Title: Summer Undergrad 2020 - Symmetries (A) - Lecture 5

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Collection: Summer Undergrad 2020 - Symmetries

Date: June 04, 2020 - 2:00 PM

URL: http://pirsa.org/20060017

Abstract: Representation theory, symmetries in quantum mechanics

Representations

Let G be a MLG. A finite dimensional complex vector space V is a $\boxed{G\text{-module}}$ if there is a continuous action of G on V such that

- $g \triangleright (\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha(g \triangleright |\psi\rangle) + \beta(g \triangleright |\varphi\rangle)$
- $g \triangleright h \triangleright |\psi\rangle = (gh) \triangleright |\psi\rangle$

the map $G \times V \to V$ is continuous
$(g, |\psi\rangle) \mapsto g \triangleright |\psi\rangle$

($g \triangleright |\psi\rangle$ denotes the action of g on $|\psi\rangle$)

Note: if V is infinite-dimensional we have to be careful about its topology!

Note 2: The action of G on V is also called a representation of G on V
(Same thing, different point of view)

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example: Let $V_n = \text{span}\{|n\rangle\}$ be a one-dimensional vector space with the action of $e^{i\theta} \in U(1)$ on $|n\rangle$ defined by

$$e^{i\theta} \triangleright |n\rangle = (e^{i\theta})^n |n\rangle = e^{in\theta}|n\rangle \qquad \text{for some fixed } n \in \mathbb{Z}$$

The action is extended to the other vectors by linearity. → $e^{i\theta} \triangleright \alpha|n\rangle$ is defined to be $\alpha(e^{i\theta}\triangleright|n\rangle)$

When $z, w \in U(1)$ we get

$$z \triangleright w \triangleright |n\rangle = z \triangleright w^n|n\rangle = z^n w^n |n\rangle = (zw)^n |n\rangle$$ → only works because $n \in \mathbb{Z}$!

<u>example:</u> Let $V_n = \mathrm{span}\{|n\rangle\}$ as before, with $n \in \mathbb{Z}$

We can extend the action defined for $U(1)$ to $\mathbb{C}\setminus\{0\} = GL(1,\mathbb{C})$ by defining

$$z \triangleright |n\rangle = z^n |n\rangle$$

• A G-module V is <u>unitary</u> if it has an inner product and G acts unitarily, that is $\overline{\langle \varphi | g \triangleright | \psi \rangle} = \boxed{\langle \psi | g^{-1} \triangleright | \varphi \rangle}$ $\forall\, |\psi\rangle, |\varphi\rangle \in V$

→ with abuse of notation, we can say "$g^\dagger = g^{-1}$"

<u>example:</u> Define an inner product on V_n as $\langle n | n \rangle = 1$.

• For the action of $U(1)$ we get $\overline{\langle n | e^{i\theta} \triangleright | n \rangle} = \overline{\langle n | e^{in\theta} | n \rangle} = \overline{e^{in\theta}} \langle n | n \rangle = e^{-in\theta}$

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- For the action of $U(1)$ we get $\overline{\langle n|e^{i\theta}\triangleright|n\rangle} = \overline{\langle n|e^{in\theta}|n\rangle} = \overline{e^{in\theta}\langle n|n\rangle} = e^{-in\theta}$

  and $\langle n|(e^{i\theta})^{-1}\triangleright|n\rangle = \langle n|e^{-i\theta}\triangleright|n\rangle = \langle n|e^{-in\theta}|n\rangle = e^{-in\theta}$

  $\Rightarrow$ V_n is unitary

- For the action of $GL(1,\mathbb{C})$ we get $\langle n|z^{-1}\triangleright|n\rangle = z^{-n}$ (same steps as above)

  but $\overline{\langle n|z\triangleright|n\rangle} = \overline{\langle n|z^n|n\rangle} = \bar{z}^n \neq z^{-n}$ in general (unless $n=0$)

  $\Rightarrow$ V_n is not unitary

Lie algebras

Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. A finite-dimensional complex vector space V is a $\mathfrak{g}$-module if there is an action of $\mathfrak{g}$ on V such that

- $X \triangleright (\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha(X \triangleright |\psi\rangle) + \beta(X \triangleright |\varphi\rangle)$
- $(\alpha X + \beta Y) \triangleright |\psi\rangle = \alpha(X \triangleright |\psi\rangle) + \beta(Y \triangleright |\psi\rangle)$
- $[X, Y] \triangleright |\psi\rangle = X \triangleright Y \triangleright |\psi\rangle - Y \triangleright X \triangleright |\psi\rangle$

- V is unitary if it has an inner product and

$$\overline{\langle\varphi| X \triangleright |\psi\rangle} = -\langle\psi| X \triangleright |\varphi\rangle \qquad \text{“} X^\dagger = -X \text{”} \text{ (action is anti-hermitian)}$$

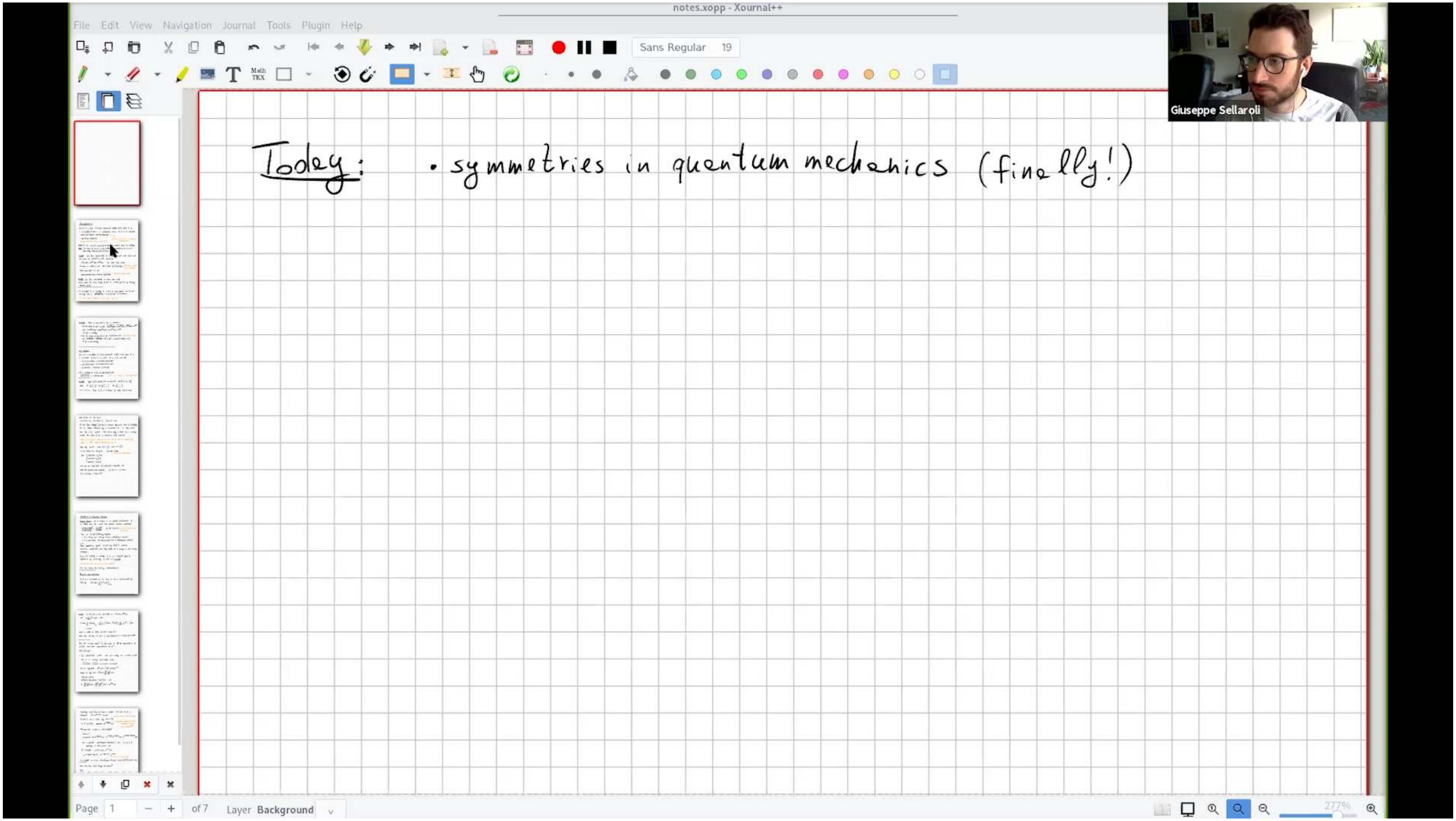
what it does to the basis:

$$[X_1, X_2] = -X_3 \quad [X_2, X_3] = -X_1 \quad [X_3, X_1] = -X_2$$

If this looks <u>almost</u> familiar is because physicists like to multiply the Lie algebra elements by $-i$ (sometimes $+i$), so they would use $J_k = -i X_k$ instead. The reason why is that for a unitary module the action of J_k is hermitian (self-adjoint).

Note: $J_k \notin \underline{su}(2)$! When we use this trick we have to "complexify" $\underline{su}(2)$ to allow scalar multiplication by $\mathbb{C}$.

Now let $V = \mathbb{C}^2$ with $|+\rangle = \binom{1}{0}$ and $|-\rangle = \binom{0}{1}$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Today: • symmetries in quantum mechanics (finally!)
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Now let $V = \mathbb{C}^2$ with $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

if we define, for $X \in su(2)$, $\quad X \triangleright |\psi\rangle = \underbrace{X|\psi\rangle}_{\text{matrix-vector multiplication}}$

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Symmetries in Quantum Mechanics

Wigner's theorem. If $T: H \to H$ is an invertible transformation of an Hilbert space into itself that preserves transition amplitudes

$$\frac{|\langle T(\psi), T(\varphi)\rangle|^2}{\|T(\psi)\|^2 \|T(\varphi)\|^2} = \frac{|\langle \psi, \varphi\rangle|^2}{\|\psi\|^2 \|\varphi\|^2}$$

for all $\psi, \varphi \in H$ (bra-ket notation doesn't work here!)

then one of the following happens:

- T is linear and unitary (up to a multiplicative constant)
- T is anti-linear and anti-unitary (up to a multiplicative constant)

Since symmetries should (at the very least!) preserve transition amplitudes, then they should act a unitary or anti-unitary

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transition amplitudes, then they should act a unitary or anti-unitary operators.

Since the identity is unitary, if G is a connected group of symmetries, by continuity G must act unitarily

(anti-unitary ones are used for time reversal)

This sure sounds like unitary representations!

Back to representations

if V is a G-module, we can make it into a $\mathrm{Lie}(G)$-module by defining
$$X \triangleright |\psi\rangle = \frac{d}{d\varepsilon}\left(e^{\varepsilon X} \triangleright |\psi\rangle\right)\Big|_{\varepsilon=0}$$

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$$X \triangleright |\psi\rangle = \frac{d}{d\varepsilon}\left(e^{\varepsilon X} \triangleright |\psi\rangle\right)\Big|_{\varepsilon=0} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(e^{\varepsilon X}\triangleright|\psi\rangle - \underbrace{e^{0X}}_{\mathbb{1}}\triangleright|\psi\rangle\right)$$

$X \in \mathrm{Lie}(G)$

$$X = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{\varepsilon X} \qquad e^{\varepsilon X}\in G \qquad \boxed{e^{\varepsilon X}|\psi\rangle}$$

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example: if $U(1)$ acts on $V_n = \text{span}\{|n\rangle\}$ as $e^{i\theta} \triangleright |n\rangle = e^{in\theta}|n\rangle$

and $i = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{i\varepsilon} \in \underline{u}(1)$, then

$$i \triangleright |n\rangle = \frac{d}{d\varepsilon} e^{i\varepsilon} \triangleright |n\rangle \Big|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(e^{i\varepsilon} \triangleright |n\rangle - e^{i0} \triangleright |n\rangle\right) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(e^{in\varepsilon} - 1\right)|n\rangle$$

$$= in|n\rangle$$

which is indeed an action of $\underline{u}(1)$ (check it!)

Note that the action of $\underline{u}(1)$ is anti-hermitian $\Rightarrow$ unitary $\underline{u}(1)$-module

---

Does the converse work? In other words, do all the representations of

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example: if $U(1)$ acts on $V_n = \mathrm{span}\{|n\rangle\}$ as $e^{i\theta} \triangleright |n\rangle = e^{in\theta}|n\rangle$

$n \in \mathbb{Z}$ fixed

and $i = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{i\varepsilon} \in \underline{u}(1)$, then

$\underline{u}(1) = \mathrm{span}\{i\} = \{i\theta \mid \theta \in \mathbb{R}\}$

$$\overset{\underline{u}(1)}{i} \triangleright |n\rangle = \frac{d}{d\varepsilon} e^{i\varepsilon} \triangleright |n\rangle \Big|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(\underbrace{e^{i\varepsilon} \triangleright |n\rangle}_{e^{in\varepsilon}|n\rangle} - \underbrace{e^{i0} \triangleright |n\rangle}_{|n\rangle}\Big) = \Big[\underbrace{\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(e^{in\varepsilon} - 1\right)}_{\frac{d}{d\varepsilon} e^{in\varepsilon}\big|_{\varepsilon=0}}\Big]|n\rangle$$

$$= \underbrace{in}_{\in \mathbb{C}} |n\rangle$$

which is indeed an action of $\boxed{\underline{u}(1)}$ (check it!)

Note that the action of $\underline{u}(1)$ is anti-hermitian $\Rightarrow$ unitary $\underline{u}(1)$-module

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Does the converse work? In other words, do all the representations of $\mathrm{Lie}(G)$ come from representations of G?

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Let's find out!

- $V_\alpha = \mathrm{Span}\{|\alpha\rangle\}$ $\quad \alpha \in \mathbb{R}$ $\quad$ with $\underline{u}(1)$ acting as $\quad i \triangleright |\alpha\rangle = i\alpha |\alpha\rangle$

  This is a unitary $\underline{u}(1)$ module, since

  $$\overline{\langle\alpha| i \triangleright |\alpha\rangle} = \overline{i\alpha \langle\alpha|\alpha\rangle} = -i\alpha \langle\alpha|\alpha\rangle = -\langle\alpha| i \triangleright |\alpha\rangle$$

Can we say that $e^{i\theta} \triangleright |\alpha\rangle =$ "$\exp(i\theta \triangleright |\alpha\rangle)$" ?

Suppose we say that $e^{i\theta} \triangleright |\alpha\rangle = \sum_{n=1}^{\infty} \frac{(i\theta)^n}{n!} \triangleright |\alpha\rangle$

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1-dim.

$\in \mathfrak{u}(1)$

• $V_\alpha = \mathrm{Span}\{|\alpha\rangle\}$ $\quad \alpha \in \mathbb{R} \quad$ with $\mathfrak{u}(1)$ acting as $i \triangleright |\alpha\rangle = i\alpha\, |\alpha\rangle$

$\in \mathbb{C}$

This is a unitary $\mathfrak{u}(1)$ module, since

$$\overline{\langle\alpha|\, i \triangleright |\alpha\rangle} = \overline{i\alpha \langle\alpha|\alpha\rangle} = -i\alpha\langle\alpha|\alpha\rangle = -\langle\alpha|\, i\triangleright|\alpha\rangle$$ anti-hermitian

Can we say that $e^{i\theta} \triangleright |\alpha\rangle =$ "$\exp(i\theta \triangleright|\alpha\rangle)$" ? reverse of

Suppose we say that $e^{i\theta} \triangleright |\alpha\rangle = \sum_{n=1}^{\infty} \frac{(i\theta)^n}{n!} \triangleright |\alpha\rangle$

$$i\theta \triangleright |\alpha\rangle = i\alpha\theta|\alpha\rangle$$

$$(i\theta)^2 \triangleright|\alpha\rangle = i\theta \triangleright i\alpha\theta|\alpha\rangle = (i\alpha\theta)^2|\alpha\rangle \qquad \text{etc.}$$

$$\rightarrow \sum_{n=1}^{\infty} \frac{(i\theta)^n}{n!} \triangleright|\alpha\rangle = \left(\sum_{n=1}^{\infty} \frac{(i\alpha\theta)^n}{n!}\right)|\alpha\rangle = e^{i\theta\alpha}|\alpha\rangle$$

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$$\overline{\langle\alpha| i\triangleright|\alpha\rangle} = \overline{i\alpha\langle\alpha|\alpha\rangle} = -i\alpha\langle\alpha|\alpha\rangle = -\langle\alpha|i\triangleright|\alpha\rangle$$ anti-hermitian

Can we say that $e^{i\theta}\triangleright|\alpha\rangle =$ "$\exp(i\theta\triangleright|\alpha\rangle)$" ?

reverse of $X\triangleright|n\rangle = \frac{d}{d\epsilon} e^{\epsilon X}\triangleright|n\rangle\Big|_{\epsilon=0}$

Suppose we say that $$e^{i\theta}\triangleright|\alpha\rangle = \sum_{n=1}^{\infty}\frac{(i\theta)^n}{n!}\triangleright|\alpha\rangle$$ $\rightarrow$ $(i\theta)^2\triangleright|\alpha\rangle \equiv i\theta\triangleright i\theta\triangleright|\alpha\rangle$

$$\boxed{i\theta\triangleright|\alpha\rangle = i\alpha\theta|\alpha\rangle}$$

$$(i\theta)^2\triangleright|\alpha\rangle = i\theta\triangleright\boxed{i\alpha\theta|\alpha\rangle} = (i\alpha\theta)^2|\alpha\rangle \qquad \text{etc.}$$

$$\rightarrow \sum_{n=1}^{\infty}\frac{(i\theta)^n}{n!}\triangleright|\alpha\rangle = \Bigg(\underbrace{\sum_{n=1}^{\infty}\frac{(i\alpha\theta)^n}{n!}}_{\in\mathbb{C}}\Bigg)|\alpha\rangle = \boxed{e^{i\theta\alpha}|\alpha\rangle}$$

$$e^{i\theta}\triangleright|\alpha\rangle \overset{?}{=} e^{i\theta\alpha}|\alpha\rangle$$

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$$\Rightarrow \sum_{n=1} \frac{(i\theta)^n}{n!} \triangleright |\alpha\rangle = \Big(\underbrace{\sum_{n=1} \frac{(i\alpha\theta)^n}{n!}}_{\in \mathbb{C}}\Big)|\alpha\rangle = \boxed{e^{i\theta\alpha}|\alpha\rangle}$$

$$e^{i\theta}\triangleright|\alpha\rangle \overset{?}{=} e^{i\theta\alpha}|\alpha\rangle$$

everything seems fine, but there's a problem: the choice of θ is ambiguous! $e^{i\theta} = e^{i\theta + i2\pi k}$ $\forall k \in \mathbb{Z}$

standard choice in complex analysis

We need to make a choice, say $\boxed{\theta \in (-\pi, \pi]}$

$\Rightarrow$ if $\boxed{z \in U(1)}$, $\boxed{z \triangleright |\alpha\rangle = e^{i\alpha \mathrm{Arg}(z)}|\alpha\rangle}$ $\rightarrow$ principal argument, returns something in $(-\pi, \pi]$ with $z = |z|e^{i\mathrm{Arg}(z)}$

$\mathrm{Arg}(e^{i3\pi}) = \pi$

All seems well. Is this a $U(1)$-module?

- linear ✓

$\ldots \rightarrow e^{i\alpha \mathrm{Arg}(z)}|\alpha\rangle = e^{i\alpha \mathrm{Arg}(z)} e^{i\alpha \mathrm{Arg}(w)}|\alpha\rangle = e^{i\alpha(\mathrm{Arg}(z) + \mathrm{Arg}(w))}|\alpha\rangle$

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but in general $\operatorname{Arg}(z) + \operatorname{Arg}(w) = \operatorname{Arg}(z+w) + 2\pi k$, $k \in \{-1, 0, 1\}$

depending on what z and w are.

For example, $e^{i\pi} \triangleright e^{i\pi} \triangleright |\alpha\rangle = e^{i\alpha 2\pi} |\alpha\rangle$

All seems well. Is this a $U(1)$-module? $\quad \mathrm{Arg}(e^{i\ldots}) = \pi$

- linear ✓
- $z \triangleright w \triangleright |\alpha\rangle = z \triangleright e^{i\alpha \mathrm{Arg}(w)} |\alpha\rangle = e^{i\alpha \mathrm{Arg}(z)} e^{i\alpha \mathrm{Arg}(w)} |\alpha\rangle = e^{i\alpha(\mathrm{Arg}(z) + \mathrm{Arg}(w))} |\alpha\rangle$

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depending on what z and w are.

For example, $e^{i\pi} \triangleright e^{i\pi} \triangleright |\alpha\rangle = e^{i\alpha 2\pi} |\alpha\rangle \overset{?}{=} (e^{i\pi} e^{i\pi}) \triangleright |\alpha\rangle \overset{?}{=} |\alpha\rangle$

$$e^{i\alpha(\mathrm{Arg}(z) + \mathrm{Arg}(w))} = e^{i\alpha \mathrm{Arg}(z+w)} \underbrace{e^{i2\pi\alpha k}}_{\neq 1 \text{ if } k \neq 0,\ \alpha \notin \mathbb{Z}}$$

for example if $\alpha = 1/2$, $\boxed{e^{i\pi} \triangleright e^{i\pi}} \triangleright |1/2\rangle = e^{i\pi} |1/2\rangle = -|1/2\rangle \neq (e^{i\pi} e^{i\pi}) \triangleright |1/2\rangle = |1/2\rangle$

Does this mean that things are broken?

$$e^{i\alpha(\mathrm{Arg}(z)+\mathrm{Arg}(w))} = e^{i\alpha\,\mathrm{Arg}(z+w)}\,\underbrace{e^{i2\pi\alpha k}}_{\neq 1 \text{ if } k\neq 0,\ \alpha\notin\mathbb{Z}}$$

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Does this mean that things are broken?

No!

Remember that physical states are only defined up to a non-zero scalar ($|\psi\rangle \sim \lambda|\psi\rangle$ if $\lambda\neq 0$)

$\rightarrow\ [e^{i\pi}\triangleright e^{i\pi}\triangleright|1/2\rangle] = [-|1/2\rangle] = [|1/2\rangle] = [e^{i2\pi}\triangleright|1/2\rangle]$

and in general $[z\triangleright w\triangleright|\alpha\rangle] = [(zw)\triangleright|\alpha\rangle]$ $\leftarrow$ projective representation in general $\quad g \triangleright h \triangleright |\psi\rangle = e^{i\omega(g,h)} (gh) \triangleright |\psi\rangle \quad$ projective rep.

→ still describes a symmetry

Nice things that happen: (assume V if finite-dimensional)

- projective reps of G are all obtained by "exponentiating" (pure) reps of Lie(G) (G connected) in general $g \triangleright h \triangleright |\psi\rangle = e^{i\omega(g,h)} (gh) \triangleright |\psi\rangle$ projective rep.

$\rightarrow$ still describes a symmetry

Nice things that happen: (assume V if finite-dimensional)

- projective reps of G are all obtained by "exponentiating" (pure) reps of Lie(G) (G connected)
  - pure reps of Lie(G) instead of proj. reps of G
  - reps of Lie(G) are easier to find then reps of G

- projective reps of G are in "one-to-one" correspondence with pure reps of $\tilde{G}$ (simply connected cover of G)

universal cover

$\hookrightarrow$ topological concept

$\widetilde{SO(3)} = SU(2)$

$\rightarrow$ instead of proj reps of $SO(3)$, we can look at reps of $SU(2)$

proj

pure reps of G (simply connected cover of G)

universal cover

$\hookrightarrow$ topological concept

$$\widetilde{SO(3)} = SU(2)$$

$\rightarrow$ instead of proj reps of $SO(3)$, we can look at reps of $SU(2)$

pure reps of $SO(3)$

labelled by $j = 0, 1, 2, 3, \ldots$

proj reps of $SO(3)$

$j = 0, 1/2, 1, 3/2, 2, \ldots$

$$\left.\begin{aligned} J_z |j m\rangle &= m |j m\rangle \\ J_\pm |j m\rangle &= C_\pm(j,m) |j m \pm 1\rangle \end{aligned}\right\} \text{rep. of } \underline{so(3)}$$

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$$\boxed{\begin{array}{l}\text{pure reps of } SU(2) \\ \text{labelled by } j = 0, 1/2, 1, 3/2, \ldots\end{array}} \longleftrightarrow$$

pure reps of $\underline{so}(3) \cong \underline{su}(2)$

$j = 0, 1/2, 1, 3/2, \ldots$