Title: Parabolic Hilbert schemes via the Dunkl-Opdam subalgebra

Speakers: Eugene Gorsky

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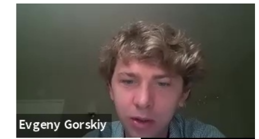
Abstract: In this note we give an alternative presentation of the rational Cherednik algebra H_c corresponding to the permutation representation of S_n. As an application, we give an explicit combinatorial basis for all standard and simple modules if the denominator of c is at least n, and describe the action of H_c in this basis. We also give a basis for the irreducible quotient of the polynomial representation and compare it to the basis of fixed points in the homology of the parabolic Hilbert scheme of points on the plane curve singularity {x^n=y^m}. This is a joint work with JosÃ© Simental and Monica Vazirani.

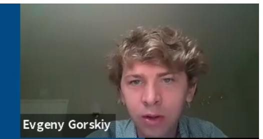


Parabolic Hilbert schemes via the Dunkl-Opdam subalgebra

Eugene Gorsky, José Simental and Monica Vazirani
University of California, Davis

June 25, 2020

Rational Cherednik algebra

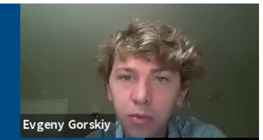


We work with the rational Cherednik algebra $H_{t,c}$ of $\mathcal{S}_n$ acting on $\mathbb{C}^n$ by permuting the coordinates. Let us recall that this is the quotient of the semidirect product algebra $\mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle \rtimes \mathcal{S}_n$ by the relations

$$[x_i, x_j] = 0, \qquad [y_i, y_j] = 0,$$
$$[y_i, x_j] = c(ij), \qquad [y_i, x_i] = t - c\sum_{j\neq i}(ij)$$

Here t and c are complex parameters. Clearly, for a nonzero complex number $a \in \mathbb{C}^*$, $H_{at,ac} \cong H_{t,c}$. For most of the talk, we will assume that $t = 1$ and write $H_c := H_{1,c}$.

Rational Cherednik algebra



The algebra H_c has a highest weight category of representations. One can define standard modules:

$$\Delta_c(\lambda) = V_\lambda \bigotimes_{\mathbb{C}[y]\rtimes S_n} H_c$$

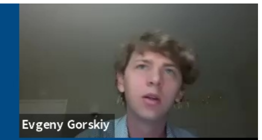
where V_λ is an irreducible representation of S_n labeled by the Young diagram λ. $\Delta_c(\lambda)$ has a unique simple quotient $L_c(\lambda)$.

Theorem (Ginzburg-Guay-Opdam-Rouquier, Berest-Etingof-Ginzburg)

- *If c is irrational or has denominator greater than n then $\Delta_c(\lambda)$ is irreducible for all λ*
- *If $c = m/n$ then $\Delta_c(\lambda)$ is irreducible unless λ is a hook*
- *If $c = m/n$ then the nontrivial morphisms between $\Delta_c(hooks)$ form the* BGG resolution*:*

$$\Delta_c(n) \leftarrow \Delta_c(n-1,1) \leftarrow \ldots \leftarrow \Delta_c(1^n).$$

Rational Cherednik algebra

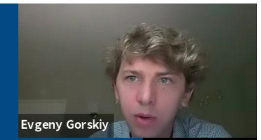


All these facts are proved using *Knizhnik-Zamolodchikov (KZ) functor* relating the representations of H_c and Hecke algebra at $q = \exp(2\pi ic)$.

We will give a new presentation of the algebra H_c to:

- Construct explicit bases in standard and simple modules
- Give a new combinatorial proof of the above theorem
- Relate $L_c(n)$ to the geometry of parabolic Hilbert schemes of points on singular surves

Affine symmetric group



The *extended affine symmetric group* $\widetilde{S}_n$ is defined as the set of n-periodic permutations of $\mathbb{Z}$, i.e. bijections $p : \mathbb{Z} \to \mathbb{Z}$ such that $p(i+n) = p(i) + n$. Such p is determined by the "window" $[p(1), \ldots, p(n)]$.

The group $\widetilde{S}_n$ has generators $s_1, \ldots, s_{n-1}$ and π such that $\pi(i) = i + 1$. We can consider *positive monoid* $\widetilde{S}_n^+$ generated by $s_1, \ldots, s_{n-1}$ and π (but not π^{-1}). A permutation p is in $\widetilde{S}_n^+$ if and only if $p(i) > 0$ for all $i > 0$.

New presentation

Evgeny Gorskiy

First, we have the family of commuting Dunkl-Opdam elements in $H_{t,c}$:

$$u_i := x_i y_i - c \sum_{j<i} (ij)$$

Let $\tau := x_1 (12 \cdots n)$, $\lambda := (12 \cdots n)^{-1} y_1$. The following is the complete set of relations between u_i, s_i, τ and λ:

$$u_i u_j = u_j u_i$$

$$s_i u_i = u_{i+1} s_i + c, \quad s_j u_i = u_i s_j \text{ if } j \neq i, i-1$$

$$\tau u_i = u_{i+1} \tau, i \neq n, \quad \tau u_n = (u_1 - t)\tau$$

$$\lambda u_i = u_{i-1} \lambda, i \neq 1, \quad \lambda u_1 = (u_n + t)\lambda$$

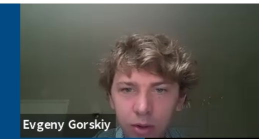
$$s_i \tau = \tau s_{i-1}, i \neq 1, \quad s_1 \tau^2 = \tau^2 s_{n-1}$$

$$s_i \lambda = \lambda s_{i+1}, i \neq n-1, \quad s_{n-1} \lambda^2 = \lambda^2 s_1$$

$$\tau \lambda = u_1, \lambda \tau = u_n + t$$

$$\lambda s_1 \tau = \tau s_{n-1} \lambda + c.$$

New presentation



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$$\tau u_i = u_{i+1}\tau, i \neq n, \quad \tau u_n = (u_1 - t)\tau$$

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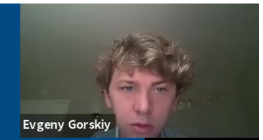
$$s_i \tau = \tau s_{i-1}, i \neq 1, \quad s_1 \tau^2 = \tau^2 s_{n-1}$$

$$s_i \lambda = \lambda s_{i+1}, i \neq n-1, \quad s_{n-1}\lambda^2 = \lambda^2 s_1$$

$$\tau\lambda = u_1, \lambda\tau = u_n + t$$

$$\lambda s_1 \tau = \tau s_{n-1}\lambda + c.$$

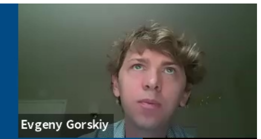
New presentation



Similar presentations of H_c appeared in the works of Griffeth, Suzuki and Webster. Several interesting subalgebras in H_c are transparent in this presentation:

- u_i and s_i generate a degenerate affine Hecke algebra
- s_i and τ generate a copy of positive affine monoid $\widetilde{\mathcal{S}_n}^+ \simeq \mathbb{C}[x_1, \ldots, x_n] \rtimes \mathcal{S}_n$. In this identification τ corresponds to π
- s_i and λ generate a copy of inverse affine monoid $(\widetilde{\mathcal{S}_n}^+)^{-1} \simeq \mathbb{C}[y_1, \ldots, y_n] \rtimes \mathcal{S}_n$. In this identification λ corresponds to π^{-1}.

Generalized eigenvalues



Theorem (GSV)

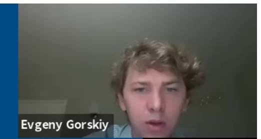
For all c, the standard module $\Delta_c(\lambda)$ has a basis labeled by pairs (a, T) where $\mathrm{a} \in \mathbb{Z}^n_{\geq 0}$ and T is an SYT of shape λ. The action of u_i in this basis is triangular with respect to a certain order, with the generalized eigenvalues

$$\mathtt{w}_i(\mathrm{a}, T) = a_i - \mathrm{ct}_T(g_{\mathrm{a}}(i))c$$

where $g_{\mathrm{a}} \in \mathcal{S}_n$ is the minimal length permutation that sorts $\mathrm{a} = (a_1, ..., a_n)$ increasingly, and $\mathrm{ct}_T(m)$ is the content of the box in T labeled by m.

The basis can be obtained by the action of the cosets $\widetilde{\mathcal{S}_n}^+ / \mathcal{S}_n$ on the eigenbasis v_T in V_λ. The order on this basis is closely related to the lexicographic order on $\widetilde{\mathcal{S}_n}^+$ in window notation.

Combinatorial representation theory

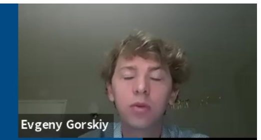


This theorem immediately implies the following:

Corollary

- *If c is irrational then $\Delta_c(\lambda)$ is irreducible for all λ*
- *If $c = a/b$ and there is a morphism between $\Delta_c(\lambda)$ and $\Delta_c(\mu)$ then λ and μ have the same b-core*
- *If $c = a/b$ and $b > n$ then $\Delta_c(\lambda)$ is irreducible*

The case $c = m/n$, standard modules



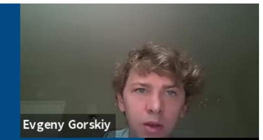
If $c = m/n$, $\gcd(m, n) = 1$ then the joint spectrum of u_i is simple, and $\Delta_c(\lambda)$ has a basis $v(\mathrm{a}, T)$ such that

$$u_i v(\mathrm{a}, T) = \mathtt{w}_i v(\mathrm{a}, T)$$
$$\tau v(\mathrm{a}, T) = v(\pi \cdot \mathrm{a}, T)$$
$$\lambda v(\mathrm{a}, T) = \mathtt{w}_1 v(\pi^{-1} \cdot \mathrm{a}, T)$$

One can compute $s_i v(\mathrm{a}, T)$ explicitly as well. Similar bases were considered by Griffeth.

In this case, we can compute all nonzero maps between standard modules (that is, all nonzero maps in BGG resolution) explicitly in these bases. This yields explicit bases in simple modules.

The case $c = m/n$, simple modules



Let us denote by $V_{\mu_\ell} := \wedge^\ell \mathbb{C}^{n-1}$ the hook representation of $\mathcal{S}_n$, so that μ_ℓ is the partition $(n-\ell, 1^\ell)$, $\ell = 0, \ldots, n-1$. Assume $0 \leq \ell < n-1$ and let $(\mathrm{a}, T) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu_\ell)$. Let us denote by i_ℓ the label of the box with smallest content of μ_ℓ under T.

Theorem

The simple module $L_c(\mu_\ell)$ has a basis indexed by pairs (a, T) such that

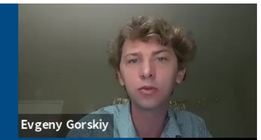
- $a_{g_{\mathrm{a}}^{-1}(n)} - m < a_{g_{\mathrm{a}}^{-1}(i_\ell)}$ *or*
- $a_{g_{\mathrm{a}}^{-1}(n)} - m = a_{g_{\mathrm{a}}^{-1}(i_\ell)}$ *and* $g_{\mathrm{a}}^{-1}(n) < g_{\mathrm{a}}^{-1}(i_\ell)$.

Example

For $\ell = 0$, $L_c(n)$ has a basis labeled by

$$\{\mathrm{a} \in \mathbb{Z}^n_{\geq 0} : a_i - a_j \leq m \text{ for every } i, j; \text{moreover, if } a_i - a_j = m \text{ then } j > i\}.$$

Hilbert schemes of singular curves



Consider the singular curve $C = \{x^m = y^n\} \subset \mathbb{C}^2$, $\gcd(m,n) = 1$. Its local ring at the origin has the basis

$$\mathcal{O} = \mathcal{O}_{C,0} = \mathbb{C}[[x]]\langle 1, y, \ldots, y^{n-1}\rangle$$

We define the Hilbert scheme

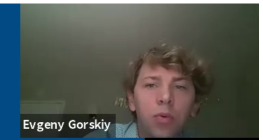
$$\mathrm{Hilb}^k(C,0) = \{J \subset \mathcal{O} : J \text{ is an ideal}, \dim \mathcal{O}/J = k\}$$

and the parabolic Hilbert scheme

$$\mathrm{PHilb}^{k,n+k}(C,0) = \{\mathcal{O} \supset J_k \supset J_{k+1} \cdots \supset J_{k+n-1} \supset J_{k+n} = xJ_k\}$$

where all J_s are ideals in $\mathcal{O}$ of codimension s.

Hilbert schemes of singular curves

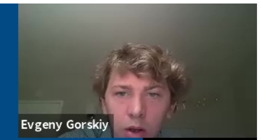


The action of $\mathbb{C}^*$ on $\mathbb{C}$ induces the action of $\mathbb{C}^*$ on Hilb and on PHilb. The fixed points are monomial ideals (resp. flags of monomial ideals):

y^2	xy^2	x^2y^2	1					
y	xy	x^2y	x^3y	x^4y	3			
1	x	x^2	x^3	x^4	x^5	x^6	2	

They are parametrized by staircases in $n \times \infty$ strip of width at most m. For example, this picture gives ideals on $C = \{x^4 = y^3\}$:
$I_{15} = \langle x^3y^2, x^5y \rangle, I_{16} = \langle x^4y^2, x^5y, x^7 \rangle, I_{17} = \langle x^4y^2, x^5y \rangle$.

Hilbert schemes of singular curves



Theorem (GSV)

There is an action of $H_{n,m}$ on the localized equivariant homology

$$U = \bigoplus_{k=0}^{\infty} H_*^{\mathbb{C}^*}(\mathrm{PHilb}_{k,n+k})$$

The corresponding representation is isomorphic to $L_{n,m}(\mathrm{triv})$.

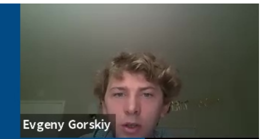
Here u_{n+1-i} correspond to the line bundles $\mathcal{L}_i = J_{k+i}/J_{k+i-1}$, $\mathcal{S}_n$ acts via certain Springer-type action. The operator τ corresponds to the map $T : \mathrm{PHilb}^{k,n+k} \to \mathrm{PHilb}^{k+1,n+k+1}$,

$$\{J_k \cdots \supset J_{k+n} = xJ_k\} \mapsto \{J_{k+1} \cdots \supset J_{k+n} = xJ_k \supset J_{k+n+1} = xJ_{k+1}\}.$$

One can check that the image of T is a zero locus of a section of a certain line bundle, and this defines an operator

$$\lambda : H_*(\mathrm{PHilb}^{k+1,n+k+1}) \to H_*(\mathrm{PHilb}^{k,n+k}).$$

Hilbert schemes of singular curves



This construction has a natural limit at $m \to \infty$ which corresponds to the Hilbert schemes of **non-reduced** curve $\{y^n = 0\}$.

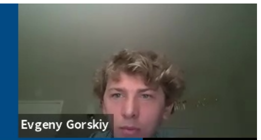
Theorem (GSV)

There is an action of $H_{0,1}$ on the localized equivariant homology

$$U = \bigoplus_{k=0}^{\infty} H_*^{\mathbb{C}^*}(\mathrm{PHilb}_{k,n+k}(\{y^n = 0\})$$

The corresponding representation is isomorphic to $\Delta_{0,1}(\mathrm{triv})$.

Coulomb branch algebra

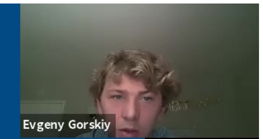


The above action is closely related to the action of *trigonometric* Cherednik algebra defined by Lusztig, Yun, Oblomkov, Varagnolo, Vasserot... in the homology of certain affine Springer fibers.

Theorem (Garner-Kivinen)

Let C be an **arbitrary** *plane curve singularity with a degree n projection to a line. Then the union $\sqcup_k \mathrm{Hilb}^k(C)$ is naturally isomorphic to a generalized affine Springer fiber in the sense of Braverman-Finkelberg-Nakajima, corresponding to the group $G = GL(n)$ and representation $V = \mathfrak{gl}_n \oplus \mathbb{C}^n$.*

Coulomb branch algebra



Given G and V as above, Braverman-Finkelberg-Nakajima defined the *Coulomb branch algebra* and its non-commutative quantization.

Theorem (Hilburn-Kamnitzer-Weekes)

The Coulomb branch algebra for (G, V) acts in the homology of (almost) all generalized Springer fibers corresponding to (G, V). If a Springer fiber is equivariant under loop rotation, then quantum Coulomb branch algebra acts in its equivariant homology.

For $G = GL(n)$ and $V = \mathfrak{gl}_n \oplus \mathbb{C}^n$ Kodera and Nakajima identified the quantum Coulomb branch algebra with the spherical rational Cherednik algebra. Garner and Kivinen have recently computed its action on the homology of $\sqcup_k \mathrm{Hilb}^k(\{x^m = y^n\})$, it would be very interesting to compare the two actions.