Title: Tate's thesis in the de Rham setting

Speakers: Sam Raskin

Collection: Geometric Representation Theory

Date: June 22, 2020 - 2:00 PM

URL: http://pirsa.org/20060027

Abstract: This is joint work with Justin Hilburn. We will explain a theorem showing that D-modules on the Tate vector space of Laurent series are equivalent to ind-coherent sheaves on the space of rank 1 de Rham local systems on the punctured disc equipped with a flat section. Time permitting, we will also describe an application of this result in the global setting. Our results may be understood as a geometric refinement of Tate's ideas in the setting of harmonic analysis. They also may be understood as a proof of a strong form of the 3d mirror symmetry conjectures: our results amount to an equivalence of A/B-twists of the free hypermultiplet and a U(1)-gauged hypermultiplet.

Tate's thesis in the de Rham setting

Sam Raskin

University of Texas at Austin

June 22, 2020

Slides available at: `http://math.utexas.edu/~sraskin/talk.pdf`

Contents

Overview

I am going to describe joint work with Justin Hilburn on an analogue of Tate's thesis in the setting of de Rham geometric Langlands.

Part of the motivation for our work comes from a body of conjectures arising from harmonic analysis for automorphic forms and separately from mathematical physics. These conjectures provide novel grist for geometric Langlands and are generically conditional on strong forms of the geometric Langlands conjectures.

My work with Justin demonstrates the most elementary (new) example in this hierarchy in the abelian case, and therefore makes this body of conjectures appear more plausible.

From the physics perspective, our work provides some of the strongest (possible?) results for $3d$ mirror symmetry conjectures, again, in a baby test case.

Attributions

The circle of ideas I'm discussing has been developed by many people. I apologize for any incompleteness and inaccuracies here, especially if I'm leaving out your contributions listener!

On the mathematical physics side, we are discussing 3d mirror symmetry and its relation with 4d S-duality. The former subject was initiated by Intriligator–Sieberg and further developed by Hanany–Witten and many others. The connections with S-duality were first considered by Gaiotto–Witten. The sharp mathematical conjectures were first considered by Hilburn-Yoo (maybe jointly with Dimofte-Gaitto) and later Braverman–Finkelberg.

In harmonic analysis, this body of conjectures originated from work of Sakellaridis–Venkatesh that provides a unified perspective on period expressions for L-functions.

Recent work of Ben-Zvi-Sakellaridis–Venkatesh joins these two perspectives and connects with the geometric Langlands program. In addition, their work emphasizes global aspects.

Disclaimer

This is new work and some parts are still in progress. The theorems I will discuss have complete proofs, but there are a number of interesting related questions that we have not settled yet.

Notation

Let k be a base field of characteristic zero.

We are interested in geometric representation theory of the non-Archimedean field $K := k((t))$.

Motivation

Roughly speaking, Tate's thesis provides a detailed analysis of the spectral decomposition of functions on K under the natural $K^\times$-action (for $k = \mathbb{F}_q$). The short summary is that for non-trivial characters χ of $K^\times$, nothing interesting happens: the character extends canonically to K and generates the corresponding eigenspace. But for the trivial character, there are singularities in the harmonic analysis giving rise to poles of ζ-functions.

Geometric side

To imitate this in geometric representation theory, it is standard to consider the category $D(K)$ of D-modules on K in place of functions. By this, I mean the following.

For Y an affine scheme, I let $Y(O)$ (resp. $Y(K)$) denote the scheme (resp. ind-scheme) such that:

$$\{\mathrm{Spec}(A) \to Y(O)\} = \{\mathrm{Spec}(A[[t]]) \to Y\}$$
$$\{\mathrm{Spec}(A) \to Y(K)\} = \{\mathrm{Spec}(A((t))) \to Y\}.$$

For $Y = \mathbb{A}^1$, I abuse notation in simply writing O or K for the corresponding scheme or indscheme. Similarly, for $Y = \mathbb{G}_m$, I write $O^\times$ or $K^\times$.

Geometric side

There are theories of D-modules on such spaces.

For example, I can think of K as an (ind-pro) affine space with coordinates a_i given by Laurent coefficients. Then a *quasi-coherent sheaf* on K is a vector space V with operators $a_i : V \to V$ $(i \in \mathbb{Z})$ such that for any vector $v \in V$, $a_i v = 0$ for $i \ll 0$. A *D-module* on K also has operators ∂_{a_i} such that for every vector v, $\partial_{a_i} v = 0$ for $i \gg 0$ and such that $[\partial_{a_i}, a_j] = \delta_{ij} \cdot \mathrm{id}$.

Such data are equivalent to modules over the Weyl (or $\beta\gamma$, or CDO on $\mathbb{A}^1$) VOA.

Geometric side

The above definition has a more categorical expression:

$$D(K) = \mathrm{colim}_n\, D(t^{-n}O) = \mathrm{colim}_n\, \mathrm{colim}_m\, D(t^{-n}O/t^m O).$$

This formula means that $D(K)$ is built out of categories of D-modules on finite-dimensional affine spaces $t^{-n}O/t^m O$ via D-module functoriality.

This definition is well-behaved on derived categories, and we take it as our definition of $D(K)$.

Geometric side

We similarly have $D(K^\times)$, which is a monoidal category under convolution. It acts canonically on $D(K)$.

Our problem is to understand the spectral decomposition of $D(K)$ as a $D(K^\times)$-module category.

Spectral side

For this problem to make sense, we recall local geometric class field theory:

Theorem (Beilinson-Drinfeld)

There is a canonical equivalence $D(K^\times) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m})$ *as symmetric monoidal categories.*

Here $\mathrm{LocSys}_{\mathbb{G}_m}$ is the moduli of rank 1 de Rham local systems $(\mathcal{L}, \nabla)$ on $\overset{o}{\mathcal{D}} := \mathrm{Spec}(K)$. By definition, we take $K^\times$ acting on Kdt via the homomorphism $d\log : K^\times \to Kdt$ and form the (stack) quotient $\mathrm{LocSys}_{\mathbb{G}_m}$.

Remark

The above result is a consequence of Contou-Carrère's duality, which Justin Campbell will speak about later in the week.

Spectral side

Now define $\mathcal{Y}$ as the moduli of data $(\mathcal{L}, \nabla, s)$ with $(\mathcal{L}, \nabla)$ a rank 1 local system on the punctured disc and $s \in \mathcal{L}$ with $\nabla(s) = 0$. I.e., $\mathcal{Y} = \mathcal{M}\text{aps}(\overset{o}{\mathcal{D}}_{dR}, \mathbb{A}^1/\mathbb{G}_m)$.

At a technical level, the structure of $\mathcal{Y}$ is given by:

Proposition

$\mathcal{Y}$ is a quotient of a classical ind-affine scheme by an action of $K^\times/(1 + tO)$.

As a consequence of this result, we can make sense of $\text{IndCoh}(\mathcal{Y})$.

Statement of the main result

Theorem (Hilburn-R.)

There is a canonical equivalence of DG categories:

$$\mathrm{IndCoh}(\mathcal{Y}) \simeq D(K)$$

compatible with local geometric class field theory.

If $(\mathcal{L}, \nabla)$ is a non-trivial local system with connection, the fiber of $\mathcal{Y} \to \mathrm{LocSys}_{\mathbb{G}_m}$ over it is just a point. So our result matches Tate's thesis in this sense.

But as a funny aside: it is actually not true (for subtle reasons) that the morphism:

$$\mathcal{Y} \underset{\mathrm{LocSys}_{\mathbb{G}_m}}{\times} \mathrm{LocSys}_{\mathbb{G}_m} \backslash \mathbb{B}\mathbb{G}_m \to \mathrm{LocSys}_{\mathbb{G}_m} \backslash \mathbb{B}\mathbb{G}_m$$

is an isomorphism.

Algebraic field theories

Recall that $3d$ TFTs assign numbers to closed 3-manifolds, vector spaces to 2-manifolds, and categories to 1-manifolds.

We have the following outline of a definition.
An (algebraic, [1,2]-extended) *3d field theory* Z on a smooth, proper algebraic curve X is ... some data:

Algebraic field theories

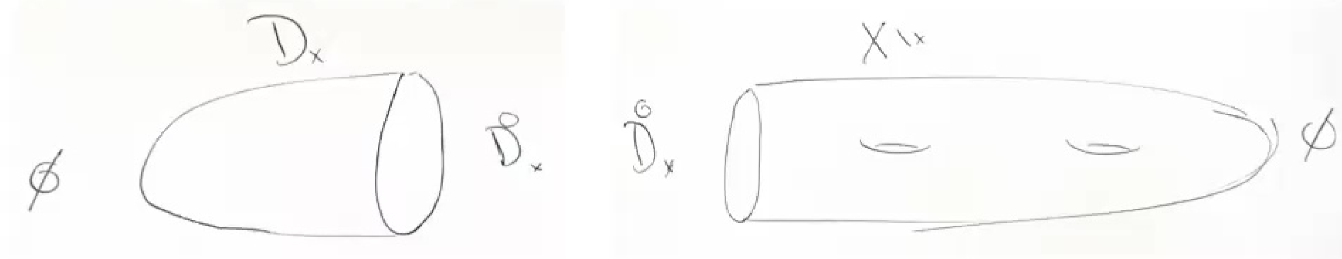
Recall that $3d$ TFTs assign numbers to closed 3-manifolds, vector spaces to 2-manifolds, and categories to 1-manifolds.

We have the following outline of a definition.
An (algebraic, [1,2]-extended) *3d field theory Z* on a smooth, proper algebraic curve X is ... some data:

First, for $x \in X$, we have a DG category $Z(\overset{o}{\mathcal{D}}_x) \in \mathrm{DGCat}$.

Algebraic field theories

Moreover, the "cobordisms:"



define a *unit* (or vacuum) object:

$$\mathbb{1}_x \in Z(\mathring{\mathcal{D}}_x)$$

and a *chiral homology* functor:

$$H_{ch,x}(-) : Z(\mathring{\mathcal{D}}_x) \to Z(\varnothing) = \text{Vect}.$$

Remark

$H_{ch,x}(\mathbb{1}_x) \in \text{Vect}$ is independent of the choice of point x, and is the vector space the theory assigns to the global curve X.

Algebraic field theories

For example, given a chiral algebra $\mathcal{A}$ on X, Beilinson-Drinfeld defined a $3d$ theory $Z_{\mathcal{A}}$.

Here $Z_{\mathcal{A}}(\overset{o}{\mathcal{D}}_x)$ is the category of $\mathcal{A}$-modules supported at x. The vacuum representation of $\mathcal{A}$ is the unit object, and chiral homology is given as defined by Beilinson-Drinfeld.

Algebraic field theories

Moreover, the "cobordisms:"

Page 66 of 105

Lagrangian $3d$ $\mathcal{N} = 4$ theories

For a stack Y, physicists say that the (non-algebraic) $3d$ σ-model Z_Y with target T^*Y has $\mathcal{N} = 4$ supersymmetry, which allows us to define two *twists* of it, which are algebraic $3d$ theories $Z_{Y,A}$ and $Z_{Y,B}$.

Lagrangian $3d$ $\mathcal{N} = 4$ theories

The basic properties of these twists are:

$$Z_{Y,A}(\mathring{\mathcal{D}}_x) = D(Y(K))$$

$$Z_{Y,B}(\mathring{\mathcal{D}}_x) = \mathrm{IndCoh}(\mathcal{M}\mathrm{aps}(\mathring{\mathcal{D}}_{x,dR}, Y)).$$

Mirror symmetry

For a $3d$ $\mathcal{N} = 4$ theory Z, there is a *abstract mirror dual theory* $Z^{\star}$. This should be the same $3d$ theory but with the supersymmetries realized in a conjugate way. By construction, $Z_A = Z_B^{\star}$ and $Z_B = Z_A^{\star}$.

Mirror symmetry in $3d$ refers to *mirror dual* pairs (Y_1, Y_2) with $Z_{Y_1} = Z_{Y_2}^{\star}$. Given the previous slide, any such pair (Y_1, Y_2) yield interesting mathematical conjectures.

Mirror symmetry

One example: $Y_1 = \mathbb{A}^1$ and $Y_2 = \mathbb{A}^1/\mathbb{G}_m$. Then the prediction $Z_{Y_1,A}(\overset{o}{\mathcal{D}}_x) \simeq Z_{Y_2,B}(\overset{o}{\mathcal{D}}_x)$ amounts to our equivalence:

$$\mathrm{IndCoh}(\mathcal{Y}) \simeq D(K)$$

from earlier.

Global results

But the perspective above suggests another compatibility. Suppose $x \in X$ is a marked smooth projective curve as before.

The chiral homology functor $D(K) \to \mathrm{Vect}$ is given by pull-push along the diagram:

$$\begin{array}{ccc} & k[X \backslash x] & \\ \swarrow & & \searrow \\ K & & *. \end{array}$$

The chiral homology functor $\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{Vect}$ is given by pull-push along the diagram:

$$\begin{array}{ccc} & \mathcal{M}\mathrm{aps}(X \backslash x, \mathbb{A}^1/\mathbb{G}_m) & \\ \swarrow & & \searrow \\ \mathcal{Y} = \mathcal{M}\mathrm{aps}(\overset{o}{\mathcal{D}}_{dR}, \mathbb{A}^1/\mathbb{G}_m) & & *. \end{array}$$

Global results

Theorem (Hilburn-R.)

The equivalence $\mathrm{IndCoh}(\mathcal{Y}) \simeq D(K)$ *is compatible with chiral homology functors.*

Global results

In fact, the above result generalizes further. We have correspondences:

$$\begin{array}{ccc} & \mathcal{M}aps(X, \mathbb{A}^1/\mathbb{G}_m) \underset{\mathrm{Bun}_{\mathbb{G}_m}}{\times} \mathrm{Bun}_{\mathbb{G}_m}^{\mathrm{level},x} & \\ \swarrow & & \searrow \\ K & & \mathrm{Bun}_{\mathbb{G}_m}^{\mathrm{level},x} \end{array}$$

and:

$$\begin{array}{ccc} & \mathcal{M}aps((X \backslash x)_{dR}, \mathbb{A}^1/\mathbb{G}_m) & \\ \swarrow & & \searrow \\ \mathcal{M}aps(\overset{o}{\mathcal{D}}_{dR}, \mathbb{A}^1/\mathbb{G}_m) & & \mathrm{LocSys}_{\mathbb{G}_m}(X \backslash x) \end{array}$$

giving functors:

$$D(K) \to D(\mathrm{Bun}_{\mathbb{G}_m}^{\mathrm{level},x})$$

$$\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathrm{LocSys}_{\mathbb{G}_m}(X \backslash x)).$$

Global results

Theorem (Hilburn-R.)

The above functors match under our equivalence and the global class field theory equivalence
$D(\mathrm{Bun}_{\mathbb{G}_m}^{\mathrm{level},x}) \simeq \mathrm{IndCoh}(\mathrm{LocSys}_{\mathbb{G}_m}(X \backslash x))$.

Remark

In physics language, our equivalence of $3d$ (algebraic) theories upgrades to an equivalence of boundary conditions for the A/B-twists of $4d$ $\mathcal{N} = 4$ SYM for $\mathbb{G}_m$, compatibly with (abelian) S-duality.

Remark

In the Ben-Zvi-Sakellaridis–Venkatesh setting, results of this type are considered as categorical analogues of identities for period integrals.

Other applications

Let $\overline{\mathbb{P}}^1 := \mathbb{A}^2/\mathbb{G}_m$.

With separate work with Sasha Braverman, we show:

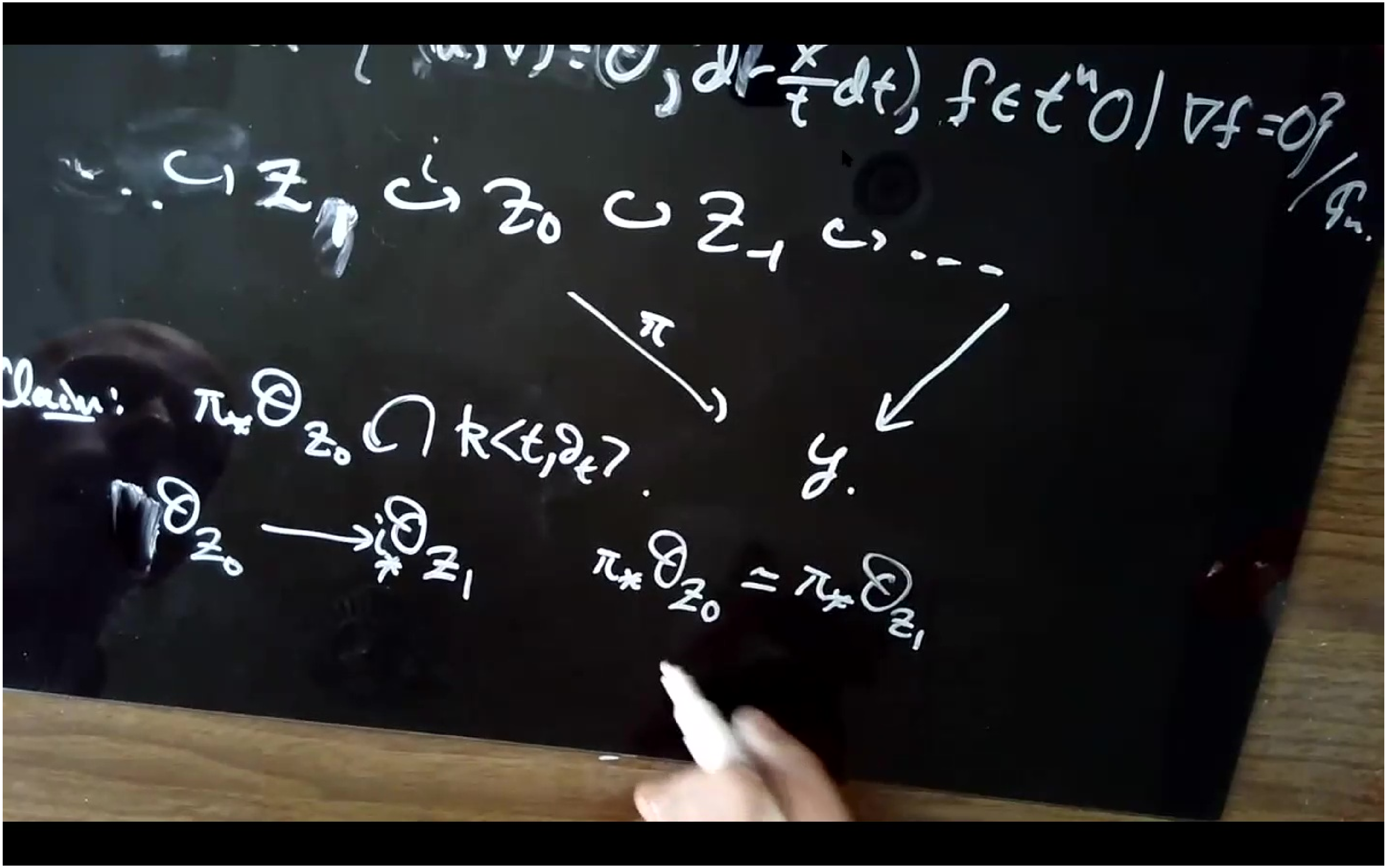
$$D(\mathrm{Gr}_{PGL_2})^{\mathrm{Hecke}} := D(\mathrm{Gr}_{PGL_2}) \underset{\mathrm{Rep}(SL_2)}{\otimes} \mathrm{Vect} \simeq D(\overline{\mathbb{P}}^1(K)).$$

From my work with Justin and separate work of mine, we deduce a mass of equivalences:

$$\widehat{\mathfrak{sl}}_{2,crit}\text{–mod}_\chi \overset{R.}{\simeq} D(\mathrm{Gr}_{PGL_2})^{\mathrm{Hecke}} \overset{Braverman-R.}{\simeq}$$
$$D_{ren}(\overline{\mathbb{P}}^1(K)) \overset{Hilburn-R.}{\simeq} \mathrm{IndCoh}(\mathcal{M}aps(\overset{o}{\mathcal{D}}_{dR}, \overline{\mathbb{P}}^1))$$

for χ an oper on the formal disc (i.e., we identify our formal disc with the completion of $\mathbb{P}^1$ at a point).

Thanks!


$\cdots \hookrightarrow Z_1 \overset{i}{\hookrightarrow} Z_0 \hookrightarrow Z_{-1} \hookrightarrow \cdots$
π
Y
Claim: $\pi_* \mathcal{O}_{Z_0} \circlearrowleft k\langle t, \partial_t \rangle$.
$\mathcal{O}_{Z_0} \longrightarrow i_* \mathcal{O}_{Z_1}$
$\pi_* \mathcal{O}_{Z_0} \simeq \pi_* \mathcal{O}_{Z_1}$

$\partial_t : i_* \mathcal{O}_{Z_1} \xrightarrow{x\cdot -} \mathcal{O}_{Z_0}$ Exercise: this is well-defined.

$\mathcal{O}_{Z_0} \xrightarrow{t(\partial_t)^{k\pm 1}} \mathcal{O}_{Z_{-1}} \to \mathcal{O}_{Z_0}$

$\to \mathcal{O}_{Z_1} \xrightarrow[x\cdot -]{\partial_t}$

A Weyl algebra $\circlearrowleft Y$.

Def: 1) $Z_n = \{ (\mathcal{E}, \nabla) = (\mathcal{O}, d - \frac{x}{t} dt), f \in t^n \mathcal{O} \mid \nabla f = 0 \} / \sim$

$\cdots \hookrightarrow Z_1 \xrightarrow{i} Z_0 \hookrightarrow Z_{-1} \hookrightarrow \cdots$

$\pi : Z_0 \to Y$, $\cdots \to Y$

Claim: $\pi_* \mathcal{O}_{Z_0} \circlearrowleft k\langle t, \partial_t \rangle$