Title: Singularities of Schubert varieties within a right cell

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Collection: Geometric Representation Theory

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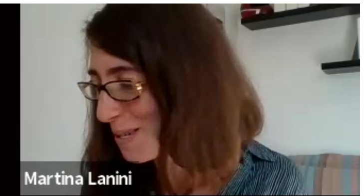
Abstract: We describe an algorithm which takes as input any pair of permutations and gives as output two permutations lying in the same Kazhdan-Lusztig right cell. There is an isomorphism between the Richardson varieties corresponding to the two pairs of permutations which preserves the singularity type. This fact has applications in the study of W-graphs for symmetric groups, as well as in finding examples of reducible associated varieties of sln-highest weight modules, and comparing various bases of irreducible representations of the symmetric group or its Hecke algebra. This is joint work with Peter McNamara.

SINGULARITIES of SCHUBERT VAR'S WITHIN A RIGHT CELL

(j/w. P. McNamara)



Key result: Let $y, w \in S_n$ be such that

- $X_w \subset \mathcal{F}l_n$ Schubert var. is singular at y

Then: $\exists\, N \geq n$ $\exists\, \bar{y}, \bar{w} \in S_N$ s.t.

(i) $\mathcal{F}l_N \supset X_{\bar{w}}$ singular at $\bar{y}$ same sing. as above

(ii) $\bar{y} \underset{R}{\sim} \bar{w}$ (belong to same KL-right cell)

SINGULARITIES of SCHUBERT VAR'S WITHIN A RIGHT CELL

(j/w. P. McNamara)

Key result: Let $y, w \in S_n$ be such that

- $X_w \subset \mathcal{Fl}_n$ Schubert var. is singular at y

Then, $\exists\, N \geq n$ $\exists\, \bar{y}, \bar{w} \in S_N$ s.t.

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Motivations

ⓐ Reducibility of associated varieties of hw modules ($\mathfrak{g}_r$ $\mathfrak{sl}_n$)
(Qu Borho-Brylinski, Joseph ~85)

ⓑ 0-1 Conjecture for (type A) KL-polys
(Qu /Obs Lascoux-Schützenberger, 81)
↳ counterexample 103: McLarnan-Warrington

ⓒ Comparison of bases of Specht modules for $\mathcal{H}(S_n)$

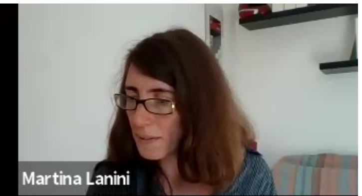
② Fix usual data:

$$\begin{array}{ll}
G \text{ ss alg gp} / \mathbb{C} & \rightsquigarrow W = N_G(T)/T = \text{Weyl gp} \\
\cup & \\
B \text{ Borel subgp} & X = G/B = \text{flag variety} \\
\cup & \quad = \bigsqcup_{w \in W} \underbrace{BwB/B}_{\text{Schubert cell}} \\
T \text{ mxl torus} & X_w := \overline{BwB/B} \quad \text{Schubert variety}
\end{array}$$

$$\rightsquigarrow$$

$$\begin{array}{l}
\mathrm{Lie}(G) =: \mathfrak{g} \\
\cup \\
\mathrm{Lie}(B) =: \mathfrak{b} \\
\cup \\
\mathrm{Lie}(T) =: \mathfrak{t} \qquad \underset{\text{root syst.}}{R} = \underset{\text{pos. roots}}{R^+} \sqcup \underset{\text{neg roots}}{R^-} \subset \mathfrak{t}^* \ni \rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha
\end{array}$$

• $w \in W \rightsquigarrow L_w := L(-w\rho - \rho)$ simple hw mod



② Fix usual data:

G ss alg gp $/\mathbb{C}$ $\rightsquigarrow$ $W = N_G(T)/T$ = Weyl gp

$\cup$

B Borel subgp $\qquad X = G/B$ = flag variety

$\cup$

T max torus $\qquad = \bigsqcup_{w \in W} \overline{BwB/B}$ $\rightsquigarrow$ Schubert cell

$\qquad X_w := \overline{BwB/B}$ Schubert variety

$\qquad \ni y \, (= yB/B)$

$\mathrm{Lie}(G) =: \mathfrak{g}$

$\cup$

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$\cup$

$\mathrm{Lie}(T) =: \mathfrak{t}$ $\qquad \underbrace{R}_{\text{root syst.}} = \underbrace{R^+}_{\text{pos. roots}} \sqcup \underbrace{R^-}_{\text{neg roots}} \subset \mathfrak{t}^* \ni \rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$

- $w \in W \rightsquigarrow L_w := L(-w\rho - \rho)$ simple h.w. $\mathcal{U}(\mathfrak{g})$ mod

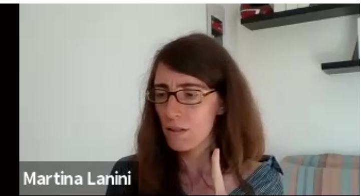
$V(L_w)$ "associated variety" $\subseteq \mathfrak{g}^*$ (ir)reducible?

(L_w it admits a filtration which is compatible with the PBW filtration on $\mathcal{U}(\mathfrak{g})$

gr L_w is a $S(\mathfrak{g})$-fin. gen. module

$S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*]$

$\rightsquigarrow$ coh. sheaf on $\mathfrak{g}^*$ $\rightsquigarrow V(L_w)$ = Supp of such a sheaf)



$V(L_w) \subseteq \mathfrak{g}^*$

$\uparrow \mu$

$Ch(L_w) \subseteq T^*X \qquad \mu(Ch(L_w)) = V(L_w)$

$Ch(L_w) \subseteq \bigcup\limits_{y \leq w} \overline{T^*_{X_y} X}$

taking multiplicities $\rightsquigarrow CC(L_w) = \sum \underbrace{m_{y,w}}_{\in \mathbb{Z}_{\geq 0}} [\overline{T^*_{X_y} X}]$

$\boxed{m_{y,w} = ?}$ HARD!

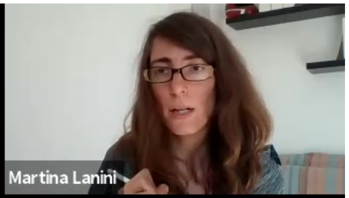
- $m_{w,w} = 1$
- $m_{y,w} = 0$ unless $\underset{ByB/B}{C_y} \subseteq X_w$ and X_w is sing at y
- sing. invt.

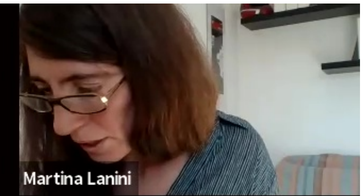
Rmk $Ch(L_w)$ irr $\Leftrightarrow m_{y,w} = 0 \ \forall y \neq w$

Conj [KL'80] $Ch(L_w)$ is irreducible in type A

False: 1997 Kashiwara-Saito $y \in X_w \subset SL_8/B$

- Braden '01
- Vilonen-Williamson ('12) $m_{y,w} \longleftrightarrow$ decomp. numbers for perv. sheaves





— BREAK! —

Main result:

Any possible behavior of invariants of Schubert variety singularities can appear within a right KL-cell

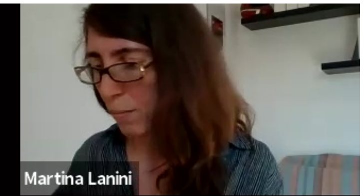
→ KL-polys, p-KL-polys, decomposition numbers for perverse sheaves, ...

Applications:

(a) Reducibility of associated varieties for $\mathfrak{sl}_n$-hw mods.

(b) Failure of weaker form of the 0-1 Conjecture for KL-polys of type A

(c) Infinitely many examples in which bases of Specht modules for $\mathcal{H}(S_n)$ constructed from p-KL-basis elements differ from the basis constructed from the KL-basis



Weaker Conj (Borho–Brylinski; Joseph) $V(L_w)$ is irr. in type A.

Fact: $V(L_w)$ irr. $\iff m_{y,w}=0 \;\; \forall y \underset{LR}{\sim} w \;\; y\neq w$

From Fact + key result + KS counterexample to Conj.

$\implies$ counterexample to Weaker Conj.

2014: Williamson showed the failure of WC by exhibiting an explicit counterexample.

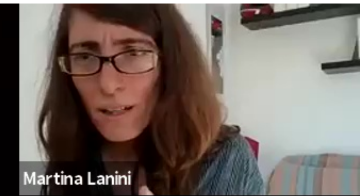
Williamson's strategy

- Pick KS counterexample: $(y,w)\in S_8\times S_8$
- Want to find $(\bar{y},\bar{w})\in S_N\times S_N$ s.t.
  - $\bar{y}\underset{R}{\sim}\bar{w}$

and such that

$$
\begin{aligned}
B_-yB/B\cap X_w &= N_{y,w}\\
&\quad\;\;\rotatebox{90}{$\cong$}\\
\tilde{B}_-\bar{y}\tilde{B}/\tilde{B}\cap\tilde{X}_{\bar{w}} &= \tilde{N}_{\bar{y},\bar{w}}
\end{aligned}
$$

- To find such pair:
  - properties of p-KL-basis
  - result of [VW]
  - computer calculations (MAGMA - Howlett - Nguyen)



KL-cells for S_n

Recall: RSK corresp. $S_n \longrightarrow SYT(n) \times SYT(n)$

$$(x_1 \dots x_n) = x \longmapsto (P(x), Q(x)) \qquad Q(x) = P(x^{-1})$$

Defn (Thm KL'79, Garsia-McLarnan 88) Let $y, w \in S_n$

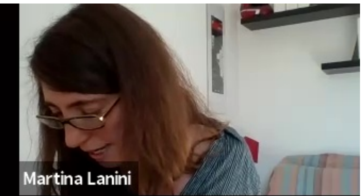
- $y \underset{R}{\sim} w$ if $P(y) = P(w)$ (belong to same right KL-cell)
- $y \underset{RL}{\sim} w$ if $P(y)$ and $P(w)$ have same shape (— " — two-sided — " —)

Rmk $y \underset{R}{\sim} w \Rightarrow y \underset{RL}{\sim} w$

Example. $y = (2\,1\,4\,3)$ $\qquad w = (4\,2\,3\,1)$

$$\emptyset \leftarrow \begin{array}{|c|}\hline 2\\\hline\end{array} \leftarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \leftarrow \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array} \leftarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} \qquad \emptyset \leftarrow \begin{array}{|c|}\hline 4\\\hline\end{array} \leftarrow \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} \leftarrow \begin{array}{|c|c|}\hline 2&3\\\hline 4\\\cline{1-1}\end{array} \leftarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}$$

$$P(y) = \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} \qquad P(w) = \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}$$



Algorithm: INPUT: $y, w \in S_n$

OUTPUT: $\bar{y}, \bar{w} \in S_N$ with $\bar{w} \underset{R}{\sim} \bar{y}$

If $P(y) = P(w) \longrightarrow \bar{y} = y \quad \bar{w} = w$

Otherwise, $\exists$ k = min'l entry which lies in diff't boxes

$$\ell := \max\{c_y(k), c_w(k)\}$$

column index of box containing k in $P(y)$

$x \in \{y, w\}$

$$x' = (k, k+1, \dots, k+\ell-2, x'(\ell), x'(\ell+1), \dots, x'(n+\ell-1))$$

$$j = 1, \dots, n \qquad x'(j+\ell-1) = \begin{cases} x(j) & \text{if } x(j) < k \\ x(j)+\ell-1 & \text{if } x(j) \geq k \end{cases}$$

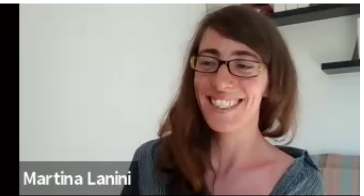
thm 1) The above algorithm terminates (in at most $n-1$ steps)

2) Let $\bar{y}, \bar{w}$ be the final output, then $m_{\bar{y},\bar{w}} = m_{y,w}$

$$(N_{y,w} \overset{?}{\cong} N_{\bar{y},\bar{w}})$$

Cor From known reducible ch. varieties $\Longrightarrow$ reducible associated var's

Exercise: KS: $y = (2\,1\,6\,5\,4\,3\,8\,7)$, $w = (6\,2\,8\,4\,5\,1\,7\,3)$ $\rightsquigarrow (\bar{y}, \bar{w})$



Example. $y = (2\,1\,4\,3)$ $\quad w = (4\,2\,3\,1)$

$$P(y) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \qquad P(w) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} 4 & \\ \cline{1-1} \end{array}$$

$k = 4 \qquad l = 2$

$y' = (4, 2, 1, 5, 3) \qquad w' = (4, 5, 2, 3, 1)$

$$P(y') = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \cline{1-1} \end{array} \;=\!=\!=\; \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \cline{1-1} \end{array} = P(w')$$

$$N_{y,w} \cong \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & b & 0 & 1 \\ c & d & 1 & 0 \end{pmatrix} \;\middle|\; ad - bc = 0 \right\}$$

$$N_{\bar{y},\bar{w}} \cong \left\{ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & a & b & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ e & c & d & 1 & 0 \end{pmatrix} \;\middle|\; \begin{array}{l} e = 0 \\ ad - bc = 0 \end{array} \right\}$$

$$x' = (k, k+1, \dots, k+l-2, x'(l), x'(l+1), \dots, x'(n+l-1))$$

$$j = 1, \dots, n \qquad x'(j+l-1) = \begin{cases} x(j) & \text{if } x(j) < k \\ x(j) + l - 1 & \text{if } x(j) \geq k \end{cases}$$

GRAZIE!