Title: Conformal blocks in genus zero, and Elliptic cohomology

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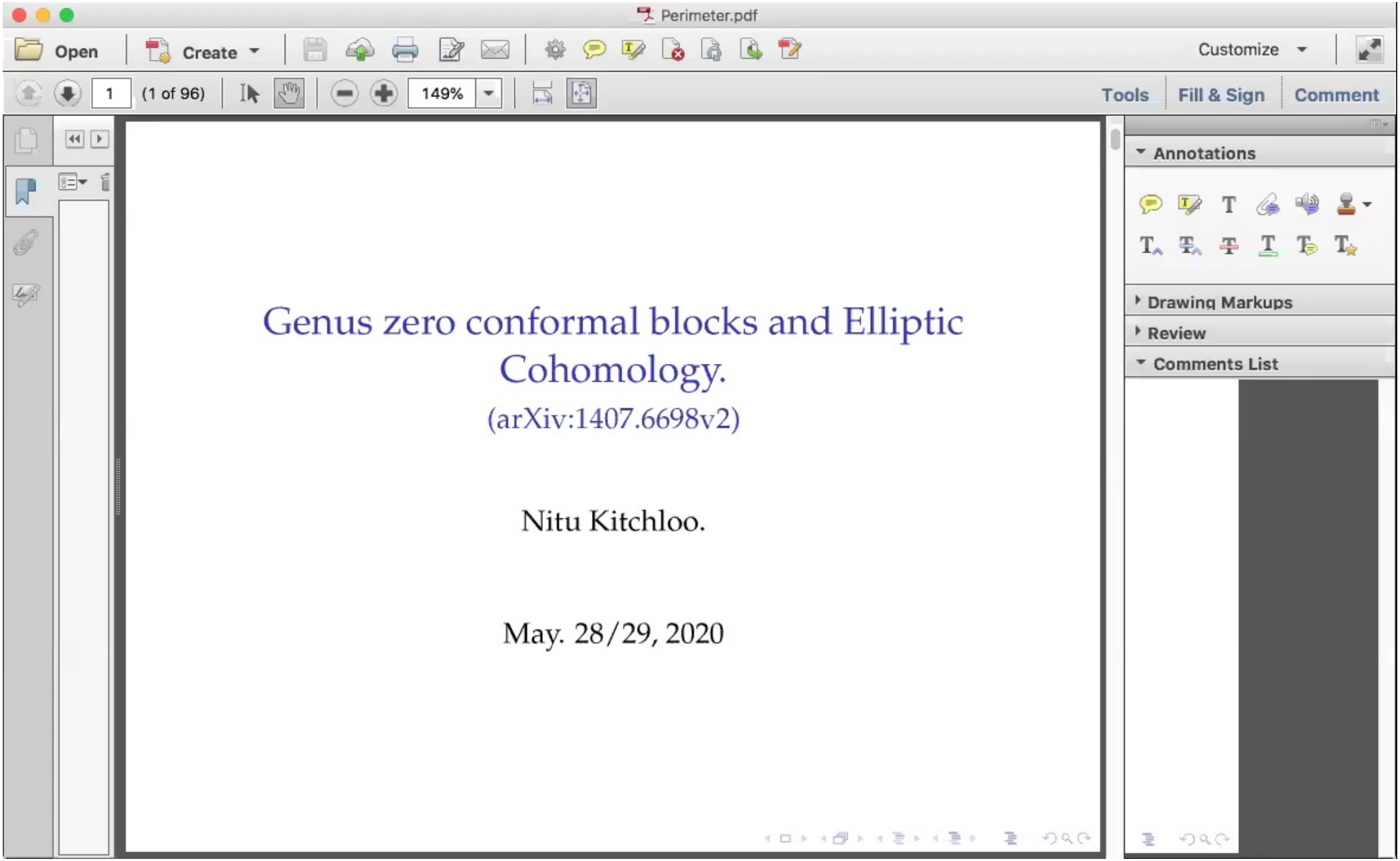
Abstract: A fundamental theorem in the theory of Vertex algebras (known as Zhuâ€™s theorem) demonstrates that the space generated by the characters of certain Vertex algebras is a representation of the modular group. We will cast this theorem in the language of homotopy theory using the language of conformal blocks. The goal of this talk is to justify the claim that equivariant elliptic cohomology, seen as a derived spectrum, is a homotopical analog of Zhuâ€™s theorem in the special case of the Affine Vacuum vertex algebra at a fixed integral level. The talk will not require knowing the definition of Vertex algebras or conformal blocks.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Genus zero conformal blocks and Elliptic Cohomology.
(arXiv:1407.6698v2)
Nitu Kitchloo.
May. 28/29, 2020

A Thought Experiment

- Assume that there is a cohomology theory $\mathcal{E}$ so that given a 2d Conformal Field Theory C, of central charge n, one gets a class $[C] \in \mathcal{E}_n$. The relation $2[C] := [C \oplus C]$ makes sense if one could think of [C] as an element in a Grothendieck group of modules over some underlying object.
- Assume furthermore that there is a (character) map:

$$\mathrm{tr}\, q^{L_0} : \mathcal{E}_n \longrightarrow \mathcal{M}_{n/2}, \quad \mathcal{M}_{n/2} = \text{modular forms of weight } n/2$$

I should really say: $\eta^n \,\mathrm{tr}\, q^{(L_0 - n/24)}$, where $\eta = q^{1/24}\prod(1-q^m)$.

The character map applied to a CFT is known as its *genus one partition function*, which is the trace of its *genus zero evolution operator* (more on this later). So we notice that the character map depends only on the genus zero information. These two observations lead us naturally to consider the category of *modules over Vertex Operator Algebras (VOAs)*.

Yongchang Zhu: Modular Invariance of Characters of Vertex Operator Algebras - Journal of the AMS. Vol. 9, No. 1, 1996.

Given a VOA that satisfies some finiteness properties, the vector space spanned by the characters of modules (which are known to be holomorphic for $|q| < 1$) is a finite dimensional representation K_0 of $SL_2(\mathbb{Z})$. We will call such a VOA *rational* (in the standard terminology, rational VOAs are bit weaker).

In particular, the Grothendieck group of representations of a rational VOA globalizes to a holomorphic bundle over the moduli of elliptic curves. Namely

$$(\mathfrak{h} \times K_0)/\!/SL_2(\mathbb{Z}) \longrightarrow \mathfrak{h}/\!/SL_2(\mathbb{Z}).$$

Our goal is to lift this observation to the K-theory (spectrum) of the category of representations of a rational VOA. In the first part of the talk, we work over $\mathbb{C}$, but at the end I'll indicate how one may extend the theory to $\mathbb{Z}$.

Modular Functors:

The $\mathbb{C}$-linear category $\mathcal{C}$ of modules over a rational vertex algebra is semi simple. The corresponding *Modular Functor* is an assignment of a functor:

$$\mathcal{C}(X) : \mathcal{C}^{\otimes I} \longrightarrow \mathcal{C}^{\otimes J}$$

for each Riemann surface X with I incoming and J outgoing parametrized circles. Glueing surfaces agrees with composing functors (projectively).

Decorating the boundaries with objects of $\mathcal{C}$, the spaces of morphisms assemble into a projectively flat bundle called *Conformal Blocks* over the moduli of surfaces.

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Consider the special case of genus zero, with two boundary components (i.e. annuli).

The upper half plane $\mathfrak{h}$ parametrizes annuli in $\mathbb{C}$ in standard position. The element $\tau \in \mathfrak{h}$ corresponds to the annulus with outer boundary being the standard unit circle, and the inner boundary parametrized by $z \mapsto q\,z$, where $q = \exp(2\pi i\tau)$.

The parametrization of annuli by $\mathfrak{h}$ therefore factors through $\mathfrak{h}/\mathbb{Z} = \{q \in \mathbb{C}^*,\ 0 < |q| < 1\}$.

The functor $\mathcal{C}(\tau) : \mathcal{C} \longrightarrow \mathcal{C}$ fixes the underlying vector space of any $V \in \mathrm{ob}(\mathcal{C})$, and is equivalent to the identity functor.

The equivalence between the identify functor and $\mathcal{C}(\tau)$ gives rise to an automorphism of V which is known as the genus zero evolution operator, $q^{L_0} : \mathfrak{h}/\mathbb{Z} \longrightarrow \mathrm{Aut}(V)$.

Concatenation of annuli corresponds to the monoid structure of $\mathfrak{h}/\mathbb{Z}$, so q^{L_0} is a homomorphism.

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□ Example: A rational VOA known as the (level k) Vacuum VOA has a category of modules that can be identified with the category $\mathcal{C}_k$ of level k, positive energy representations of LG.

- Let G denote a simple, simply connected compact Lie group, with maximal torus T, and Weyl group W. Let LG denote the group of smooth loops in G. The group LG admits a universal central S^1 extension $\tilde{LG}$.
- Let $\mathbb{T}$ denote the circle acting on LG by rotation: $e^{i\theta}.\psi(z) = \psi(e^{i\theta}z)$. Then the $\mathbb{T}$ action on LG lifts to $\tilde{LG}$, and we denote $\mathcal{G} := (\mathbb{T} \ltimes \tilde{LG})$.
- The sub group $(\mathbb{T} \times S^1 \times T) \subset (\mathbb{T} \ltimes \tilde{LG})$ is a maximal torus in $(\mathbb{T} \ltimes \tilde{LG})$, and its Weyl group $\tilde{W}$ is called the *Affine Weyl group*. $\tilde{W}$ is isomorphic to $W \ltimes \pi_1(T)$.

Given a fixed integer $k > 0$, define the category $\mathcal{C}_k$ with objects given by unitary representations of $\mathcal{G} = (\mathbb{T} \ltimes \tilde{LG})$ so that the center acts by a constant character of degree k, and the negative modes of $\mathbb{T}$ are finite dimensional.

$\mathcal{C}_k$ can be identified with the representations of a certain rational VOA known as the level k vacuum (actually its simple quotient).

Consider the *Rigidification Functor* that identifies a $\mathcal{G}$-representation V with its underlying Hilbert space

$$\rho_k : \mathcal{C}_k \longrightarrow \text{Hilb}.$$

The *Hilbert space* $\mathbb{H}_{\rho_k}$ *dominated by* ρ_k is defined as the smallest Hilbert space that contains each irreducible in $\mathcal{C}_k$ with infinite multiplicity.

Dominant K-theory

The *Dominant K-theory* $K(\rho_k)$ is defined as the K-theory spectrum whose underlying infinite-loop space is the space of Fredholm operators on $\mathbb{H}_{\rho_k}$, endowed with a conjugation $(\mathbb{T} \ltimes LG)$-action

$$\mathrm{Fred}(\mathbb{H}_{\rho_k}) = \Omega^\infty(K(\rho_k)).$$

An important property of $K(\rho_k)$ is that for any *compact subgroup* $H \subset (\mathbb{T} \ltimes LG)$, the spectrum $K(\rho_k)$ restricts to a module over the equivariant K-theory spectrum of H.

Another way to say this is that $\mathbb{H}_{\rho_k}$, as a representation of the central extension $\tilde{H}$ induced from $H \subset (\mathbb{T} \ltimes LG)$ is closed under tensoring with arbitrary H-representations.

Hence we have a Rep(H)-module structure on $\pi_0(K(\rho_k)^H)$.

Extending the moduli of annuli by the Phase space

The action of the moduli space of annuli $\mathfrak{h}/\mathbb{Z}$, on a $\mathcal{G}$-rep V belongs to the complexification of the action of the rotation circle $\mathbb{T}$. Therefore $\mathfrak{h}$ should be understood as an open domain in the complexification of the Lie algebra of $\mathbb{T}$.

It is therefore natural to extend $\mathfrak{h}$ to the $(\mathbb{C}^* \ltimes \mathrm{LG}_{\mathbb{C}})$-space $(\mathfrak{h} \times \mathcal{G}_{\mathbb{C}})$, where $\mathcal{G}_{\mathbb{C}}$ denotes the complexified Lie algebra of LG.

The space $\mathcal{G}_{\mathbb{C}}$ can be understood in another way. It can be seen as the cotangent bundle of the space of G-connections on S^1, namely classical 2d-gauge fields along the annuli, the so-called *Classical Phase Space*.

The GIT quotient $\mathscr{A} := (\mathfrak{h} \times \mathcal{G}_{\mathbb{C}})/\!/(\mathbb{C}^* \ltimes \mathrm{LG}_{\mathbb{C}})$ is an analytic space, so that the local form is given by the neighbourhood of zero in $\mathcal{H}_{\mathbb{C}}/\!/\mathrm{H}_{\mathbb{C}}$, for some compact connected subgroup H in $(\mathbb{T} \ltimes \mathrm{LG})$. Here $\mathcal{H}_{\mathbb{C}}$ denotes the adjoint action of $\mathrm{H}_{\mathbb{C}}$ on its Lie algebra.

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Genus one blocks, and the derived line bundle $\mathscr{K}(\rho_k)$

At this point it is clear how to construct a derived sheaf of spectra $\mathscr{K}(\rho_k)$ on the analytic space $\mathscr{A} = (\mathfrak{h} \times \mathscr{G}_{\mathbb{C}})/\!/(\mathbb{C}^* \ltimes LG_{\mathbb{C}})$.

The value of $\mathscr{K}(\rho_k)$ is determined by demanding that the spectrum associated to $\mathscr{H}_{\mathbb{C}}/\!/H_{\mathbb{C}}$ is given by inducing $K(\rho_k)^H$ up along the inclusion

$$\mathrm{Rep}(H) \subseteq \mathscr{O}(\mathscr{H}_{\mathbb{C}})^H := \mathscr{O}(\mathscr{H}_{\mathbb{C}}/\!/H_{\mathbb{C}}).$$

In characteristic zero $\mathscr{K}(\rho_k)$ is in fact a (derived) line bundle over $\mathscr{A}$, which we can describe as follows

Since we are working over $\mathbb{C}$, there is a global description of the space $\mathscr{A}$ as the GIT quotient $(\mathfrak{h} \times \mathscr{T}_{\mathbb{C}})/\!/\tilde{W}$, where $\tilde{W}$ the Affine Weyl group, and with $\mathscr{T}_{\mathbb{C}}$ denoting the complexified Lie algebra of the maximal torus T of G.

The action of $\tilde{W}$ on $\mathfrak{h} \times \mathscr{T}_{\mathbb{C}}$ extends to a larger discrete group $\tilde{W}_2(\mathbb{Z})$ defined as

$$\tilde{W}_2(\mathbb{Z}) := (SL_2(\mathbb{Z}) \times W) \ltimes (\pi_1(T) \oplus \pi_1(T)).$$

The GIT quotient $\mathfrak{M} := (\mathfrak{h} \times \mathscr{T}_{\mathbb{C}})/\!/\tilde{W}_2(\mathbb{Z})$ fits in a diagram

$$\begin{array}{ccc} \mathscr{A} & \longrightarrow & \mathfrak{M} \\ \downarrow & & \downarrow \\ \mathfrak{h}/\!/\mathbb{Z} & \longrightarrow & \mathfrak{h}/\!/SL_2(\mathbb{Z}) \end{array}$$

where $\mathfrak{M}$ can be identified with the moduli space of principal G-connections on the universal elliptic curve. It supports a (pre-quantum) line bundle $\mathscr{L}$ called the *Looijenga Line Bundle*, whose (fiberwise or curvewise) holomorphic sections forms the bundle of genus one blocks of $\mathcal{C}_k$ over $\mathfrak{h}/\!/SL_2(\mathbb{Z})$.

The homotopy of the sheaf $\mathscr{K}(\rho_k)$ is the sheaf of holomorphic sections of the pullback of $\mathscr{L}^{\otimes k}$ to $\mathscr{A}$.

A double cover of $\tilde{W}_2(\mathbb{Z})$ acts on this pullback line bundle $\mathscr{L}_{\mathscr{A}}^{\otimes k}$. The $W \ltimes (\pi_1(T) \oplus \pi_1(T))$-invariant sections of $\mathscr{L}_{\mathscr{A}}^{\otimes k}$ are *W-invariant Theta Functions*, which form a basis of the level k-representations of the loop group, with a residual $SL_2(\mathbb{Z})$-action.

In other words, we have recovered Zhu's theorem as a statement about the homotopy groups of $\mathscr{K}(\rho_k)$.

Note that the subspace invariant under the double cover of the full group $\tilde{W}_2(\mathbb{Z})$ contains an interesting class denoted by [WZW], representing the chiral WZW-model. More on this later.

Introducing a background manifold M

What we have described so far should perhaps be seen as a *Derived Categorical Quantization procedure* for the modular functor $\mathcal{C}_k$ namely, we construct a derived pre-quantum line bundle $\mathscr{K}(\rho_k)$ over the universal phase space $\mathscr{A}$.

This allowed us to detect a canonical class (namely the chiral WZW-model) with $\mathcal{C}_k$ as the underlying modular functor.

So if we now introduce a background G-manifold M, our analogy demands that we replace $\mathscr{A}$ by $(\mathscr{A} \times \mathrm{LM})$, where LM is the loop space of M, seen as a topological $\mathcal{G}$-space.

Alternatively, we can replace $\mathscr{K}(\rho_k)$ by the spectrum of maps: $\mathrm{Maps}(\mathrm{LM}, \mathscr{K}(\rho_k))$, which is the derived sheaf over $\mathscr{A}$ whose value on $\mathscr{H}_{\mathbb{C}}/\!/\mathrm{H}_{\mathbb{C}}$, is given by inducing up $\mathrm{Maps}(\mathrm{LM}, \mathrm{K}(\rho_k))^{\mathrm{H}}$.

The homotopy stalks of the sheaf Maps(LM, $\mathscr{K}(\rho_k)$) localize about a subspace of LM called "ghost loops" in LM. Equivalently, they can be identified with *genus one Symplectic Vortex Equations (i.e gauged J-holomorphic curves) in TM.*

These equations are local in M and so Maps(LM, $\mathscr{K}(\rho_k)$) is a cohomological functor of M. This functor is equivalent to G-Equivariant Elliptic cohomology as studied by Grojnowski, Ando, Rosu, Rezk, Spong, Berwick Evans-Tripathy and others.

Note: A torus T is certainly not simply connected, so our construction does not give a T-equivariant cohomology theory.

If $T \subset G$ is a maximal torus for some G. Then for a T-space X, we could take as proxy the G-equivariant elliptic cohomology of the induced G-space $M = G \times_T X$.

Setting $M = *$, the answer one gets is indeed all theta functions of level k and for the induced (Cartan-Killing) lattice on $\pi_1(T)$.

Work in progress with D. Gepner

In this talk, I assumed that all analytic spaces were defined over $\mathbb{C}$. It is obviously desirable to ask if one may construct our objects over $\mathbb{Z}$?

On thinking a bit more about this question, one quickly realizes that the right first step is to replace the *linear phase space*, $(\mathfrak{h} \times \mathcal{G}_{\mathbb{C}})$, by the underlying *non-linear phase space*: $(\mathfrak{h}/\mathbb{Z} \ltimes \mathrm{LG}_{\mathbb{C}})$.

The space $(\mathfrak{h}/\mathbb{Z} \ltimes \mathrm{LG}_{\mathbb{C}})$ has an algebraic variant given by the Kac-Moody group: $\mathbb{C}^* \ltimes \mathrm{G}_{\mathbb{C}}(\mathcal{O}(\mathbb{C}^*))$, which admits a $\mathbb{Z}$-form.

The GIT quotient $\mathcal{M}$ of the Kac-Moody group may locally be written as $\mathcal{M}(\mathrm{H}_{\mathbb{C}}) := \mathrm{H}_{\mathbb{C}}/\!/\mathrm{H}_{\mathbb{C}}$, for suitable *split algebraic groups* $H_{\mathbb{C}}$ *defined over* $\mathbb{Z}$. So we obtain a $\mathbb{Z}$-form $\mathcal{M}_{\mathbb{Z}}$.

Since $\mathcal{O}(\mathcal{M}_{\mathbb{Z}}(\mathrm{H}_{\mathbb{C}})) = \mathrm{Rep}(\mathrm{H})$, the derived (pre-quantum) line bundle $\mathcal{K}(\rho_k)$ extends to a $\mathbb{Z}$-form $\mathcal{K}_{\mathbb{Z}}(\rho_k)$ over $\mathcal{M}_{\mathbb{Z}}$.

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But the wise say that the right bundle over $\mathscr{M}_{\mathbb{Z}}$ we should work with is the *Metaplectically* corrected derived bundle $\mathscr{K}_{\mathbb{Z}}(\rho_{k+\check{h}})$.

Computing the homotopy groups of $\mathscr{K}_{\mathbb{Z}}(\rho_{k+\check{h}})$, we see that it lives entirely in (sheaf) cohomological degree n, where n is the rank of G.

This homotopy agrees integrally with the Verlinde algebra of level k or the integral Grothendieck group of $\mathcal{C}_k$. Invoking Serre duality, this recovers the result of Freed-Hopkins-Teleman.

☐ The complex points of $\mathscr{M}_{\mathbb{Z}}$ over any point $\tau \in \mathfrak{h}/\mathbb{Z}$, is the moduli space of flat G-connections on the elliptic curves $E(\tau)$, but it incorporates the W-action by definition, and so is *not* a Jacobian variety.

☐ The automorphism group of the Affine Dynkin diagram for G is expected to act on the entire construction of $\mathscr{K}_{\mathbb{Z}}(\rho_k)$ (maybe after extending to fractional powers of q). So we can define twisted versions.

Example: $\mathbb{C}^* \ltimes \mathrm{SL}_2(\mathscr{O})$

The $\mathbb{Z}$-form for the GIT quotient $\mathscr{M}_{\mathbb{Z}}$ of $\mathbb{C}^* \ltimes \mathrm{SL}_2(\mathscr{O})$ is described as a pushout of affine schemes over $\mathbb{Z}$

$$\begin{array}{ccc} \mathscr{M}_{\mathbb{Z}}(\mathrm{GL}_1 \times \mathrm{GL}_1) & \longrightarrow & \mathscr{M}_{\mathbb{Z}}(\mathrm{GL}_1 \times \mathrm{SL}_2) \\ \downarrow & & \downarrow \\ \mathscr{M}_{\mathbb{Z}}(\mathrm{GL}_2) & \longrightarrow & \mathscr{M}_{\mathbb{Z}} \end{array}$$

On the level of functions, the above diagram is given by

$$\begin{array}{ccc} \mathbb{Z}[q^{\pm}, x^{\pm}] & \longleftarrow & \mathbb{Z}[q^{\pm}, (x + x^{-1})] \\ \uparrow & & \uparrow \\ \mathbb{Z}[q^{\pm}, (x + qx^{-1})] & \longleftarrow & \mathscr{O}(\mathscr{M}_{\mathbb{Z}}) \end{array}$$

We see from this that $\mathscr{M}_{\mathbb{Z}}$ is a $\mathbb{P}^1$-bundle over the punctured affine line $\mathrm{Spec}(\mathbb{Z}[q^{\pm}])$ that extends to the affine line $\mathrm{Spec}(\mathbb{Z}[q])$.

An interesting application?

Let us consider the map that extends coefficients from $\mathbb{Z}$ to $\mathbb{C}$

$$\mathrm{Maps}(\mathrm{LM}, \mathscr{K}_{\mathbb{Z}}(\rho_{\mathsf{k}+\check{h}})) \longrightarrow \mathrm{Maps}(\mathrm{LM}, \mathscr{K}_{\mathbb{C}}(\rho_{\mathsf{k}+\check{h}})).$$

We think of the image of the left hand side as a "lattice" inside the right hand side.

By work of the previously mentioned authors, the right hand side admits an $\mathrm{SL}_2(\mathbb{Z})$-action (this is not obvious from our construction). The lattice is likely *not* invariant for arbitrary M. So the interesting question that arises is

What are the $\mathrm{SL}_2(\mathbb{Z})$-invariant elements in the lattice? Does this say anything about the large-volume limit of 2d-sigma models with target M? More precisely, does it offer viable obstructions?

I'm afraid I don't even know the answer for $\mathrm{M} = *$, but surely someone here does....THANK YOU!