Title: Global Demazure modules

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Abstract: The Beilinson-Drinfeld Grassmannian of a simple complex algebraic group admits a natural stratification into "global spherical Schubert varieties". In the case when the underlying curve is the affine line, we determine algebraically the global sections of the determinant line bundle over these global Schubert varieties as modules over the corresponding Lie algebra of currents. The resulting modules are the global Weyl modules (in the simply laced case) and generalizations thereof. This is a joint work with Ilya Dumanski and Evgeny Feigin.

Global Demazure modules

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Demazure modules

- $\mathfrak{g}$ complex simple Lie algebra, $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ Cartan and Borel subalgebras, $\mathfrak{n} \subset \mathfrak{b}$ the nilpotent radical, W the Weyl group, $G \supset B$ the corresponding simply connected group with Borel subgroup, $\mathcal{B} = G/B$ the flag variety, $\mathcal{B}^w \subset \mathcal{B}$ the Schubert subvariety corresponding to $w \in W$ (the closure of a B-orbit), θ a dominant weight, $L(\theta)$ finite dimensional irreducible $\mathfrak{g}$-module.
- Take $w \in W$ and an extremal vector $v_{w\theta} \in L(\theta)_{w\theta}$. *Demazure module* $D_{w\theta} \subset L(\theta)$ is the $\mathfrak{b}$-submodule generated by $v_{w\theta}$.
- Borel-Weil: $L(\theta)^* = \Gamma(\mathcal{B}, \mathcal{L}_{-\theta})$ (line bundle induced from B-character $-\theta$).
- Demazure: $D^*_{w\theta} = \Gamma(\mathcal{B}^w, \mathcal{L}_{-\theta})$.

Generator and relations

- Joseph: There is a surjection $U(\mathfrak{b}) \twoheadrightarrow D_{w\theta},\ 1 \mapsto v_{w\theta}$, with kernel $K \subset U(\mathfrak{b})$ generated by
  (a) The kernel of the character $w\theta\colon U(\mathfrak{h}) \to \mathbb{C}$;
  (b) The positive root vectors e_α such that $w^{-1}\alpha$ is also positive;
  (c) The powers of positive root vectors $e_\alpha^{1-w\theta(h_\alpha)}$ such that $w^{-1}\alpha$ is negative.
- Mathieu: The same is true for Demazure modules in integrable modules over an arbitrary Kac-Moody Lie algebra, only
  (b) has to be reformulated as the augmentation ideal $U^+(\mathfrak{n}(w))$, where $\mathfrak{n}(w)$ is spanned by the positive root subspaces $\mathfrak{g}_\alpha$ such that $w^{-1}\alpha$ is still positive.

Affine Lie algebras

- Affine Lie algebra $\mathfrak{g}_{\mathrm{aff}} = \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$.
  Integrable level 1 modules are dual to global sections of the very ample determinant line bundle $\mathcal{L}$ on connected components of the affine Grassmannian $\mathrm{Gr} = \mathrm{Gr}_{G^{\mathrm{ad}}}$ of the adjoint group G^{ad}. The extremal weights are numbered by the extended affine Weyl group orbits of the highest weights $W^{\mathrm{ext}}/W \cong P$: the *coweight* lattice of G^{ad}.
- So the level 1 Demazure modules will be numbered by P, $D^*_{1,\lambda} = \Gamma(\mathrm{Gr}^{\lambda}, \mathcal{L})$ (sections over a spherical Schubert variety). They are modules over the current algebra $\mathfrak{g}[t]$.
- The $\mathfrak{g}$-component of the lowest $\mathfrak{g}_{\mathrm{aff}}$-weight of $D_{1,\lambda}$ is $\iota(\lambda)_{\mathrm{low}}$. Here $\iota\colon P \hookrightarrow P^{\vee}$ is the "level 1" embedding into the *weight* lattice of $G = G^{\mathrm{sc}}$, and $\iota(\lambda)_{\mathrm{low}}$ is the lowest weight in the W-orbit of $\iota(\lambda)$.
- The $\mathfrak{g}$-component of the highest $\mathfrak{g}_{\mathrm{aff}}$-weight of $D_{1,\lambda}$ is 0 if λ lies in the coroot lattice $Q \subset P$, and it is the minuscule representative of $\iota(\lambda)$ otherwise.

Global Weyl modules

- For a dominant $\mathfrak{g}$-weight θ, the *global Weyl module* $\mathbb{W}(\theta)$ is the quotient of $U(\mathfrak{g}[t])$ modulo the left ideal generated by
  (a) The kernel of the character $\theta\colon U(\mathfrak{h}) \to \mathbb{C}$;
  (b) The augmentation ideal $U^+(\mathfrak{n}[t])$;
  (c) The powers of the negative root vectors $f_\alpha^{1+\theta(h_\alpha)}$.
- $\mathbb{W}(\theta)$ is the maximal G-integrable quotient of $\mathrm{Ind}_{\mathfrak{n}[t]\oplus\mathfrak{h}}^{\mathfrak{g}[t]} \mathbb{C}_\theta$.
- Let $\mathfrak{g}$ be simply laced. For a fundamental weight $\omega_i^\vee = \iota(\omega_i)$, the global Weyl module $\mathbb{W}(\omega_i^\vee)$ is related to the Demazure module D_{1,ω_i}. Namely, $\mathbb{W}(\omega_i^\vee) = D_{1,\omega_i} \otimes \mathbb{C}[t]$ with the $\mathfrak{g}[t]$-action

$$xt^m(v \otimes t^k) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (xt^j \cdot v) \otimes t^{m+k-j}$$

for $m, k \in \mathbb{N},\ x \in \mathfrak{g},\ v \in D_{1,\omega_i}$.

Polynomial algebra $\mathcal{A}_\lambda$

- For arbitrary dominant weight θ, the polynomial algebra $U(\mathfrak{h}[t])$ acts on $\mathbb{W}(\theta)$ on the right via

$$(u \cdot w_\theta)a = (ua) \cdot w_\theta$$

  for $u \in U(\mathfrak{g}[t]),\ a \in U(\mathfrak{h}[t]),\ w_\theta$ the highest vector of $\mathbb{W}(\theta)$.
- This action factors through a free action of $\mathcal{A}_\theta = \mathbb{C}[\mathbb{A}^\theta]$, where $\theta = \sum_{i\in I} d_i \omega_i^\vee$, and $\mathbb{A}^\theta = \prod_{i\in I} \mathbb{A}^{(d_i)}$.
- When $\mathfrak{g}$ is simply laced, and $\theta = \iota(\lambda)$ for a dominant coweight $\lambda = \sum_{i\in I} d_i\omega_i$, then $\mathcal{A}_\theta = \mathcal{A}_\lambda = \mathbb{C}[\mathbb{A}^\lambda]$, and the fiber of $\mathbb{W}(\iota(\lambda))$ at $0 \in \mathbb{A}^\lambda$ is the Demazure module $D_{1,\lambda}$.
- Conversely, when $\mathfrak{g}$ is simply laced, $\mathbb{W}(\iota(\lambda)) = \bigodot_{i\in I} D_{1,\omega_i}[t]^{\odot d_i}$, where for cyclic $\mathfrak{g}[t]$-modules M_1, M_2 with generators v_1, v_2 we define the *cyclic* product $M_1 \odot M_2 := U(\mathfrak{g}[t]) \cdot (v_1 \otimes v_2) \subset M_1 \otimes M_2$. If v is the cyclic vector of D_{1,ω_i}, then $v \otimes 1$ is the cyclic vector of $D_{1,\omega_i}[t]$.

Beilinson-Drinfeld Grassmannian

- $k \in \mathbb{N} \implies \mathrm{Gr}_{\mathbb{A}^k}$ is the moduli space of the data
  (a) A k-tuple of points $\mathbf{c} = (c_1, \dots, c_k)$ in the affine line;
  (b) a G^{ad}-bundle $\mathcal{F}$ on $\mathbb{A}^1$;
  (c) a trivialization $\sigma\colon \mathcal{F}|_{\mathbb{A}^1 \setminus \mathbf{c}} \simeq \mathcal{F}_{\mathrm{triv}}$.
- The global coordinate t on $\mathbb{A}^1$ gives rise to $\mathrm{Gr}_{\mathbb{A}^1} \cong \mathrm{Gr} \times \mathbb{A}^1$.
- The *factorization property* with respect to the addition of effective divisors $\mathbb{A}^k \times \mathbb{A}^m \to \mathbb{A}^{k+m}$:

$$(\mathrm{Gr}_{\mathbb{A}^k} \times \mathrm{Gr}_{\mathbb{A}^m})|_{(\mathbb{A}^k \times \mathbb{A}^m)_{\mathrm{disj}}} \cong \mathrm{Gr}_{\mathbb{A}^{k+m}} \times_{\mathbb{A}^{k+m}} (\mathbb{A}^k \times \mathbb{A}^m)_{\mathrm{disj}}.$$

- Away from all diagonals in $\mathbb{A}^k$, the fiber of $\mathrm{Gr}_{\mathbb{A}^k}$ is $(\mathrm{Gr})^k$, but the fiber over $0 \in \mathbb{A}^k$ is just Gr. (sic!)
- The affine Grassmannian $\mathrm{Gr} = G^{\mathrm{ad}}(\!(t)\!)/G^{\mathrm{ad}}[\![t]\!]$ is acted upon by $G[\![t]\!] = \lim\limits_{\leftarrow} G(\mathbb{C}[t]/t^n)$. Similarly, $\mathrm{Gr}_{\mathbb{A}^k}$ is acted upon by a group scheme $\mathbb{G}(k)$ over $\mathbb{A}^k$ whose fiber over $\mathbf{c}$ is

$$\lim_{\leftarrow} G(\mathbb{C}[t]/P_{\mathbf{c}}(t)^n), \; P_{\mathbf{c}}(t) := (t - c_1) \cdots (t - c_k).$$

Global spherical Schubert varieties

- Homomorphism $G[t] \to \Gamma(\mathbb{A}^k, \mathbb{G}(k)),\ t \mapsto t \pmod{P_{\mathbf{c}}^n(t)}$.
- For a dominant coweight $\lambda = \sum_{i \in I} d_i \omega_i$, we set $d = \sum_{i \in I} d_i$, and take the closure $\mathrm{Gr}^\lambda_{\mathbb{A}^d}$ of $(\mathbb{A}^d \setminus \Delta) \times \prod_{i \in I} (\mathrm{Gr}^{\omega_i})^{d_i}$ in $\mathrm{Gr}_{\mathbb{A}^d}$.
- $\mathrm{Gr}^\lambda_{\mathbb{A}^d}$ is the closure of a $\mathbb{G}(d)$-orbit in the generic fiber of $\mathrm{Gr}_{\mathbb{A}^d}$.
- This closure is acted upon by $S_\lambda := \prod_{i \in I} S_{d_i}$, and we denote the categorical quotient by $\mathrm{Gr}^\lambda_{\mathbb{A}^\lambda} := \mathrm{Gr}^\lambda_{\mathbb{A}^d}/S_\lambda \subset \mathrm{Gr}_{\mathbb{A}^\lambda} := \mathrm{Gr}_{\mathbb{A}^d}/S_\lambda$.
- The fiber of $\mathrm{Gr}^\lambda_{\mathbb{A}^\lambda}$ over $0 \in \mathbb{A}^\lambda$ is $\mathrm{Gr}^\lambda \subset \mathrm{Gr}$.
- The determinant line bundle on $\mathrm{Gr}_{\mathbb{A}^d}$ is S_λ-equivariant and descends to a relatively very ample line bundle $\mathcal{L}$ on $\mathrm{Gr}^\lambda_{\mathbb{A}^\lambda}$.
- The action of $\mathbb{G}(d)$ also descends to an action on $\mathrm{Gr}^\lambda_{\mathbb{A}^\lambda}$, and $\mathcal{L}$ is $\mathbb{G}(d)$-equivariant.
- In particular, $G[t]$ and $\mathfrak{g}[t]$ act on $\Gamma(\mathrm{Gr}^\lambda_{\mathbb{A}^\lambda}, \mathcal{L})$.

Global Demazure modules

- Theorem: $\mathfrak{g}$ a simple Lie algebra, $\lambda = \sum_{i \in I} d_i \omega_i$ a dominant coweight.
  (a) $\Gamma\big(\mathrm{Gr}^{\lambda}_{\mathbb{A}^{\lambda}}, \mathcal{L}\big)$ is a free module over $\mathbb{C}[\mathbb{A}^{\lambda}] = \mathcal{A}_{\lambda}$;
- (b) The dual module $\Gamma\big(\mathrm{Gr}^{\lambda}_{\mathbb{A}^{\lambda}}, \mathcal{L}\big)^{\vee} := \mathrm{Hom}_{\mathcal{A}_{\lambda}}\big(\Gamma\big(\mathrm{Gr}^{\lambda}_{\mathbb{A}^{\lambda}}, \mathcal{L}\big), \mathcal{A}_{\lambda}\big)$ is the cyclic product $\bigodot_{i \in I} D_{1,\omega_i}[t]^{\odot d_i} =: \mathbb{D}_{1,\lambda}$;
- (c) In particular, if $\mathfrak{g}$ is simply laced, the global Demazure module $\mathbb{D}_{1,\lambda} \simeq \mathbb{W}(\iota(\lambda))$ coincides with the global Weyl module. □
- Higher level $r > 1$: Demazure module $D_{r,\lambda} = \Gamma(\mathrm{Gr}^{\lambda}, \mathcal{L}^{r})^{*} = D_{1,\lambda}^{\odot r}$.
- Theorem: (a) $\Gamma\big(\mathrm{Gr}^{\lambda}_{\mathbb{A}^{\lambda}}, \mathcal{L}^{r}\big)$ is a free module over $\mathbb{C}[\mathbb{A}^{\lambda}] = \mathcal{A}_{\lambda}$;
  (b) The dual module $\Gamma\big(\mathrm{Gr}^{\lambda}_{\mathbb{A}^{\lambda}}, \mathcal{L}^{r}\big)^{\vee}$ is the cyclic product $\bigodot_{i \in I} D_{r,\omega_i}[t]^{\odot d_i} =: \mathbb{D}_{r,\lambda}$. □

Diagonal strata of configuration space

- Recall that $\mathrm{Gr}^{\lambda}_{\mathbb{A}^d}$ is the closure of a $\mathbb{G}(d)$-orbit in the generic fiber of $\mathrm{Gr}_{\mathbb{A}^d}$ corresponding to a partition $\lambda = \sum \omega_i$. Most general $\mathbb{G}(k)$-orbit in the generic fiber of $\mathrm{Gr}_{\mathbb{A}^k}$ corresponds to a partition $\lambda = \lambda_1 + \ldots + \lambda_k$ into a sum of dominant coweights. We write $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k)$.
- The addition of effective divisors gives rise to a finite morphism of configuration spaces $\mathrm{add}\colon \mathbb{A}^{\lambda_1} \times \ldots \times \mathbb{A}^{\lambda_k} \to \mathbb{A}^{\lambda}$.
- Each $\mathbb{A}^{\lambda_m}$ contains the main diagonal $\mathbb{A}^{(\lambda_m)} \simeq \mathbb{A}^1$, and the closed stratum $\mathbb{A}^{(\underline{\lambda})}$ is the image of $\mathrm{add}\colon \mathbb{A}^k = \mathbb{A}^{(\lambda_1)} \times \ldots \times \mathbb{A}^{(\lambda_k)} \to \mathbb{A}^{\lambda}$.
- The closed embedding $\mathbb{A}^{(\underline{\lambda})} \hookrightarrow \mathbb{A}^{\lambda} \implies \mathcal{A}_{\lambda} = \mathbb{C}[\mathbb{A}^{\lambda}] \twoheadrightarrow \mathbb{C}[\mathbb{A}^{(\underline{\lambda})}] =: \mathcal{A}(\underline{\lambda})$.
- Explicitly, $\mathcal{A}(\underline{\lambda}) \subset \mathcal{A}(\lambda_1) \otimes \ldots \otimes \mathcal{A}(\lambda_k) = \mathbb{C}[z_1, \ldots, z_k]$, and the image is generated by the weighted power sums

$$\langle \iota(\lambda_1), h \rangle z_1^p + \ldots + \langle \iota(\lambda_k), h \rangle z_k^p,\ p \geq 1,\ h \in \mathfrak{h}.$$

Remarks

- The closed stratum $\mathbb{A}^{(\underline{\lambda})}$ usually is *not* normal, but often is Cohen-Macaulay.
- Some algebras $\mathcal{A}(\underline{\lambda})$ appeared as the algebras of integrals of motion of deformed Calogero-Moser-Sutherland quantum integrable systems [Veselov, Chalykh, M.Feigin, Sergeev,...] (algebras of *quasi-invariants*).
- The cyclic product $D_{1,\lambda_1}[t] \odot \ldots \odot D_{1,\lambda_k}[t]$ is equipped with the right action of $U(\mathfrak{h}[t])$ commuting with the left $U(\mathfrak{g}[t])$-action. The action of $U(\mathfrak{h}[t])$ on the cyclic vector factors through $U(\mathfrak{h}[t]) \twoheadrightarrow \mathcal{A}(\underline{\lambda})$.

Beilinson-Drinfeld spherical Schubert varieties

- $\mathrm{Gr}^{\lambda}_{\mathbb{A}^k} \subset \mathrm{Gr}_{\mathbb{A}^k}$ is the closure of the following $\mathbb{G}(k)$-orbit in the generic fiber. This orbit fiber over $(c_1, \ldots, c_k) \in \mathbb{A}^k$ with c_m all distinct, is $\mathrm{Gr}^{\lambda_1} \times \ldots \times \mathrm{Gr}^{\lambda_k}$. And the fiber of $\mathrm{Gr}^{\lambda}_{\mathbb{A}^k}$ over $0 \in \mathbb{A}^k$ is Gr^{λ}.
- Recall $\lambda_1 + \ldots + \lambda_k = \lambda = \sum_{i \in I} d_i \omega_i, \; d = \sum_{i \in I} d_i$.
  We have the finite cover
  $\mathbb{A}^k = \mathbb{A}^{(\lambda_1)} \times \ldots \times \mathbb{A}^{(\lambda_k)} \to \mathbb{A}^{(\underline{\lambda})} \subset \mathbb{A}^{\lambda} = \mathbb{A}^d / S_{\lambda}$.
  There are cartesian squares

$$\begin{array}{ccccccc}
\mathrm{Gr}^{\lambda}_{\mathbb{A}^k} & \longrightarrow & \mathrm{Gr}^{(\underline{\lambda})}_{\mathbb{A}^{(\underline{\lambda})}} & \longrightarrow & \mathrm{Gr}^{\lambda}_{\mathbb{A}^{\lambda}} & = & \mathrm{Gr}^{\lambda}_{\mathbb{A}^d}/S_{\lambda} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{A}^k & \longrightarrow & \mathbb{A}^{(\underline{\lambda})} & \longrightarrow & \mathbb{A}^{\lambda} & = & \mathbb{A}^d/S_{\lambda}
\end{array}$$

Global sections of $\mathcal{L}^r$ over spherical BD Schubert varieties

- Theorem: (a) $\Gamma\big(\mathrm{Gr}^{(\lambda)}_{\mathbb{A}^{(\lambda)}}, \mathcal{L}^r\big)$ is a free module over $\mathbb{C}[\mathbb{A}^{(\lambda)}] = \mathcal{A}(\underline{\lambda})$;
- (b) The dual module $\Gamma\big(\mathrm{Gr}^{(\lambda)}_{\mathbb{A}^{(\lambda)}}, \mathcal{L}^r\big)^\vee := \mathrm{Hom}_{\mathcal{A}(\underline{\lambda})}\Big(\Gamma\big(\mathrm{Gr}^{(\lambda)}_{\mathbb{A}^{(\lambda)}}, \mathcal{L}\big), \mathcal{A}(\underline{\lambda})\Big)$ is the cyclic product

$$D_{r,\lambda_1}[t] \odot \ldots \odot D_{r,\lambda_k}[t] =: \mathbb{D}(r, \underline{\lambda})$$

- (c) $\Gamma\big(\mathrm{Gr}^{\underline{\lambda}}_{\mathbb{A}^k}, \mathcal{L}^r\big) = \mathbb{D}(r, \underline{\lambda})^\vee \otimes_{\mathcal{A}(\underline{\lambda})} \mathbb{C}[\mathbb{A}^k]$ (a free $\mathbb{C}[\mathbb{A}^k]$-module).
- (d) $\mathrm{Gr}^{(\lambda)}_{\mathbb{A}^{(\lambda)}} = \mathrm{Proj}\Big(\bigoplus_{r\geq 0} \mathbb{D}(r, \underline{\lambda})\Big)$, where $\mathbb{D}(0, \underline{\lambda}) := \mathcal{A}(\underline{\lambda})$. □