Title: The ``Springer'' representation of the DAHA

Speakers: Monica Vazirani

Collection: Geometric Representation Theory

Date: June 24, 2020 - 2:00 PM

URL: http://pirsa.org/20060033

Abstract: The Springer resolution and resulting Springer sheaf are key players in geometric representation theory. While one can construct the Springer sheaf geometrically, Hotta and Kashiwara gave it a purely algebraic reincarnation in the language of equivariant $D(\mathfrak{g})$-modules. For $G = GL_N$, the endomorphism algebra of the Springer sheaf, or equivalently of the associated D-module, is isomorphic to $\mathbb{C}[\mathcal{S}_n]$ the group algebra of the symmetric group. In this talk, I'll discuss a quantum analogue of this.
In joint work with Sam Gunningham and David Jordan, we define quantum Hotta-Kashiwara D-modules HK_χ, and compute their endomorphism algebras.
In particular $\mathrm{End}_{\mathcal{D}_q(G)}(\mathrm{HK}_0) \simeq \mathbb{C}[\mathcal{S}_n]$. This is part of a larger program to understand the category of strongly equivariant quantum D-modules.
Our main tool to study this category is Jordan's elliptic Schur-Weyl
duality functor to representations of the double affine Hecke algebra
(DAHA).
When we input HK_0 into Jordan's functor,
the endomorphism algebra over the DAHA of the output is
$\mathbb{C}[\mathcal{S}_n]$ from which we deduce the result above.

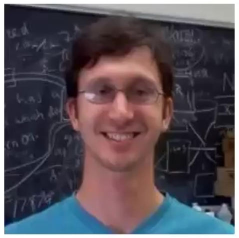
From studying the output of all the HK_χ, we are
able to compute that for input a distinguished projective
generator of the category
the output is the DAHA module generated by the sign idempotent.
This is joint work with Sam Gunningham and David Jordan.

2020GJV-Springer-twin.pdf - Drawboard PDF

The "Springer" representation of the DAHA
DAHA = Double affine Hecke algebra

Monica Vazirani, UC Davis
Joint with David Jordan, U Edinburgh and
Sam Gunningham, Kings College London / Montana State U

oops!



twinned Geometric Representation Theory
Max Planck and Perimeter Institute June 22-26, 2020

goal of talk

Describe $\mathcal{D}_q(G)$ and the category $\mathcal{D}_q(G)$-str-mod of strongly equivariant $\mathcal{D}_q(G)$-modules, for $G = \mathrm{GL}_N$ or SL_N.
In particular study the Hotta-Kashiwara quantum D-modules $\mathbf{HK}_\chi$ via their images $\mathcal{F}^N(\mathbf{HK}_\chi)$ in DAHA-mod.

Outline

1. Background, motivation
2. Classical picture - from Springer sheaf to HK_χ
3. quantum Hotta-Kashiwara modules $\mathbf{HK}_\chi$
4. sketch $\mathcal{D}_q(G)$ and $\mathcal{D}_q(G)$-str-mod
5. Jordan's family of functors $\mathcal{F}^d : \mathcal{D}_q(G)$-str-mod $\to$ DAHA -mod
6. Describe $\mathcal{F}^N$(Springer), $\mathcal{F}^N(\mathbf{HK}_\chi)$ and hence $\mathrm{End}_{\mathcal{D}_q(G)}(\mathbf{HK}_\chi)$ as well as $\mathrm{End}_{\mathcal{D}_q(G)}(\mathbf{Dist})$.

2020GJV-Springer-twin.pdf - Drawboard PDF

goal of talk

Describe $\mathcal{D}_q(G)$ and the category $\mathcal{D}_q(G)$-str-mod of strongly equivariant $\mathcal{D}_q(G)$-modules, for $G = \mathrm{GL}_N$ or SL_N.
In particular study the Hotta-Kashiwara quantum D-modules $\mathbf{HK}_\chi$ via their images $\mathcal{F}^N(\mathbf{HK}_\chi)$ in DAHA-mod.

Outline

1. Background, motivation
2. Classical picture - from Springer sheaf to $\mathtt{HK}_\chi$
3. quantum Hotta-Kashiwara modules $\mathbf{HK}_\chi$
4. sketch $\mathcal{D}_q(G)$ and $\mathcal{D}_q(G)$-str-mod
5. Jordan's family of functors $\mathcal{F}^d : \mathcal{D}_q(G)\text{-str-mod} \to \text{DAHA-mod}$
6. Describe $\mathcal{F}^N(\text{Springer})$, $\mathcal{F}^N(\mathbf{HK}_\chi)$ and hence $\mathrm{End}_{\mathcal{D}_q(G)}(\mathbf{HK}_\chi)$ as well as $\mathrm{End}_{\mathcal{D}_q(G)}(\mathbf{Dist})$.

2020GJV-Springer-twin.pdf - Drawboard PDF

Monica Vazirani

disclaimers

Everything today is type A (mostly GL_N)

q is generic

For geometric questions, I may defer to experts in the audience

2020GJV-Springer-twin.pdf - Drawboard PDF

mathematical goals

$\mathcal{D}_q(G)$

Understand the category of strongly equivariant $\mathcal{D}_q(G)$-modules, where $\mathcal{D}_q(G)$ is the algebra of quantum differential operators on G, $G = \mathrm{GL}_N, \mathrm{SL}_N$

How? By using Jordan's functor (Schur-Weyl)

$$\mathcal{F}^d : \mathcal{D}_q(G)\text{-str-mod} \to \text{DAHA -mod}$$
$$\mathcal{M} \mapsto (V^{\otimes d} \bigotimes \mathcal{M})^{\mathrm{inv}}$$

where $V =$ the N-dimensional defining representation of G

2020GJV-Springer-twin.pdf - Drawboard PDF

related work

Calaque - Enriquez - Etingof
Arakawa - Suzuki
Arakawa - Suzuki - Tsuchiya
Lyubashenko - Majid
Reshetikhin
Semenov-Tian-Shansky
Varagnolo - Vasserot
Ben-zvi - Brochier - Jordan
Alekseev - Schomerus
Backelin - Kremnizer
Jordan - White
Gunningham - Jordan - Safronov

Attribution

Feel free to speak up if I left yours and others' names off, and apologies if so!

2020GJV-Springer-twin.pdf - Drawboard PDF

(2) Classical

Springer theory to build irreps of $\mathcal{S}_N$.
Riemann-Hilbert correspondence: work with $\mathcal{D}$-modules

Hotta-Kashiwara

Springer sheaf or Hotta-Kashiwara $\mathcal{D}$-module

$$\mathrm{HK}_0 = \mathcal{D}(\mathfrak{g})/\langle \mathcal{D}(\mathfrak{g})\mathrm{ad}(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})J_0 \rangle$$

where $\mathrm{ad}(\mathfrak{g})$ is vector space of vector fields whose tangent vector at x is $[A, x]$, $A \in \mathfrak{g}$ (invariance),
J_0 cuts out $\mathcal{N}$ by specifying invariant polynomial functions, i.e. elementary symmetric functions in eigenvalues to be 0.

$$\mathrm{End}_{\mathcal{D}(\mathfrak{g})}(\mathrm{HK}_0) = \mathbb{C}[\mathcal{S}_N]$$

2020GJV-Springer-twin.pdf - Drawboard PDF

Classical, Hotta-Kashiwara

$$\mathtt{HK}_0 = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\,(\mathrm{ad}(\mathfrak{g}) + J_0)$$

$$\mathrm{End}_{\mathcal{D}(\mathfrak{g})}(\mathtt{HK}_0) = \mathbb{C}[W] = \mathbb{C}[\mathcal{S}_N]$$

$$\mathtt{HK}_\chi = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\,(\mathrm{ad}(\mathfrak{g}) + J_\chi)$$

$$\mathrm{End}_{\mathcal{D}(\mathfrak{g})}(\mathtt{HK}_\chi) = \mathbb{C}[W_\chi] = \mathbb{C}[\mathrm{Stab}(\chi)]$$

$$\mathtt{HK}_{\mathrm{univ}} = \mathrm{Dist} = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\mathrm{ad}(\mathfrak{g})$$

$$\mathrm{End}_{\mathcal{D}(\mathfrak{g})}(\mathrm{Dist}) = (\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\mathrm{ad}(\mathfrak{g}))^G = \mathcal{D}(\mathfrak{t})^{\mathcal{S}_N}$$

$= $ spherical RCA at parameter 0

Monica Vazirani

2020GJV-Springer-twin.pdf - Drawboard PDF

Classical to quantum

	eigenvalues
nilpotent	$0, 0, \cdots, 0$
unipotent	$1, 1, \cdots, 1$
qunipotent	$1, q^{-2}, q^{-4}, q^{-6} \cdots$

2020GJV-Springer-twin.pdf - Drawboard PDF

Classical to quantum

		eigenvalues	elementary symmetric functions in e-vals
$\mathcal{D}(\mathfrak{g})$	nilpotent	$0, 0, \cdots, 0$	0
$\mathcal{D}(G)$	unipotent	$1, 1, \cdots, 1$	
$\mathcal{D}_q(G)$	**q**unipotent	$1, q^{-2}, q^{-4}, \cdots$	"higher" Casimirs

2020GJV-Springer-twin.pdf - Drawboard PDF

(3) quantum Hotta-Kashiwara

Theorem (Gunningham-Jordan-V, in progress, $G = \mathrm{GL}_N$)

$$\mathbf{Spr}_q = \mathbf{HK}_0 = \mathcal{D}_q(G)/\langle \mathcal{D}_q(G)\mu_q(I_\epsilon) + \mathcal{D}_q(G)\partial_\triangleleft J_0 \rangle$$

$$\mathrm{End}_{\mathcal{D}_q(G)}(\mathbf{HK}_0) = \mathbb{C}[W] = \mathbb{C}[\mathcal{S}_N]$$

$$\mathbf{HK}_\chi = \mathcal{D}_q/\mathcal{D}_q\left(\mu_q(I_\epsilon) + \partial_\triangleleft J_\chi\right)$$

$$\mathrm{End}_{\mathcal{D}_q}(\mathbf{HK}_\chi) = \mathbb{C}[W_\chi] = \mathbb{C}[\mathrm{Stab}_\rho(\chi)]$$

$\mathbf{Dist} := \mathbf{HK}_{\mathrm{univ}} = \mathcal{D}_q/\mathcal{D}_q\mu_q(I_\epsilon)$ *is the "universal" strongly equivariant* $\mathcal{D}_q(G)$*-module*

$$\mathrm{End}_{\mathcal{D}_q}(\mathbf{Dist}) = \textit{anti-spherical DAHA at parameters } t = q^{-2}$$

μ_q = quantum moment map
$\mathcal{D}_q(G)\mu_q(I_\epsilon)$ = moment map (left) ideal
I_ϵ = kernel of augmentation map

2020GJV-Springer-twin.pdf - Drawboard PDF

(4) $\mathcal{D}_q(G)$

	$\mathcal{D}_q(G)$	$\mathcal{O}_q(G)$	DAHA	
			$X^{\pm}$	$Y^{\pm}$
	$\mathcal{D}(G)$	$\mathcal{O}(G)$		
	$\mathcal{D}(\mathfrak{g})$	$\mathbb{C}[\mathfrak{g}]$	RCA	
			x	y
			creation	Dunkl
			acts freely	
Toy	$\mathbb{C}[x, \partial_x]$	$\mathbb{C}[x]$	x	∂_x
(later)	b_{ij}, a_{ij}			

13

268%

2020GJV-Springer-twin.pdf - Drawboard PDF

$\mathcal{D}_q(G)$

$G = \mathrm{GL}_N, \mathrm{SL}_N \quad \mathfrak{g} = \mathfrak{gl}_N, \mathfrak{sl}_N$

$\mathcal{O}_q(G)$

$\mathcal{O}_q(G)$ = quantum coordinate algebra
= quantum deformation of $\mathcal{O}(G)$ polynomial functions on G
= algebra of matrix coefficients
$\simeq$ "locally finite subalgebra" $\mathcal{U}_q(\mathfrak{g})^{\mathrm{locfin}}$

$\mathcal{O}_q(G)$ quantizes the Semenov-Tian-Shansky bracket, so is conjugation equivariant (vs Sklyanin bracket L/R)

$$\mathcal{D}_q(G) = \mathcal{O}_q(G) \otimes \mathcal{O}_q(G)$$

as vector spaces. Carries 3 actions of $\mathcal{U}_q(\mathfrak{g})^{\mathrm{locfin}}$ — ad, R, L

2020GJV-Springer-twin.pdf - Drawboard PDF

$\mathcal{D}_q(G)$

Notation

$$\mathcal{D}_q(G) = \quad \mathcal{O}_q(G) \quad \otimes \quad \mathcal{O}_q(G)$$

$$A = [a_{ij}] \qquad B = [b_{ij}]$$

While coordinate functions in $\mathcal{O}(G)$ commute, relations in $\mathcal{O}_q(G)$ involve R-matrices. (Also invert $\det$ or set $\det = 1$.)
$\mathcal{O}_q(\mathrm{Mat})$ = reflection equation algebra: $R_{21}A_1R_{12}A_2 = A_2R_{21}A_1R_{12}$.

2020GJV-Springer-twin.pdf - Drawboard PDF

$\mathcal{D}_q(G)$-str-mod

Strongly equivariant quantum $\mathcal{D}$-modules are those which are locally finite for the moment map action.

$$\textbf{Dist} := \textbf{HK}_{\text{univ}} = \mathcal{D}_q(G)/\mathcal{D}_q(G)\mu_q(I_\epsilon)$$

is the "universal" strongly equivariant $\mathcal{D}_q(G)$-module.

μ_q = quantum moment map
$\mathcal{D}_q(G)\mu_q(I_\epsilon)$ = moment map ideal (left ideal)
I_ϵ = kernel of augmentation map

quotienting forces some extra commuting relations on A, B

268%

2020GJV-Springer-twin.pdf - Drawboard PDF

Monica Vazirani

$\mathbf{HK}_\chi$

$$\mathbf{HK}_\chi = \mathcal{D}_q(G)/\langle \mathcal{D}_q(G)\mu_q(I_\epsilon) + \mathcal{D}_q(G)\partial_\triangleleft J_\chi \rangle$$

Choice of $\chi \iff$ classical: specifies invariants $\mathrm{tr}(A), \mathrm{tr}(A^2), \mathrm{tr}(A^3), \ldots$
or coefficients of characteristic polynomial
quantum: "higher" Casimirs $\mathbf{c}_k$

take $\chi = \epsilon \iff$ "**q**unipotent" cone (**q**-nipotent)

$\mathbf{HK}_\chi$ is a universal "category $\mathcal{O}$" type module among the strongly equivariant ones. $\mathcal{F}^d(\mathbf{HK}_\chi)$ is in Category $\mathcal{O}$ for DAHA.
Motivates our definition.

$\mathbf{HK}_\chi$

$$\mathbf{HK}_\chi = \mathcal{D}_q(G)/\langle \mathcal{D}_q(G)\mu_q(I_\epsilon) + \mathcal{D}_q(G)\partial_{\triangleleft} J_\chi \rangle$$

Choice of $\chi \iff$ classical: specifies invariants $\mathrm{tr}(A), \mathrm{tr}(A^2), \mathrm{tr}(A^3), \ldots$
or coefficients of characteristic polynomial
quantum: "higher" Casimirs $\mathbf{c}_k$

take $\chi = \epsilon \iff$ "**q**unipotent" cone (q-nipotent)

$\mathbf{HK}_\chi$ is a universal "category $\mathcal{O}$" type module among the strongly equivariant ones. $\mathcal{F}^d(\mathbf{HK}_\chi)$ is in Category $\mathcal{O}$ for DAHA.
Motivates our definition.

2020GJV-Springer-twin.pdf - Drawboard PDF

(5) the functor $\mathcal{F}^d$ intermission?

$$\mathcal{F}^d(\mathcal{M}) := \left(V^{\otimes d} \bigotimes \mathcal{M}\right)^{\mathrm{inv}} \circlearrowleft \quad \mathrm{DAHA}_d$$

where $\mathcal{M}$ is a strongly equivariant $\mathcal{D}_q(G)$-module and V is the N-dimensional defining representation of G.
The moment map μ_q and strong equivariance gives the $\mathcal{U}_q(\mathfrak{g})$ action by which we take inv.

Calaque-Enriquez-Etingof 0702670

$$F^d(M) := \left(V^{\otimes d} \bigotimes M\right)^{\mathrm{inv}} \circlearrowleft \quad \mathrm{RCA}$$

where M is a $\mathcal{D}(\mathfrak{g})$-module.

(Schur-Weyl)

2020GJV-Springer-twin.pdf - Drawboard PDF

HK$_\chi$ is a universal "category $\mathcal{O}$" type module among the strongly equivariant ones. $\mathcal{F}^d(\mathbf{HK}_\chi)$ is in Category $\mathcal{O}$ for DAHA. Motivates our definition.

Levasseur - Stafford

e(triv) D(A e(triv) vs. e(sgn) D(A e(sgn))

Monica Vazirani | Springer representation of DAHA | June 24, 2020 | 16 / 30

(5) the functor $\mathcal{F}^d$ intermission?

$$\mathcal{F}^d(\mathcal{M}) := \left(V^{\otimes d} \bigotimes \mathcal{M}\right)^{\mathrm{inv}} \circlearrowleft \quad \mathrm{DAHA}_d$$

where $\mathcal{M}$ is a strongly equivariant $\mathcal{D}_q(G)$-module and V is the N-dimensional defining representation of G.
The moment map μ_q and strong equivariance gives the $\mathcal{U}_q(\mathfrak{g})$ action by which we take inv.

Calaque-Enriquez-Etingof 0702670

2020GJV-Springer-twin.pdf - Drawboard PDF

DAHA

$\mathcal{F}^d(\mathcal{M})$ is a DAHA$_d$-module
DAHA $= \mathbb{H}_d$ is the double affine Hecke algebra of type GL_d or SL_d.

parameters $q, t \in \mathcal{K}$

$$\mathbb{H}_d(GL) = \mathcal{K}[Y_1^{\pm 1}, \ldots, Y_d^{\pm 1}] \otimes H_d^{\mathrm{fin}} \otimes \mathcal{K}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$$

$$= \underbrace{\mathcal{K}[Y_1^{\pm 1}, \ldots, Y_d^{\pm 1}]}_{\mathcal{Y}} \otimes \mathcal{H}_d^{\mathrm{aff}}$$

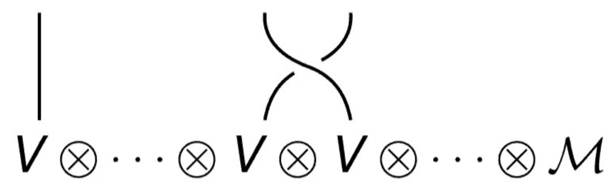
We specialize $t = q^{-2k}$, $d = kN$, swapping usual q, t notation to favor $\mathcal{U}_q(\mathfrak{g})$ over $\mathcal{U}_t(\mathfrak{g})$.

2020GJV-Springer-twin.pdf - Drawboard PDF

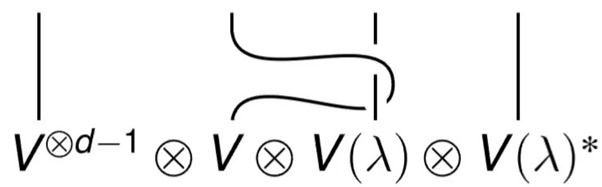
DAHA action (Jordan 0805.2766) on $\mathcal{M} = \mathcal{O}_q(G)$

Using Peter-Weyl decomp $\mathcal{O}_q(G) \simeq \bigoplus_{\lambda \in P^+} V(\lambda) \otimes V(\lambda)^*$ as $\mathcal{U}_q(\mathfrak{g})$-modules. Note $\mathbf{HK}_0 \twoheadrightarrow \mathcal{O}_q(G)$

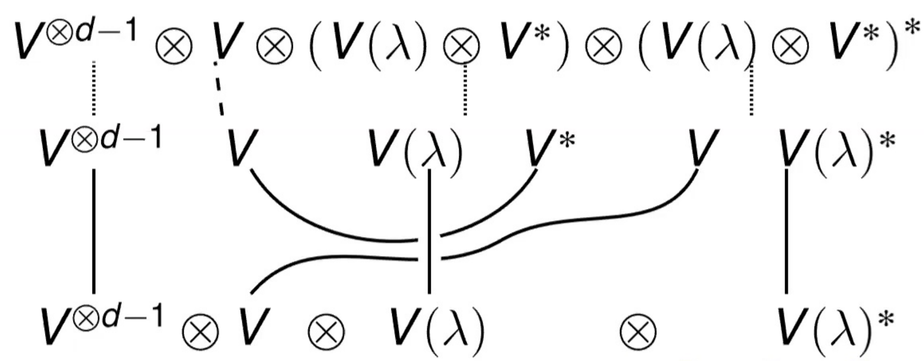
$T_i \sim R$ matrix



Y_1 double braiding



X_1



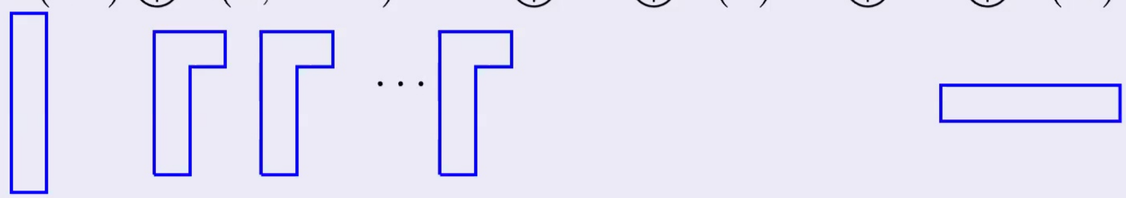
Monica Vazirani | Springer representation of DAHA | June 24, 2020 | 19 / 30

2020GJV-Springer-twin.pdf - Drawboard PDF

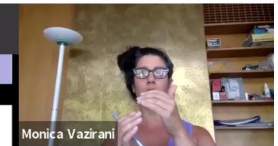
(6) $\mathcal{F}^N(\mathbf{HK}_\chi)$ results

Theorem (Gunningham-Jordan-V, in progress)

1. $\mathcal{F}^N(\mathbf{HK}_0) \simeq \mathrm{Ind}_{\mathcal{Y}}^{\mathbb{H}} q^0 \boxtimes q^{-2} \boxtimes \cdots \boxtimes q^{-2(N-1)}$
$\simeq L(1^N) \bigoplus L(2, 1^{N-2})^{\oplus N-1} \bigoplus \cdots \bigoplus L(\lambda)^{\oplus f_\lambda} \bigoplus \cdots \bigoplus L(N)$



2. $\mathrm{End}_{\mathcal{D}_q(G)}(\mathbf{HK}_0) \simeq \mathrm{End}_{\mathrm{DAHA}}(\mathrm{Ind}_{\mathcal{Y}}^{\mathbb{H}} q^{-\rho}) \simeq \mathcal{K}[\mathcal{S}_N]$
3. $\mathrm{End}_{\mathcal{D}_q(G)}(\mathbf{HK}_\chi) \simeq \mathcal{K}[W_\chi]$
4. $\mathcal{F}^N(\mathbf{Dist}) = \mathbb{H}\varepsilon$ *and* $\mathrm{End}_{\mathcal{D}_q(G)}(\mathbf{Dist}) \simeq \mathrm{End}_{\mathbb{H}}(\mathbb{H}\varepsilon) \simeq \varepsilon\mathbb{H}\varepsilon$ *the anti-spherical DAHA.*



idea

$F^N(HK_0)$

$\shortparallel$

Analyze End in DAHA.

$$\mathrm{Ind}_{H\otimes\mathcal{S}ym}^{\mathbb{H}}\, \mathrm{sgn} \boxtimes \chi_0 \simeq \mathrm{Ind}_{\mathcal{Y}}^{\mathbb{H}}\, q^0 \boxtimes q^{-2} \boxtimes \cdots \boxtimes q^{-2(N-1)}$$

where $H = H_N^{\mathrm{fin}}$ = finite Hecke algebra,
$\mathcal{S}\mathrm{ym} = \mathcal{K}[Y_1^{\pm 1}, \ldots, Y_d^{\pm 1}]^{\mathcal{S}_N} = \mathcal{Y}^{\mathcal{S}_N}$
sgn = the sign representation of H_N^{fin} corresponding to ε
χ_0 describes one-dim $\mathcal{S}$ym representation corresp to $q^{-\rho}$

More generally $\mathrm{Ind}_{H\otimes\mathcal{S}ym}\, \mathrm{sgn} \boxtimes \chi \simeq \mathrm{Ind}_{\mathcal{Y}}^{\mathbb{H}}(1\text{-}\mathrm{dim})$
and as $\mathbb{H} \otimes \mathcal{S}$ym-modules $\mathrm{sgn} \boxtimes \chi \iff \bigwedge^N V \otimes \mathbb{1} = \varepsilon(V^{\otimes N} \otimes \mathbb{1}) \in \varepsilon(V^{\otimes N} \otimes \mathbf{HK}_\chi)^{\mathrm{inv}} = \bigwedge^N V \otimes (\mathbf{HK}_\chi)^{\mathrm{inv}}$

For **Dist** move from generic χ to no χ. Observe $\mathbb{H}\varepsilon = \mathrm{Ind}_H^{\mathbb{H}}\, \mathrm{sgn}$.

2020GJV-Springer-twin.pdf - Drawboard PDF

Diagrammatic sketch

Reshetikhin: $\mathbf{c}_k$ higher Casimirs, value determined by χ
(picture from [Jordan-White 1709.09149])

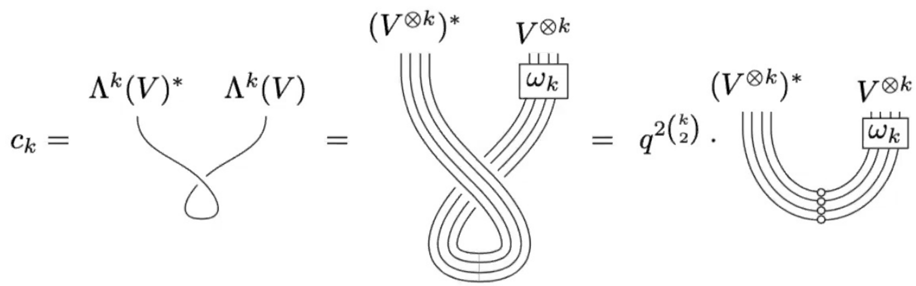
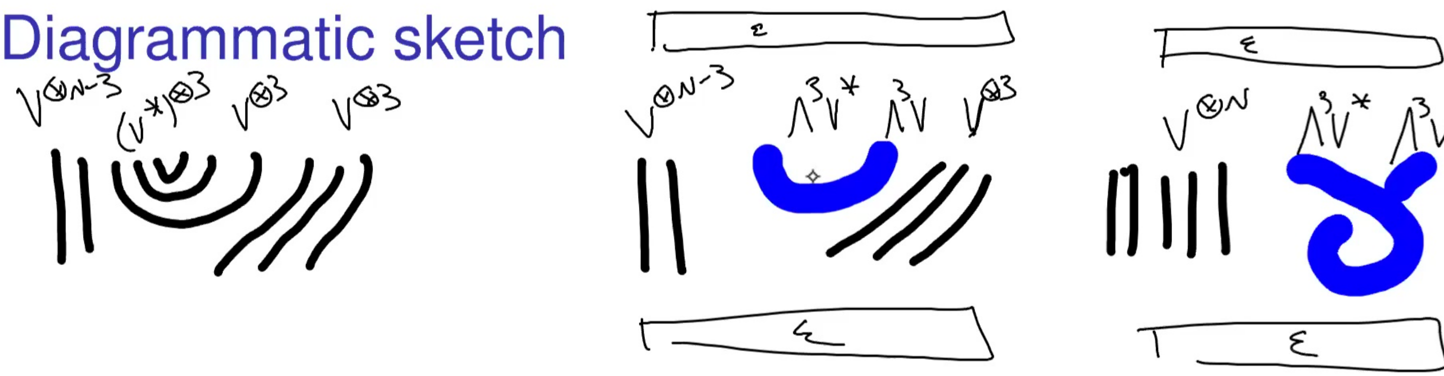


FIGURE 7. The canonical central elements are the quantum traces of the kth exterior power of the defining representation, which may be embedded into $C_{V^{\otimes k}}$ via the corresponding projector ω_k. See Lemma 2.18.

$$\varepsilon \mathbf{c}_k(A)\varepsilon = \varepsilon\, \mathrm{elem}_k(\mathcal{Y})\, \varepsilon = \mathrm{const}\, \varepsilon Y_1 Y_2 \cdots Y_k \varepsilon$$

Diagrammatic sketch



$Y_1 Y_2 Y_3$

$\varepsilon Y_1 Y_2 Y_3 \varepsilon$

$= \text{const} \cdot \varepsilon Y_2 Y_7 Y_8 \varepsilon$

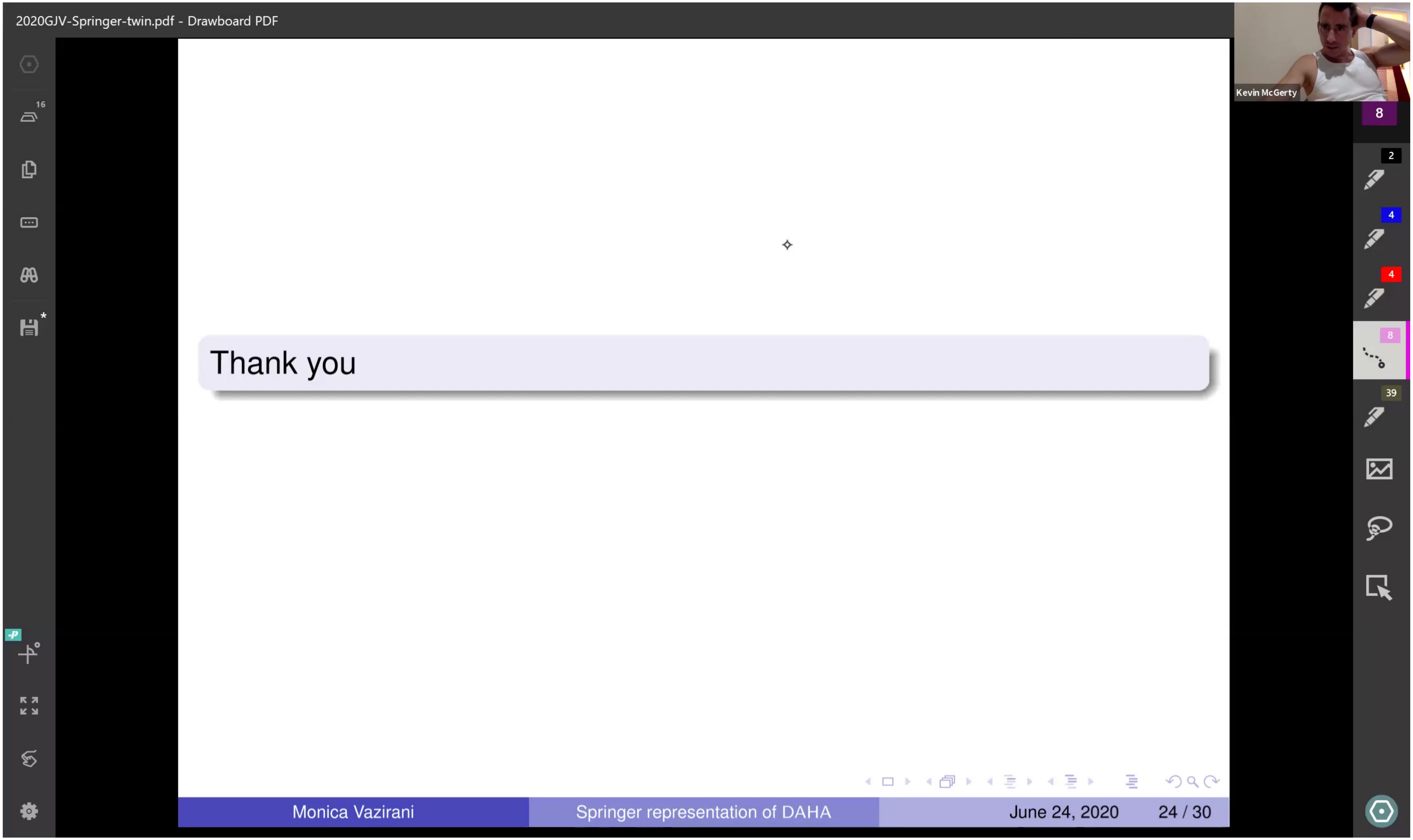
$= \text{const} \cdot \varepsilon \, \text{elem}_3(y) \, \varepsilon$

$\varepsilon \, C_k(A) \, \varepsilon$

$$\varepsilon \, \text{elem}_k(y) \, \varepsilon = \varepsilon \, C_k(A) \, \varepsilon$$

$\updownarrow$ $\qquad$ $\updownarrow$

$\text{sgn} \boxtimes \mathcal{X}$ $\qquad$ $\text{HK}\,\mathcal{X}$


2020GJV-Springer-twin.pdf - Drawboard PDF
Kevin McGerty
Thank you
Monica Vazirani
Springer representation of DAHA
June 24, 2020
24 / 30