Title: Type D quiver representation varieties, double Grassmannians, and symmetric varieties

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Abstract: Since the 1980s, mathematicians have found connections between orbit closures in type A quiver representation varieties and Schubert varieties in type A flag varieties. For example, singularity types appearing in type A quiver orbit closures coincide with those appearing in Schubert varieties in type A flag varieties (Bobinski-Zwara); combinatorics of type A quiver orbit closure containment is governed by Bruhat order on the symmetric group (follows from work of Zelevinsky, Kinser-R); and multiple researchers have produced formulas for classes of type A quiver orbit closures in equivariant cohomology and K-theory in terms of Schubert polynomials, Grothendieck polynomials, and related objects.
After recalling some of this type A story, I will discuss joint work with Ryan Kinser on type D quiver representation varieties. I will describe explicit embeddings which completes a circle of links between orbit closures in type D quiver representation varieties, B-orbit closures (for a Borel subgroup B of GL_n) in certain symmetric varieties GL_n/K, and B-orbit closures in double Grassmannians Gr(a, n) x Gr(b, n). I will end with some geometric and combinatorial consequences, as well as a brief discussion of joint work in progress with Zachary Hamaker and Ryan Kinser on formulas for classes of type D quiver orbit closures in equivariant cohomology.

Type D quiver representation varieties, double Grassmannians, and symmetric varieties

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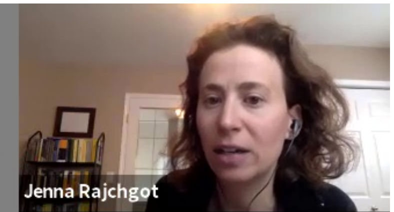
Determinantal varieties

Consider the affine space of matrices

$$\mathrm{Mat}(d_0, d_1) = \{V \mid \mathbb{C}^{d_0} \xrightarrow{V} \mathbb{C}^{d_1}\}.$$

Let $X = (x_{ij})$ be a $d_0 \times d_1$ matrix of variables.

- There is a natural (right) action of $\mathrm{GL}(d_0) \times \mathrm{GL}(d_1)$ on $\mathrm{Mat}(d_0, d_1)$ by $V \cdot (g_0, g_1) = g_0^{-1} V g_1$.
- Orbits are determined by matrix rank r.
- Orbit closures are determinantal varieties, defined by the prime ideals
$$I_r = \langle (r+1 \times r+1) \text{ minors of } X \rangle \subseteq \mathbb{C}[x_{ij}]$$
- Orbit closures normal and Cohen-Macaulay, have nice Gröbner bases, orbit closure containment understood, multigraded Hilbert series understood...



Quiver representations

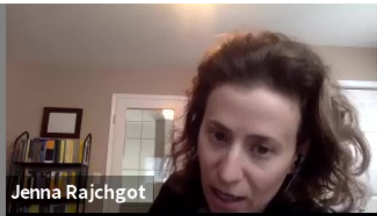
- A quiver Q is a finite directed graph.

$$\bullet \to \bullet \qquad \bullet \to \bullet \leftarrow \bullet \qquad \begin{tikzcd} \bullet \arrow[dr] & \bullet \arrow[d] & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd}$$

- A representation of Q is an assignment of vector space to each vertex and linear map to each arrow.

$$\mathbb{C}^2 \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \mathbb{C}^3 \qquad \mathbb{C} \xrightarrow{A_1} \mathbb{C}^2 \xleftarrow{A_2} \mathbb{C} \qquad \begin{tikzcd} \mathbb{C} \arrow[dr, "A"'] & \mathbb{C} \arrow[d, "B"'] & \mathbb{C}^2 \arrow[dl, "C"] \\ & \mathbb{C}^3 & \end{tikzcd}$$

- A dimension vector $\mathbf{d} = (d_0, d_1, \ldots, d_n)$ for Q assigns vector space $\mathbb{C}^{d_i}$ to vertex of Q with label i.



- A representation space $\mathrm{rep}_Q(\mathbf{d})$ is the space of all representations for Q with fixed dimension vector.
- $\mathbf{GL}(\mathbf{d}) := GL(d_0) \times GL(d_1) \times \cdots \times GL(d_n)$ acts on $\mathrm{rep}_Q(\mathbf{d})$ by conjugation.
- A quiver locus is a $\mathbf{GL}(\mathbf{d})$ orbit closure.

Example

Let $Q = \bullet \to \bullet$, and let $\mathbf{d} = (2, 3)$.

- $\mathrm{rep}_Q(\mathbf{d}) = 2 \times 3$ matrices.
- There are three $\mathbf{GL}(\mathbf{d}) = GL(2) \times GL(3)$ orbits (determined by matrix rank):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \mathbf{GL}(\mathbf{d}), \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \mathbf{GL}(\mathbf{d}), \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \mathbf{GL}(\mathbf{d})$$

- Quiver loci are determinantal varieties.

Dynkin quiver loci

- For Q connected, $\text{rep}_Q(\mathbf{d})$ has finitely many orbits for all $\mathbf{d}$ if and only if Q has Dynkin type A, D, or E. (Gabriel's Theorem)
- Orbit closures for Dynkin quivers Q are determined set-theoretically by minors of certain somewhat complicated matrices. (Bongartz)

Example (Equioriented A_4)

$$\mathbb{C}^{d_0} \xrightarrow{V_1} \mathbb{C}^{d_1} \xrightarrow{V_2} \mathbb{C}^{d_2} \xrightarrow{V_3} \mathbb{C}^{d_3}$$

Orbits determined by ranks of: $V_1, V_2, V_3, V_1V_2, V_2V_3, V_1V_2V_3$

Example (Not Dynkin)

$$\begin{tikzcd} \mathbb{C}^d \arrow[loop above, "V"] \end{tikzcd}$$

$\mathbf{GL}(\mathbf{d})$ orbits characterized by Jordan canonical forms of $d \times d$ matrices (thus, infinitely many orbits).

A few motivations and connections

Motivations from commutative algebra and algebraic geometry

- up to radical, Dynkin quiver loci are generalized determinantal varieties, and include some classically studied varieties
- naturally arise in study of degeneracy loci of vector bundles

Another connection: representation theory of algebras

- The category rep(Q) of reps. of Q is equivalent to category of right modules over the path algebra kQ.
- connections between geometry of quiver loci to rep. theory of Q

Eg. Degeneration order

- For $V, W \in \mathrm{rep}_Q(\mathbf{d})$, define $V \leqslant_{deg} W$ iff $\overline{\mathbf{GL}(\mathbf{d}) \cdot V} \supseteq \overline{\mathbf{GL}(\mathbf{d}) \cdot W}$.
- $V \leqslant_{deg} W$ iff $\dim \mathrm{Hom}_Q(V, X) \leqslant \dim \mathrm{Hom}_Q(W, X)$ for all indecomposable reps. X. (Bongartz)

Degeneracy loci and quiver polynomials

Degeneracy loci:
Let Y be a non-singular algebraic variety. Let V, W be vector bundles over Y of ranks m, n and consider $\varphi : V \to W$. Let $r \leqslant \min(m, n)$.

- degeneracy locus: $Y_r = \{y \in Y \mid \operatorname{rank} \varphi_y \leqslant r\}$
- defined locally by vanishing of $(r+1) \times (r+1)$ minors of a matrix

To quiver loci:

- For general φ, the cohomology class $[Y_r]$ is expressible in terms of a Schur function. (Giambelli-Thom-Porteous formula)
- Buch-Fulton: Found an analogous formula for sequences of bundle maps $V_1 \to V_2 \to \cdots \to V_n$.
- Knutson-Miller-Shimozono: more formulas in the Buch-Fulton setting by computing *multidegrees* of equioriented type A quiver loci (quiver polynomials)
- more formulas were subsequently found (in both cohomology and K-theory) in this and related settings (Buch, Fehér, Kinser, Knutson, Kresch, Miller, R, Rimányi, Shimozono, Tamvakis, Yong)

Part II: Type A quiver loci and Schubert varieties

Schubert varieties

Let $G = \mathrm{GL}(n)$ and let $B_+ \leqslant G$ (resp. $B_- \leqslant G$) be subgroups of upper triangular (resp. lower triangular) matrices.

- Flag variety: $B_-\backslash G$
  This space is naturally identified with the space of complete flags: $0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq \mathbb{C}^n$, $\dim V_i = i$.
- Schubert cell: B_+-orbit in $B_-\backslash G$.
- Schubert variety: closure of a Schubert cell.
- Schubert cells and varieties are indexed by permutations in S_n.

Woo-Yong, building on Kazhdan-Lusztig: can study local properties (eg. singularities) via generalized determinantal varieties

Kazhdan-Lusztig varieties

The Kazhdan-Lusztig variety $X_{v,w}$ is a generalized determinantal variety associated to two permutations $v, w \in S_n$.

- v determines a matrix of 0s, 1s, and variables.
- w determines a collection of minors of the matrix assoc. to v.

Each Kazhdan-Lusztig variety is isomorphic, up to an affine space factor, to an open patch of the Schubert variety X_w.

Example

$$\begin{pmatrix} a & b & c & 1 \\ d & e & 1 & 0 \\ f & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

matrix from $v = 4321$ $\qquad$ $w = 1423$ $\qquad$ NW ranks of w

$$X_{v,w} = \mathbb{V}\left(\det \begin{bmatrix} a & b & c \\ d & e & 1 \\ f & 1 & 0 \end{bmatrix}, (2 \times 2) - \text{minors of } \begin{bmatrix} a & b & c \\ d & e & 1 \end{bmatrix}\right)$$

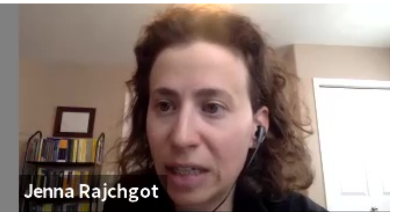
Type A quiver loci and Schubert varieties

$$\mathrm{rep}_Q(\mathbf{d}) = \{(V_1, V_2, V_3) \mid \mathbb{C}^{d_0} \xrightarrow{V_1} \mathbb{C}^{d_1} \xrightarrow{V_2} \mathbb{C}^{d_2} \xrightarrow{V_3} \mathbb{C}^{d_3}\}$$

Orbits determined by ranks of: $V_1, V_2, V_3, V_1V_2, V_2V_3, V_1V_2V_3$.
The equioriented Zelevinsky map is defined to be:

$$\mathrm{rep}_Q(\mathbf{d}) \to \begin{bmatrix} * & * & * & I_{d_0} \\ * & * & I_{d_1} & 0 \\ * & I_{d_2} & 0 & 0 \\ I_{d_3} & 0 & 0 & 0 \end{bmatrix}, \quad (V_1, V_2, V_3) \to \begin{bmatrix} 0 & 0 & V_1 & I_{d_0} \\ 0 & V_2 & I_{d_1} & 0 \\ V_3 & I_{d_2} & 0 & 0 \\ I_{d_3} & 0 & 0 & 0 \end{bmatrix}$$

- Equioriented type A quiver loci are isom. to open subvarieties of Schubert varieties (Musili-Seshadri, Zelevinsky, Lakshmibai-Magyar)
- Consequences for these quiver loci: prime defining ideals, singularity results, F-splitting, orbit closure containment via Bruhat order
- **In general:** combinatorial and algebro-geometric aspects of arbitrary type A quiver loci are governed by corresponding aspects of Schubert varieties (Bobiński-Zwara, Kinser-R)



Multigraded Hilbert series

Set-up:

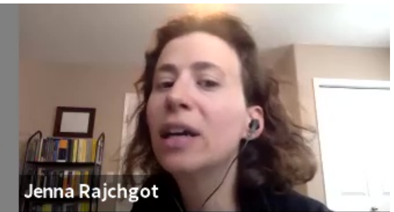
- Let $S = \mathbb{C}[x_1, \ldots, x_n]$ be positively $\mathbb{Z}^d$-graded.
- Let M be a finitely generated $\mathbb{Z}^d$-graded module over S.

Definition

The multigraded Hilbert series of M is

$$H(M;\mathbf{t}) = \sum_{\mathbf{a}\in\mathbb{Z}^d} \dim_{\mathbb{C}}(M_{\mathbf{a}})\mathbf{t}^{\mathbf{a}} = \frac{\mathcal{K}(M;\mathbf{t})}{\prod_{i=1}^{n}(1-\mathbf{t}^{\mathbf{a_i}})}, \quad \deg(x_i) = \mathbf{a_i}$$

- K-polynomial of M: the numerator $\mathcal{K}(M;\mathbf{t})$
- multidegree of M: sum of lowest deg. terms of $\mathcal{K}(M;\mathbf{1}-\mathbf{t})$.



Multigraded Hilbert series for determinantal varieties

$$\mathrm{Mat}_{m\times n}(\mathbb{C}) = \{M \mid \mathbb{C}^m \xrightarrow{M} \mathbb{C}^n\}.$$

- Have a (right) action of $(\mathbb{C}^\times)^m \times (\mathbb{C}^\times)^n$ on $\mathrm{Mat}_{m\times n}(\mathbb{C})$:

$$M \cdot (g_0, g_1) = g_0^{-1} M g_1.$$

- It induces a $\mathbb{Z}^{m+n}$-grading on coord. ring $\mathbb{C}[\mathrm{Mat}_{m\times n}(\mathbb{C})]$.
- The K-polynomial of the determinantal variety $X_r \subseteq \mathrm{Mat}_{m\times n}(\mathbb{C})$ is a double Grothendieck polynomial.
- The multidegree is a double Schubert polynomial.

This has been generalized, in multiple ways, to general type A quiver loci.

- Eg: the K-poly. of a type A quiver locus is a ratio of specialized double Grothedieck polynomials (Knutson-Miller-Shimozono, Kinser-Knutson-R)

Another generalization: type A quiver component formulas

In 2005, Buch and Rimányi proved a positive multidegree formula for type A quiver loci and conjectured an alternating K-polynomial formula.

Theorem (Kinser-Knutson-R)

The Buch-Rimányi conjecture is true.

Proof sketch: We reduce to the bipartite case where the goal is to show:

$$\mathcal{K}(\Omega; \mathbf{t}, \mathbf{s}) = \sum_{\mathbf{w} \in KW(\Omega)} (-1)^{|\mathbf{w}| - \mathrm{codim}(\Omega)} \mathfrak{G}_{\mathbf{w}}(\mathbf{t}; \mathbf{s}).$$

We give a geometric proof of this combining:

- degenerations of bipartite quiver loci;
- a bipartite version of the Zelevinsky map (Kinser-R);
- properties of Kazhdan-Lusztig ideals;
- combinatorial arguments.

Remark: proofs in *equioriented* type A came earlier (Knutson-Miller-Shimozono, Buch, Miller, Yong)

5 minute break: summary slide

General:

- We are interested in quiver loci of Dynkin quivers a.k.a. closures of $\mathbf{GL}(\mathbf{d})$-orbits of representations of Dynkin quivers.
- Our main motivation is to find explicit formulas for *quiver polynomials*.

Type A:

- Geometry and combinatorics of type A quiver loci is closely tied to that of Schubert varieties in type A flag varieties.
- Formulas for quiver polynomials are expressible in terms of well-known polynomials from Schubert calculus.
- Helpful idea: reduce all orientations of type A quivers to the bipartite (source-sink) orientation.

After the break: type D

Part III: Type D quiver representation varieties, double Grassmannians, and symmetric varieties

Type D quiver loci

Let $G = GL(a+b)$ and $K = GL(a) \times GL(b)$ embedded as two blocks along the diagonal. Then $K\backslash G$ is a variety. It can be embedded in the double Grassmannian $Gr(a,n) \times Gr(b,n)$ as an open subvariety.

Theorem (Kinser-R)

There is a series of links

$$\boxed{\begin{array}{c}\textit{B-orbit closures}\\ \textit{in } K\backslash G\end{array}} \hookrightarrow \boxed{\begin{array}{c}\textit{B-orbit closures}\\ \textit{in } \mathbf{Gr}(a,n)\times\mathbf{Gr}(b,n)\end{array}} \hookrightarrow \boxed{\begin{array}{c}\mathbf{GL}(\mathbf{d})\textit{-orbit closures}\\ \textit{in } \mathrm{rep}_Q(\mathbf{d}),\ Q \textit{ type } D\end{array}} \hookleftarrow \boxed{\begin{array}{c}B'\textit{-orbit closures}\\ \textit{in } K'\backslash G'\end{array}}$$

each of which allows for transfer of various algebro-geometric and combinatorial properties from target to source.

$\Longrightarrow$ results on sings. (recovering work of Bobiński-Zwara), type D poset as a subposet of clans poset (latter poset structure described by Wyser)

Remark: unlike in type A, don't get prime defining ideals

Main idea

To a type D quiver

$$Q = \begin{tikzcd}[row sep=small] & 1 & & & \\ & & 3 \arrow[ul] \arrow[r] & 4 \arrow[r] & 5 & 6 \arrow[l] \\ & 2 \arrow[ur] & & & \end{tikzcd},$$

with fixed dimension vector $\mathbf{d}$, associate a (non-type D) bipartite quiver of the form

$$Q^* = \begin{tikzcd}[row sep=small] & y_0^3 \arrow[dl, "\beta_0"'] \arrow[dr, dashed, red, "\alpha_1"] & & & & & & & \\ x_0^1 & & x_1^3 & y_1^3 \arrow[l, dashed, red, "\beta_1"] \arrow[r, "\alpha_2"'] & x_2^4 & y_2^4 \arrow[l, dashed, red, "\beta_2"] \arrow[r, "\alpha_3"'] & x_3^5 & y_3^6 \arrow[l, "\beta_3"] \\ & y_0'^2 \arrow[ur, "\alpha_1'"'] & & & & & & \end{tikzcd}$$

and an associated dimension vector $\mathbf{d}^*$ where $\mathbf{d}^*(v^m) = \mathbf{d}(m)$.

- Define the open subvariety $X_Q \subseteq \mathrm{rep}_{Q*}(\mathbf{d}^*)$ where red arrows are invertible.
- Realize each quiver locus in $\mathrm{rep}_Q(\mathbf{d})$, up to smooth factor, as an orbit closure in X_Q.

Embed X_Q into a slice $S(\mathbf{d}^*)$ of $K\backslash G$:

$$
S(\mathbf{d}^*):\quad
\begin{array}{c|ccccccccccc}
 & x_2 & x_1 & x_0 & x_0^s & x_1^s & x_2^s & y_2 & y_1 & y_0+y_0' & y_1^s & y_2^s \\
\hline
y_0 & \cdot & \cdot & \cdot & * & * & * & \cdot & \cdot & [\,J\ \cdot\,] & \cdot & \cdot \\
y_1 & \cdot & \cdot & \cdot & * & * & * & \cdot & J & \cdot & \cdot & \cdot \\
y_2 & \cdot & \cdot & \cdot & * & * & * & J & \cdot & \cdot & \cdot & \cdot \\
x_0 & \cdot & \cdot & J & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x_1 & \cdot & J & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x_2 & J & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
y_0' & \cdot & \cdot & \cdot & \star & \star & \star & \cdot & \cdot & [\,\cdot\ 1\,] & \cdot & \cdot \\
y_1 & \cdot & \cdot & \cdot & \star & \star & \star & \cdot & J & \cdot & 1 & \cdot \\
y_2 & \cdot & \cdot & \cdot & \star & \star & \star & J & \cdot & \cdot & \cdot & 1 \\
x_0 & \cdot & \cdot & J & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x_1 & \cdot & J & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x_2 & J & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

$$
\mathbf{V} \mapsto
\left[\begin{array}{ccccccccccc}
\cdot & \cdot & \cdot & V_{\beta_0} & V_{\alpha_1} & 0 & \cdot & \cdot & [\,J\ \cdot\,] & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0 & V_{\beta_1} & V_{\alpha_2} & \cdot & J & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0 & 0 & V_{\beta_2} & J & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & J & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & J & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
J & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & 0 & V_{\alpha_1'} & 0 & \cdot & \cdot & [\,\cdot\ 1\,] & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0 & V_{\beta_1} & V_{\alpha_2} & \cdot & J & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & 0 & 0 & V_{\beta_2} & J & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & J & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & J & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
J & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]
$$

Identify each orbit closure in X_Q with an open subvariety of $\overline{\mathcal{O}} \cap S(\mathbf{d}^*)$, for some P-orbit closure $\overline{\mathcal{O}}$ in $K\backslash G$. Then show that the latter is isomorphic, up to smooth factor, to an open subvariety of $\overline{\mathcal{O}}$.

A very small example

$$Q = \begin{tikzcd} y_0 \arrow[dr] & & \\ & x_1 & y_1 \arrow[l] \\ y_0' \arrow[ur] & & \end{tikzcd}, \quad \mathbf{d}(y_0) = \mathbf{d}(y_0') = 0, \mathbf{d}(x_1) = \mathbf{d}(y_1) = 1$$

Then $Q^* = Q$, $\mathbf{d}^* = \mathbf{d}$, and

$$S(\mathbf{d}^*) : \begin{array}{c} \\ y_1 \\ x_1 \\ y_1 \\ x_1 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_1^s & y_1 & y_1^s \end{array} \\ \left[\begin{array}{cccc} \cdot & a & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \hline \cdot & b & 1 & 1 \\ 1 & 1 & \cdot & \cdot \end{array}\right] \end{array}$$

- $\mathrm{rep}_Q(\mathbf{d})$ and the closed orbit $\{0\} \subseteq \mathrm{rep}_Q(\mathbf{d})$ embed in $S(\mathbf{d}^*)$ as the vanishing sets of $\langle a - b \rangle$ and $\langle a, b \rangle$ respectively.
- corresponding clans: $1 - +1$ and 1122 respectively.

Example continued

$$Q = \begin{tikzcd}[row sep=small] y_0 \arrow[dr] & & \\ & x_1 & y_1 \arrow[l] \\ y_0' \arrow[ur] & & \end{tikzcd}, \quad \mathbf{d}(y_0) = \mathbf{d}(y_0') = 0, \mathbf{d}(x_1) = \mathbf{d}(y_1) = 1$$

$T = \mathbb{C}^\times \times \mathbb{C}^\times$ acts on $\mathrm{rep}_Q(\mathbf{d})$ by conjugation, which induces a grading on $\mathbb{C}[a] := \mathbb{C}[\mathrm{rep}_Q(\mathbf{d})]$, with $\deg(a) = t - s$.

We can compute this as a ratio of specialized Wyser-Yong polynomials, $\Upsilon_\gamma(C; D)$:

$$\Upsilon_{1-+1}(c_1, c_2, c_3, c_4; d_1, d_2, d_3, d_4) = c_1 - d_1 + c_2 - d_2,$$

$$\Upsilon_{1122}(c_1, c_2, c_3, c_4; d_1, d_2, d_3, d_4) = (c_1 + c_2 - d_3 - d_4)(c_1 - d_1 + c_2 - d_2)$$

The quiver polynomial of the closed orbit is:

$$\frac{\Upsilon_{1122}(-s, -s, -t, -t; -t, -s, -t, -s)}{\Upsilon_{1-+1}(-s, -s, -t, -t; -t, -s, -t, -s)}$$

Generalizing this is work in progress with Z. Hamaker and R. Kinser.

Thanks for listening!