Title: Quasi-elliptic cohomology theory and the twisted, twisted Real theories

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Abstract: Quasi-elliptic cohomology is closely related to Tate K-theory. It is constructed as an object both reflecting the geometric nature of elliptic curves and more practicable to study than most elliptic cohomology theories. It can be interpreted by orbifold loop spaces and expressed in terms of equivariant K-theories. We formulate the complete power operation of this theory. Applying that we prove the finite subgroups of Tate curve can be classified by the Tate K-theory of symmetric groups modulo a certain transfer ideal.

In addition, we define twisted quasi-elliptic cohomology. They can be related to a twisted equivariant version Devotoâ€™s elliptic cohomology via a Chern character map. Moreover, we construct twisted Real quasi-elliptic cohomology and the Chern character map in this case. This is joint work with Matthew Spong and Matthew Young.

Quasi-elliptic cohomology

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Overview

Plan.

- Quasi-elliptic cohomology
  - Motivation, Definition, Loop space construction
  - Strickland's theorem
- twisted Quasi-elliptic cohomology
  - construction
  - Relation with twisted equivariant elliptic cohomology: a Chern character map
- twisted Real Quasi-elliptic cohomology
  - construction
  - a twisted Real Chern character map

Quasi-elliptic cohomology

Explicit Definition

$$QEll_G^*(X) := \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g)$$

- G^{tors}_{conj}: a set of representatives of $G-$conjugacy classes in G^{tors};
- $\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle$;
- $x \cdot [a, t] = x \cdot a$, for all $[a, t] \in \Lambda_G(g)$, $x \in X^g$.

$QEll_G^0(X)$ is an $\mathbb{Z}[q^\pm]-$algebra

$$1 \longrightarrow C_G(g) \longrightarrow \Lambda_G(g) \xrightarrow{\pi} \mathbb{T} \longrightarrow 0$$

$$\mathbb{Z}[q^\pm] = K^0_{\mathbb{T}}(\text{pt}) \xrightarrow{\pi^*} K^0_{\Lambda_G(g)}(\text{pt}) \longrightarrow K^0_{\Lambda_G(g)}(X^g)$$

Relation with Tate K-theory

$$QEll_G^*(X) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) \cong K^*_{Tate}(X /\!\!/ G).$$

Basic Properties of Quasi-elliptic cohomology

Representation theory

Restriction map: $RG \longrightarrow RH$;

Induction map: $RH \longrightarrow RG$.

Equivariant K-theory

Restriction map: $K_G^*(X) \longrightarrow K_H^*(X)$;

Induction map: $K_H^*(X) \longrightarrow K_G^*(X)$;

Quasi-elliptic cohomology

Restriction map: $QEll_G^*(X) \longrightarrow QEll_H^*(X)$;

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Restriction map: $RG \longrightarrow RH$;

Induction map: $RH \longrightarrow RG$.

$RG \otimes RH \longrightarrow R(G \times H)$.

Equivariant K-theory

Restriction map: $K^*_G(X) \longrightarrow K^*_H(X)$;

Induction map: $K^*_H(X) \longrightarrow K^*_G(X)$;

Künneth map: $K^*_G(X) \otimes K^*_H(Y) \longrightarrow K^*_{G\times H}(X \times Y)$;

Quasi-elliptic cohomology

Restriction map: $QEll^*_G(X) \longrightarrow QEll^*_H(X)$;

Induction map: $QEll^*_H(X) \longrightarrow QEll^*_G(X)$;

Künneth map: $QEll^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]}QEll^*_H(Y) \longrightarrow QEll^*_{G\times H}(X \times Y)$.

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Künneth map: $K_G^*(X) \otimes K_H^*(Y) \longrightarrow K_{G\times H}^*(X \times Y)$;

Change-of-group isomorphism: $K_G^*(Y \times_H G) \xrightarrow{\cong} K_H^*(Y)$;

Quasi-elliptic cohomology

Restriction map: $QEll_G^*(X) \longrightarrow QEll_H^*(X)$;

Induction map: $QEll_H^*(X) \longrightarrow QEll_G^*(X)$;

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$K^*_G(-)$ can be represented by an orthogonal $G-$spectrum;

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$K^*_G(-)$ can be represented by an orthogonal $G-$spectrum;

Global K-theory exists.

Quasi-elliptic cohomology

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Transfer ideal

Motivating Example: K-theory

$$I_{tr} := \sum_{\substack{i+j=N,\\ N>j>0}} \mathrm{Image}[I^{\Sigma_N}_{\Sigma_i\times\Sigma_j} : K_{\Sigma_i\times\Sigma_j}(\mathrm{pt}) \longrightarrow K_{\Sigma_N}(\mathrm{pt})].$$

I_{tr} is the smallest ideal such that the quotient

$$\mathbb{P}_N/I_{tr} : K(\mathrm{pt}) \xrightarrow{\mathbb{P}_N} K_{\Sigma_N}(\mathrm{pt}) \to K_{\Sigma_N}(\mathrm{pt})/I_{tr}$$

is a map of commutative rings.

Transfer Ideal

$$I^{Tate}_{tr} := \sum_{\substack{i+j=N,\\ N>j>0}} \mathrm{Image}[I^{\Sigma_N}_{\Sigma_i\times\Sigma_j} : K_{Tate}(\mathrm{pt}/\!\!/\Sigma_i \times \Sigma_j) \longrightarrow K_{Tate}(\mathrm{pt}/\!\!/\Sigma_N)]$$

$$\mathcal{I}^{QEll}_{tr} := \sum_{\substack{i+j=N,\\ N>j>0}} \mathrm{Image}[\mathcal{I}^{\Sigma_N}_{\Sigma_i\times\Sigma_j} : QEll(\mathrm{pt}/\!\!/\Sigma_i \times \Sigma_j) \longrightarrow QEll(\mathrm{pt}/\!\!/\Sigma_N)]$$

Relation between elliptic cohomology and Loop spaces

An old idea by Witten [Landweber]

$$LX = \mathbb{C}^{\infty}(\mathbb{T}, X),$$

$$Ell^*(X) \overset{?}{\leftrightsquigarrow} K^*_{\mathbb{T}}(LX)$$

Relation between elliptic cohomology and Loop spaces

An old idea by Witten [Landweber]

$$LX = \mathbb{C}^\infty(\mathbb{T}, X),\ \mathbb{T} \circlearrowright \mathbb{T} \quad X \circlearrowright G$$

$$Ell^*(X) \overset{?}{\leftrightsquigarrow} K^*_{\mathbb{T}}(LX)$$

It's **SURPRISINGLY** difficult to make this idea precise.

Relevant Work [Devoto][Ganter]

2007, $G-$equivariant Tate K-theory for finite groups G is modelled on the loop space of a global quotient orbifold.

Question

How can we construct elliptic cohomology theories from loop spaces?

What is "Loop"?

Review: Free Loop Space

$$LX = C^\infty(\mathbb{T}, X).$$

$\mathbb{T}-$action: $\gamma \cdot t = (s \mapsto \gamma(s+t))$.

$LG-$action: $\gamma \cdot \delta = (s \mapsto \gamma(s) \cdot \delta(s))$.

$LG \rtimes \mathbb{T}-$action: $\gamma \cdot (\delta, t) = (s \mapsto \gamma(s+t) \cdot \delta(s+t))$.

$$(\delta_1, t_1) \cdot (\delta_2, t_2) = (s \mapsto \delta_1(s)\delta_2(s+t_1), t_1 + t_2).$$

Interpretation of the $LG \rtimes \mathbb{T}-$action

LG: the group of gauge transformations.

$LG \rtimes \mathbb{T}$: the extended gauge group

$$\begin{tikzcd} G \times \mathbb{T} \arrow[r, "{(g,s)\mapsto(\delta(s)g,s+t)}"] \arrow[d] & G \times \mathbb{T} \arrow[d] \\ \mathbb{T} \arrow[r, "s\mapsto s+t"] & \mathbb{T} \end{tikzcd}$$

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Interpretation of the $LG \rtimes \mathbb{T}-$action

LG: the group of gauge transformations.

$LG \rtimes \mathbb{T}$: act on loops

$$\begin{tikzcd} G \times \mathbb{T} \arrow[r] \arrow[d] & G \times \mathbb{T} \arrow[r, "\widetilde{\gamma}"] \arrow[d] & X \\ \mathbb{T} \arrow[r] & \mathbb{T} & \end{tikzcd}$$

The Answer: What is "Loop"?

New Definition of Equivariant loops $Loop(X/\!\!/G)$ [Rezk]

Objects:

$$\mathbb{T} \xleftarrow{\pi} P \xrightarrow{f} X$$

- π : principal $G-$bundle over $\mathbb{T}$
- f: $G-$equivariant;

Morphism (α, t) : $\{\ \mathbb{T} \xleftarrow{\pi} P' \xrightarrow{f'} X\ \} \longrightarrow \{\ \mathbb{T} \xleftarrow{\pi} P \xrightarrow{f} X\ \}$:

$$\begin{tikzcd}
P' \arrow[r, "\alpha"] \arrow[d] \arrow[rr, bend left, "f'"] & P \arrow[r, "f"] \arrow[d] & X \\
\mathbb{T} \arrow[r, "t"] & \mathbb{T} &
\end{tikzcd}$$

Relation with Bibundles

$Bibun(\mathbb{T}/\!\!/*, X/\!\!/G)$

same objects;

morphisms: (α, Id). No rotations.

The Answer: What is "Loop"?

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Relation with Bibundles

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same objects;

morphisms: (α, Id). No rotations.

Loop construction of Quasi-elliptic cohomology

$\Lambda(X /\!/ G)$: a subgroupoid of $Loop(X /\!/ G)$ consisting of constant loops.

$$\Lambda(X /\!/ G) \cong \coprod_{g \in G^{tors}_{conj}} X^g /\!/ \Lambda_G(g)$$

$$QEll^*_G(X) = K^*_{orb}(\Lambda(X /\!/ G))$$

Further Question

Can we use this new definition of loop spaces to construct elliptic cohomology theories?

Classification problems on the formal group

[Stricklands, Hopkins-Kuhn-Ravenel, 1990s]

	Complex K-theory	Morava E-theory E_n
Formal group	$\mathbb{G}_m$	$\mathbb{G}_u$
$\mathrm{Hom}(A^*, \mathbb{G})$	RA	$E_n^0(BA)$
Subgroups	$R\Sigma_{p^k}/I_{tr}$	$E_n^0(B\Sigma_{p^k})/I_{tr}$
A^*-Level structures	RA/I_A	$E_n^0(BA)/I_A$

Classification of the subgroups of Tate curve

Theorem (Huan)

The Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve.

$$K_{Tate}(pt/\!\!/\Sigma_N)/I_{tr}^{Tate} \cong \prod_{N=de} \mathbb{Z}((q))[q_s'^{\pm}]/\langle q^d - q_s'^{e}\rangle,$$

where I_{tr}^{Tate} is the transfer ideal and q'_s is the image of q under the stringy power operation, the product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

Relation between representations $K_{C_G(g)}(\mathrm{pt}) \otimes_{\mathbb{Z}} K_{\mathbb{T}}(\mathrm{pt}) \cong K_{\Lambda_G(g)}(\mathrm{pt})$

$$\begin{array}{ccc} \text{a } R\mathbb{T}\text{-basis of } R\Lambda_G(g) & & \rho \\ \Big\downarrow {\scriptstyle 1-1} & & \big\downarrow \\ \{\text{irreducible representations of } C_G(g)\} & & \rho|_{C_G(g)} \end{array}$$

Classification problems on generalized $E-$theories

[Schlank, Stapleton, 2015], [Ganter, Huan, 2018], [Huan, Stapleton, 2020]

	Quasi-elliptic cohomology $K^*_{orb}(\Lambda(-))$	$E^*_n(\mathcal{L}^h(X/\!\!/G))$
Formal group	$\mathbb{G}_m \oplus \mathbb{Q}/\mathbb{Z}$	$\mathbb{G}_u \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^h$
$\mathrm{Hom}(A^*, \mathbb{G})$	$K^*_{orb}(\Lambda(\mathrm{pt}/\!\!/A))$	$E^0_n(\mathcal{L}^h BA)$
Subgroup	$K_{orb}(\Lambda(\mathrm{pt}/\!\!/\Sigma_{p^k}))/I_{tr}$	$E^0_n(\mathcal{L}^h B\Sigma_{p^k})/I_{tr}$
A^*-Level structures	$K^*_{orb}(\Lambda(\mathrm{pt}/\!\!/A))/I_A$	$E^0_n(\mathcal{L}^h BA)/I_A$

Twisted Quasi-elliptic cohomology [Huan, Spong]

G finite. Twist $QEll_G^*(-)$ by $\alpha \in H^3(BG; U(1))$.

Transgression $\tau : H^3(BG; U(1)) \longrightarrow H^2(\mathrm{Map}(S^1, BG); U(1))$

$$BG \xleftarrow{ev} S^1 \times \mathrm{Map}(S^1, BG) \xrightarrow{proj} \mathrm{Map}(S^1, BG)$$

$$\begin{tikzcd} H^3(BG; U(1)) \arrow[r, "evaluation^*"] & H^3(S^1 \times \mathrm{Map}(S^1, BG); U(1)) \arrow[d, "projection_*"] \\ & H^2(\mathrm{Map}(S^1, BG); U(1)) \end{tikzcd}$$

$$H^2(\mathrm{Map}(S^1, BG); U(1)) \cong \prod_{[g]} H^2(BC_G(g); U(1))$$

$$\tau(\alpha) = \prod_{[g]} \theta_g .$$

Twisted Quasi-elliptic cohomology

Twisted equivariant K-theory

$$X \circlearrowleft G, \quad \theta \in H^2(BG; U(1)), \quad 1 \longrightarrow U(1) \longrightarrow G^\theta \longrightarrow G \longrightarrow 1$$

$K_G^{*+\theta}(X)$:=Grothendieck group of $\theta-$ twisted $G-$vector bundles over X.
$U(1) \subset G^\theta$ acts on each fiber as complex multiplication.

$$K_G^{*+\theta}(X) \subseteq K^*_{G^\theta}(X).$$

$$1 \longrightarrow U(1) \longrightarrow C_G^{\theta_g}(g) \longrightarrow C_G(g) \longrightarrow 1$$

$$\Lambda_G^{\theta_g}(g) := \mathbb{R} \times C_G^{\theta_g}(g)/\langle(-1,(0,g))\rangle,$$

$$QEll_G^{*+\alpha}(X) := \prod_{[g]} K^{*+\theta_g}_{\Lambda_G(g)}(X^g).$$

Comparison theorem v.s. Chern Character map

$$RG \otimes_{\mathbb{Z}} \mathbb{C} \cong \text{the ring of class functions on } G$$

[1989 Atiyah Segal]

$$K_G^*(X) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_{[g]} (K^*(X^g) \otimes \mathbb{C})^{C_G(g)} \xrightarrow{Chern} \bigoplus_{[g]} (H^*(X^g) \otimes \mathbb{C})^{C_G(g)}$$

[2001 Adem Ruan]

$$K_G^{*+\theta}(X) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_{[g]} (H^*(X^g) \otimes \chi_g^\theta)^{C_G(g)}$$

$$\chi_g^\theta : C_G(g) \longrightarrow U(1), \quad h \mapsto \theta(h, g)\theta(h, g^{-1})$$

$$\bigoplus_{[g]} K^{*+\theta_g}_{C_G(g)}(X^g) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_{[g],[h]} (H^*(X^{g,h}) \otimes \chi_h^{\theta_g})^{C_G(g,h)}$$

Chern Character of twisted *QEll* [Huan, Spong]

$$
\begin{aligned}
QEll_G^{*+\alpha}(X)\otimes\mathbb{C} &= \prod_{[g]} K^{*+\theta_g}_{\Lambda_G(g)}(X^g)\otimes\mathbb{C} \\
&\xrightarrow{c^*} \prod_{[g]} K^{*+\theta_g}_{\mathbb{T}\times C_G(g)}(X^g)\otimes\mathbb{C} \\
&\xrightarrow{\cong} \prod_{[g]} K^{*+\theta_g}_{C_G(g)}(X^g)\otimes\mathbb{C}\otimes\mathbb{Z}[q^{\pm}] \\
&\xrightarrow{Chern} \prod_{[g,h]} (H^*(X^{g,h})\otimes\chi_h^{\theta_g})^{C_G(g,h)}\otimes\mathbb{Z}[q^{\pm}] \\
&\subseteq Ell_G^{*+\alpha}(X)
\end{aligned}
$$

$Ell_G^{*+\alpha}(-)$: a twisted version of Devoto's equivariant elliptic cohomology with complex coefficients [Evans-Berwick].

Real quasi-elliptic cohomology [Huan, Young]

$\pi : \hat{\mathrm{G}} \to \mathbb{Z}/2\mathbb{Z}$. $\mathrm{G} = Ker\pi$. finite.

The Real centralizer of $g \in \hat{\mathrm{G}}$

$$C^R_{\hat{\mathrm{G}}}(g) = \{\omega \in \hat{\mathrm{G}} \mid \omega g^{\pi(\omega)} \omega^{-1} = g\} \leqslant \hat{\mathrm{G}}.$$

Involution: Reflection of the circle

Sign representation $\epsilon : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$:

$$\mathbb{R} \circlearrowleft \mathbb{Z} \quad t \cdot n = t\epsilon(n).$$

Fix $\omega \in C^R_{\hat{\mathrm{G}}}(g) \setminus C_{\mathrm{G}}(g)$.
$\mathbb{R} \rtimes_\epsilon \mathbb{Z}$ acts on $C_{\mathrm{G}}(g)$ by $h \cdot (t, n) = \omega^{-n} h \omega^n$.

The enhanced Real stabilizer of $g \in \mathrm{G}$

$$\Lambda^R_{\hat{\mathrm{G}}}(g) = (C_{\mathrm{G}}(g) \rtimes_\omega (\mathbb{R} \rtimes_\epsilon \mathbb{Z})) / \langle (g, -1, 0), (\omega^2, 0, 2) \rangle.$$

Real quasi-elliptic cohomology

$\pi : \hat{\mathfrak{X}} \longrightarrow */\!\!/(\mathbb{Z}/2\mathbb{Z})$: a $\mathbb{Z}/2\mathbb{Z}-$graded orbifold.
$\mathfrak{X} \longrightarrow \hat{\mathfrak{X}}$: the double cover classified by π.

$\Lambda\hat{\mathfrak{X}}$: quotient loop groupoid of $\mathfrak{X}$

Objects: $(x, \gamma) \in \hat{\mathfrak{X}}_0 \times Aut_{\hat{\mathfrak{X}}}(x)$;

$$\begin{aligned} Mor((x_1, \gamma_1), (x_2, \gamma_2)) = &\{(g, t) \in Mor_{\hat{\mathfrak{X}}}(x_1, x_2) \times \mathbb{R} | \\ &\gamma_2 = g\gamma_1 g^{-1}; (\gamma_2 g, t+1) = (g, t))\}. \end{aligned}$$

Real quasi-elliptic cohomology

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$\mathfrak{X} \longrightarrow \hat{\mathfrak{X}}$: the double cover classified by π.

$\hat{\Lambda}\hat{\mathfrak{X}}$: unoriented quotient loop groupoid of $\mathfrak{X}$

Objects: $(x, \gamma) \in \hat{\mathfrak{X}}_0 \times Aut_{\hat{\mathfrak{X}}}(x)$;

$$Mor((x_1, \gamma_1), (x_2, \gamma_2)) = \{(g, t) \in Mor_{\hat{\mathfrak{X}}}(x_1, x_2) \times \mathbb{R} | \\ \gamma_2 = g\gamma_1^{\pi(g)}g^{-1}; (\gamma_2 g, t+1) = (g, t))\}.$$

Involution on $\hat{\Lambda}\hat{\mathfrak{X}}$

objects: $(x, \gamma) \mapsto (\omega^{-1}x\omega, \omega^{-1}\gamma^{-1}\omega)$

morphisms: $(g, t) \mapsto (\omega^{-1}g\omega, -t)$

Real quasi-elliptic cohomology [Huan, Young]

Real quasi-elliptic cohomology of $\hat{\mathfrak{X}}$

$$QEllR^*(\hat{\mathfrak{X}}) = K^*_{\mathrm{orb}}(\hat{\Lambda}\hat{\mathfrak{X}}).$$

The global quotient case

$$QEllR^*_{\mathrm{G}}(X) \cong \prod_{g \in \pi_0(\mathrm{G}^{\mathrm{tor}} /\!\!/_R \hat{\mathrm{G}})} K^*_{\Lambda^R_{\hat{\mathrm{G}}}(g)}(X^g).$$

twisted Real quasi-elliptic cohomology [Huan, Young]

Start from $\hat{\alpha} \in H^{3+\pi}(B\hat{\mathrm{G}}; U(1))$, a lift of $\alpha \in H^3(B\mathrm{G}; U(1))$.

Twisted loop transgression map [Noohi, Young]

$$\hat{\tau} : H^{3+\pi}(B\hat{\mathrm{G}}; U(1)) \longrightarrow H^2(\mathrm{Map}(S^1, B\hat{\mathrm{G}})/(\mathbb{Z}/2\mathbb{Z}); U(1))$$

$$\hat{\tau}(\hat{\alpha}) = \prod_{[g]} \hat{\theta}_g \in H^2(BC^R_{\hat{\mathrm{G}}}(g); U(1)).$$

twisted Real quasi-elliptic cohomology

$\pi : \hat{\mathfrak{X}} \longrightarrow * /\!\!/ (\mathbb{Z}/2\mathbb{Z})$: a $\mathbb{Z}/2\mathbb{Z}-$graded orbifold.
$\mathfrak{X} \longrightarrow \hat{\mathfrak{X}}$: the double cover classified by π.

The $\hat{\alpha}$-twisted Real quasi-elliptic cohomology of $\hat{\mathfrak{X}}$

$$QEll R^{*+\hat{\alpha}}(\hat{\mathfrak{X}}) = K_{\mathrm{orb}}^{*+\hat{\tau}(\hat{\alpha})}(\hat{\Lambda}\hat{\mathfrak{X}}).$$

The global quotient case

$$QEll R_{\mathrm{G}}^{*+\hat{\alpha}}(X) \cong \prod_{g \in \pi_0(\mathrm{G}^{\mathrm{tor}} /\!\!/_R \hat{\mathrm{G}})} K_{\Lambda_{\hat{\mathrm{G}}}^R(g)}^{*+\hat{\theta}_g}(X^g).$$

${}^{\hat{\theta}_g}\Lambda_{\hat{\mathrm{G}}}^R(g)$: The twisted enhanced Real stabilizer of $g \in \mathrm{G}$

$$\left({}^{\theta_g} C_{\mathrm{G}}(g) \rtimes_\omega (\mathbb{R} \rtimes_\epsilon \mathbb{Z})\right) / \langle ((g,1), -1, 0), ((\omega^2, \hat{\theta}_g([\omega^{-1}|\omega]), 0, 2) \rangle.$$

The representation theories

$$
\begin{tikzcd}
K^{*}_{C_G(g)}(\mathrm{pt}) \otimes_{\mathbb{Z}} K_{\mathbb{T}}(\mathrm{pt}) \arrow[r, "\cong"] & K^{*}_{\Lambda_G(g)}(\mathrm{pt}) \\
K^{*+\theta_g}_{C_G(g)}(\mathrm{pt}) \otimes_{\mathbb{Z}} K_{\mathbb{T}}(\mathrm{pt}) \arrow[r, "\cong"] \arrow[u] & K^{*+\theta_g}_{\Lambda_G(g)}(\mathrm{pt}) \arrow[u] \\
K^{*+\hat{\theta}_g}_{C^R_{\hat{G}}(g)}(\mathrm{pt}) \otimes_{\mathbb{Z}} K_{\mathbb{T}}(\mathrm{pt}) \arrow[r, "\cong"] \arrow[u] & K^{*+\hat{\theta}_g}_{\Lambda^R_{\hat{G}}(g)}(\mathrm{pt}) \arrow[u]
\end{tikzcd}
$$

All the vertical maps are restrictions.

twisted Real Chern character map [Huan, Young]

$$\begin{tikzcd}
QEllR_{\hat{G}}^{*+\hat{\alpha}}(X) \otimes \mathbb{C} \arrow[r, "Chern"] \arrow[d] & \prod\limits_{[g,\omega]\in\pi_0(\hat{G}^{\langle 2\rangle}/\!/\hat{G})} (H^*(X^{g,\omega}) \otimes \chi_{\omega}^{\hat{\theta}_g})^{C^R_{\hat{G}}(g,\omega)} \otimes \mathbb{Z}[q^{\pm}] \arrow[d] \\
QEll_{G}^{*+\alpha}(X) \otimes \mathbb{C} \arrow[r, "Chern"] & \prod\limits_{[\sigma,\tau]\in\pi_0(G^{\langle 2\rangle}/\!/G)} (H^*(X^{\sigma,\tau}) \otimes \chi_{\tau}^{\theta_\sigma})^{C_G(\sigma,\tau)} \otimes \mathbb{Z}[q^{\pm}]
\end{tikzcd}$$

Further problems: quasi-elliptic cohomology

Twisted Power Operation

Definition of twisted equivariant power operation:
Power operation for twisted quasi-elliptic cohomology;
Power operation for twisted Morava E-theory.

2-equivariant elliptic cohomology

Define quasi-elliptic cohomology of 2-group;
compare it with 2-equivariant elliptic cohomology.

Thank you.