\mathrm{Coh}^{G}(\mathcal{N})$ in the quantum case

Further study of



Vector bundles on orbits

For $\mathscr{O} \subset \mathcal{N}$, $x \in \mathscr{O}$, recall

$$\mathrm{Coh}^{G}(\mathscr{O}) \cong \mathrm{Rep}(G^{x}) \quad \supset \quad \mathrm{Rep}(G^{x}_{\mathrm{red}})$$

G^{x}_{red} = reductive quotient of G^{x}.

A vector bundle $\mathcal{F} \in \mathrm{Coh}^{G}(\mathscr{O})$ is

- **irreducible** if corresp to irred G^{x}- or G^{x}_{red}-rep
- **tilting** if corresp to tilting G^{x}_{red}-rep

Thm (Bezrukavnikov)

In the quantum case:
Assume $\mathrm{H}^{\bullet}_{\mathfrak{g}}(\mathrm{T}(\lambda)) \neq 0$ and $\mathrm{supp}\, \mathrm{H}^{\bullet}_{\mathfrak{g}}(\mathrm{T}(\lambda)) = \overline{\mathscr{O}}$.

$\mathrm{H}^{\bullet}_{\mathfrak{g}}(\mathrm{T}(\lambda))|_{\mathscr{O}}$ = an **irreducible** G-eqvt vec bdle

The assignment $\lambda \mapsto \mathrm{H}^{\bullet}_{\mathfrak{g}}(\mathrm{T}(\lambda))|_{\mathscr{O}}$ realizes a bijection

$$W_{\mathrm{f}} \backslash W_{\mathrm{ext}} / W_{\mathrm{f}} \xleftrightarrow{\sim} \left\{ \left(\begin{matrix} \text{nilp} \\ \text{orbit} \end{matrix}, \begin{matrix} \text{irred} \\ \text{vec bdle} \end{matrix} \right) \right\}$$

called the **Lusztig–Vogan bijection**.

But...

In the reductive group case, the thm above is **false**, already for $\mathrm{G} = \mathrm{SL}_2$. Instead, examples suggest:

$\mathrm{H}^{\bullet}_{\mathfrak{g}}(\mathrm{T}(\lambda))|_{\mathscr{O}}$ = an **indecomp tilting** G-eqvt vec bdle

Tilting vector bundles on nilpotent orbits

Quantum case

Thm (Bezrukavnikov)

Assume $\mathsf{H}^\bullet_{\mathfrak{g}}(\mathsf{T}(\lambda)) \neq 0$ and $\operatorname{supp} \mathsf{H}^\bullet_{\mathfrak{g}}(\mathsf{T}(\lambda)) = \overline{\mathscr{O}}$.

$$\mathsf{H}^\bullet_{\mathfrak{g}}(\mathsf{T}(\lambda))|_{\mathscr{O}} = \text{an } \textbf{irreducible } \boldsymbol{G}\text{-eqvt vec bdle}$$

The assignment $\lambda \mapsto \mathsf{H}^\bullet_{\mathfrak{g}}(\mathsf{T}(\lambda))|_{\mathscr{O}}$ realizes a bijection

$$W_{\mathrm{f}} \backslash W_{\mathrm{ext}} / W_{\mathrm{f}} \stackrel{\sim}{\longleftrightarrow} \left\{ \left(\begin{array}{c} \text{nilp} \\ \text{orbit} \end{array}, \begin{array}{c} \text{irred} \\ \text{vec bdle} \end{array} \right) \right\}$$

called the Lusztig–Vogan bijection.

Reductive group case

Conj (A.–Hardesty–Riche)

Assume $\mathsf{H}^\bullet_{\mathfrak{g}}(\mathsf{T}(\lambda)) \neq 0$ and $\operatorname{supp} \mathsf{H}^\bullet_{\mathfrak{g}}(\mathsf{T}(\lambda)) = \overline{\mathscr{O}}$.

$$\mathsf{H}^\bullet_{\mathfrak{g}}(\mathsf{T}(\lambda))|_{\mathscr{O}} = \text{an } \textbf{indecomp tilting } \boldsymbol{G}\text{-eqvt vec bdle}$$

The assignment $\lambda \mapsto \mathsf{H}^\bullet_{\mathfrak{g}}(\mathsf{T}(\lambda))|_{\mathscr{O}}$ realizes a bijection

$$W_{\mathrm{f}} \backslash W_{\mathrm{ext}} / W_{\mathrm{f}} \stackrel{\sim}{\longleftrightarrow} \left\{ \left(\begin{array}{c} \text{nilp} \\ \text{orbit} \end{array}, \begin{array}{c} \text{indecomp tilting} \\ \text{vec bdle} \end{array} \right) \right\}$$

matching the Lusztig–Vogan bijection. **Moreover,**

$$\langle \mathsf{T}(\lambda) \rangle_{\otimes} = \langle \mathsf{T}(\mu) \rangle_{\otimes} \quad \text{in } \mathrm{Tilt}(\mathsf{G})$$

if and only if $\operatorname{supp} \mathsf{H}^\bullet_{\mathfrak{g}}(\mathsf{T}(\lambda)) = \operatorname{supp} \mathsf{H}^\bullet_{\mathfrak{g}}(\mathsf{T}(\mu))$ and

$$\langle \mathsf{H}^\bullet_{\mathfrak{g}}(\mathsf{T}(\lambda))|_{\mathscr{O}} \rangle_{\otimes} = \langle \mathsf{H}^\bullet_{\mathfrak{g}}(\mathsf{T}(\mu))|_{\mathscr{O}} \rangle_{\otimes} \quad \text{in } \mathrm{Tilt}^{\boldsymbol{G}}(\mathscr{O})$$

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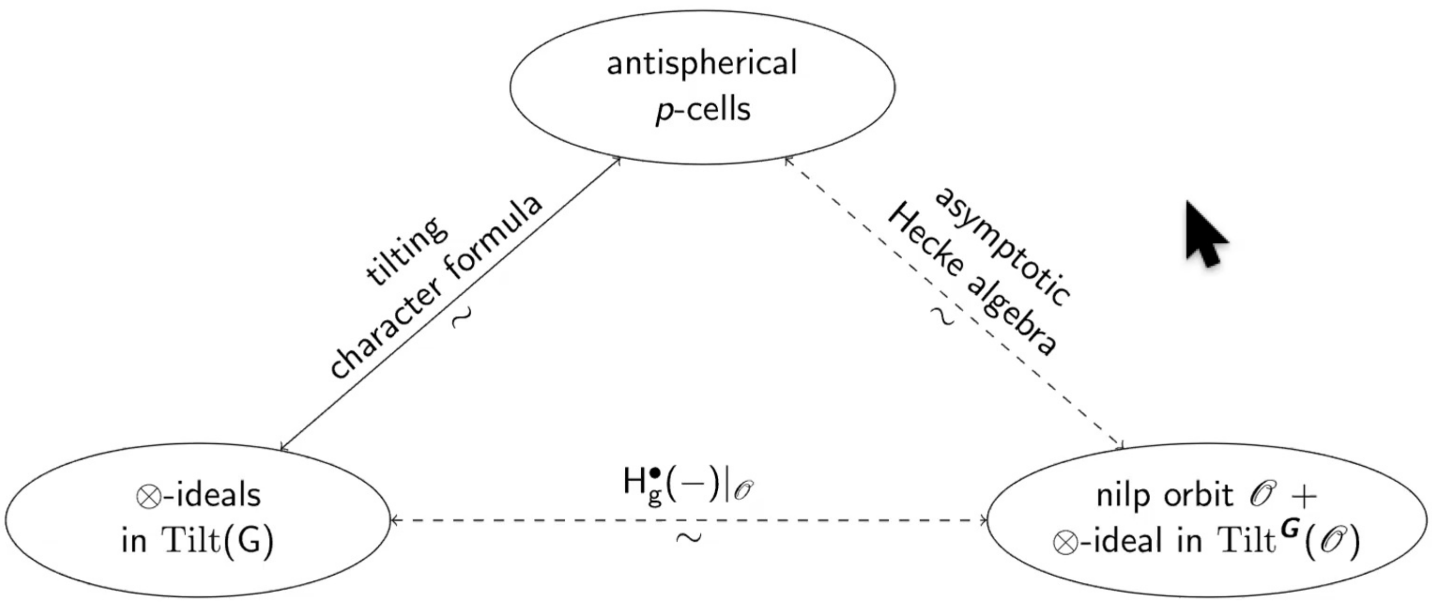
Main conjecture

Conj (A.–Hardesty–Riche)

There is a commutative diagram of three bijections



True for SL_2, SL_3, SO_5.

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Tensor categories attached to antispherical KL-cells

Quantum case

$\mathbf{c}$ = an antispherical KL-cell

Lusztig defined a category

$$\mathscr{A}_{\mathbf{c}}$$

as a subquotient of semisimple obj in $\mathrm{Perv}_I(\mathrm{Fl})$.

- monoidal, under "truncated convolution" $\odot$
- categorifies (part of) **asymptotic Hecke alg** (modify struc consts for $\mathcal{H}_{\mathrm{ext}}$ in KL-basis).

Thm (Bezrukavnikov, Finkelberg, Ostrik)

Suppose $\mathbf{c}$ corresp to $\mathscr{O} \ni x$.

$$(\mathscr{A}_{\mathbf{c}}, \odot) \cong (\mathrm{Rep}(\boldsymbol{G}^x_{\mathrm{red}}), \otimes).$$

Reductive group case

$\mathbf{c}$ = an antispherical KL-cell (**not** a p-cell)

Hope: still possible to define

$${}^p\mathscr{A}_{\mathbf{c}}$$

as a subquotient of parity sheaves on Fl, such that:

- "truncated convolution" $\odot$ still makes sense
- still categorifies the same part of the asymptotic Hecke algebra

Hope

Suppose $\mathbf{c}$ corresp to $\mathscr{O} \ni x$.

$$({}^p\mathscr{A}_{\mathbf{c}}, \odot) \cong (\mathrm{Tilt}(\boldsymbol{G}^x_{\mathrm{red}}), \otimes).$$

Combinatorics for GL_n

Quantum case

Lusztig, Ostrik, Bezrukavnikov $\Longrightarrow$

$$\left\{\begin{matrix}\otimes\text{-ideals}\\ \text{in } \mathrm{Tilt}(\mathrm{GL}_n)\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{nilpotent}\\ \text{orbits}\end{matrix}\right\}$$

$$\overset{\sim}{\longleftrightarrow} \{\text{partitions } \boldsymbol{\pi} \vdash n\}$$

For $\boldsymbol{\pi} \vdash n$, let $\mathscr{O}_{\boldsymbol{\pi}} =$ corresp orbit

If $\boldsymbol{\pi} = [p_1^{a_1}, \ldots, p_k^{a_k}]$, let $\mathrm{mult}(\boldsymbol{\pi}) = (a_1, \ldots, a_k)$.

$$x \in \mathscr{O}_{\boldsymbol{\pi}} \quad \Longrightarrow \quad (\mathrm{GL}_n)^x_{\mathrm{red}} \cong \mathrm{GL}_{\mathrm{mult}(\boldsymbol{\pi})}.$$

Reductive group case

Conjecture $\Longrightarrow$

$$\left\{\begin{matrix}\otimes\text{-ideals}\\ \text{in } \mathrm{Tilt}(\mathrm{GL}_n)\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\left(\begin{matrix}\text{nilpotent}\\ \text{orbit}\end{matrix}, \begin{matrix}\otimes\text{-ideal in}\\ \mathrm{Tilt}(\mathrm{Coh}^{\boldsymbol{G}}(\mathscr{O}))\end{matrix}\right)\right\}$$

$$\overset{\sim}{\longleftrightarrow} \{(\boldsymbol{\pi}, \otimes\text{-ideal in } \mathrm{Tilt}(\mathrm{GL}_{\mathrm{mult}(\boldsymbol{\pi})}))\}$$

Iterate:

$$\left\{\begin{matrix}\otimes\text{-ideals}\\ \text{in } \mathrm{Tilt}(\mathrm{GL}_n)\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{sequences of multipartitons}\\ (\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots,)\end{matrix}\right\}$$

where:

- $\boldsymbol{\pi}_1 \vdash n$
- For $k > 1$, $\boldsymbol{\pi}_k \vdash \mathrm{mult}(\boldsymbol{\pi}_{k-1})$
- For $k \gg 1$, $\boldsymbol{\pi}_k$ is trivial (all 1's)