Title: Elliptic characteristic classes, bow varieties, 3d mirror duality

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Collection: Elliptic Cohomology and Physics

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Abstract: We study elliptic characteristic classes of natural subvarieties in some ambient spaces, namely in homogeneous spaces and in Nakajima quiver varieties. The elliptic versions of such characteristic classes display an unexpected symmetry: after switching the equivariant and the Kahler parameters, the classes of varieties in one ambient space ``coincideâ€• with the classes of varieties in another ambient space. This duality gets explained as â€œ3d mirror dualityâ€• if we regard our ambient spaces as special cases of Cherkis bow varieties. I will report on a work in progress with Y. Shou, based on earlier related works with G. Felder, A. Smirnov, V. Tarasov, A. Varchenko, A. Weber, Z. Zhou.

Richárd Rimányi (UNC)

Elliptic characteristic classes, bow varieties, 3d mirror symmetry

Elliptic cohomology and physics
Perimeter Institute for Theoretical Physics
May 2020

Based on

- work in progress with Y. Shou
- R-Weber: Elliptic classes of Schubert varieties via Bott-Samelson resolution
- R-Smirnov-Varchenko-Zhou: 3d mirror symmetry and elliptic stable envelopes
- R-Smirnov-Varchenko-Zhou: Three dimensional mirror self symmetry of the cotangent bundle of the full flag variety
- discussions with L. Rozansky

In the land of

characteristic classes of singularities

a beautiful structure is revealed, if we study

<u>elliptic</u> characteristic classes.

\begin{walkthrough}

Characteristic class of (singular) subvariety.

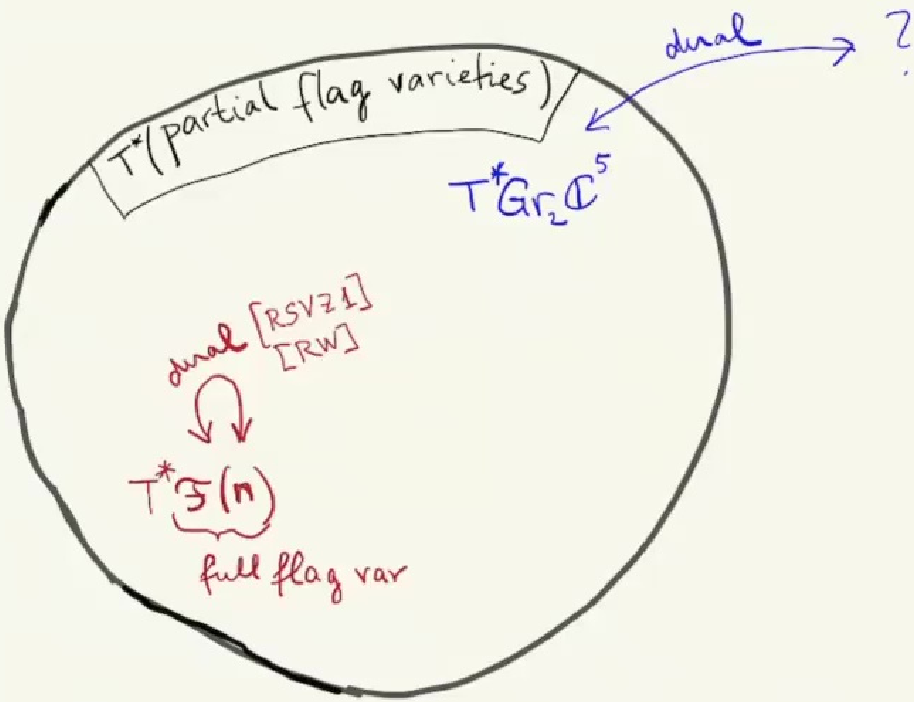
$$\Sigma \subset \underset{\text{smooth}}{M} \rightsquigarrow [\Sigma] \in H^{\operatorname{codim}(\Sigma)}(M)$$

"right settings"

- equivariant $\left(T \circlearrowright (\Sigma \subset M),\ H_T^* \ldots \right)$
- $\hbar$ - deformed $\quad \operatorname{class}(\Sigma \subset M) = [\Sigma] + \text{h.o.t.}$

  (c.f. $M \rightsquigarrow T^*M \ \ldots$)
- $H \to K \to Ell$

$$\Longrightarrow E(\Sigma \subset M)\left(\begin{smallmatrix}\text{equivariant}\\ \text{variables}\end{smallmatrix},\ \begin{smallmatrix}\text{dynamical}\\ \text{(aka Kähler) var's}\end{smallmatrix},\ \hbar\right) \in Ell_T(M)$$



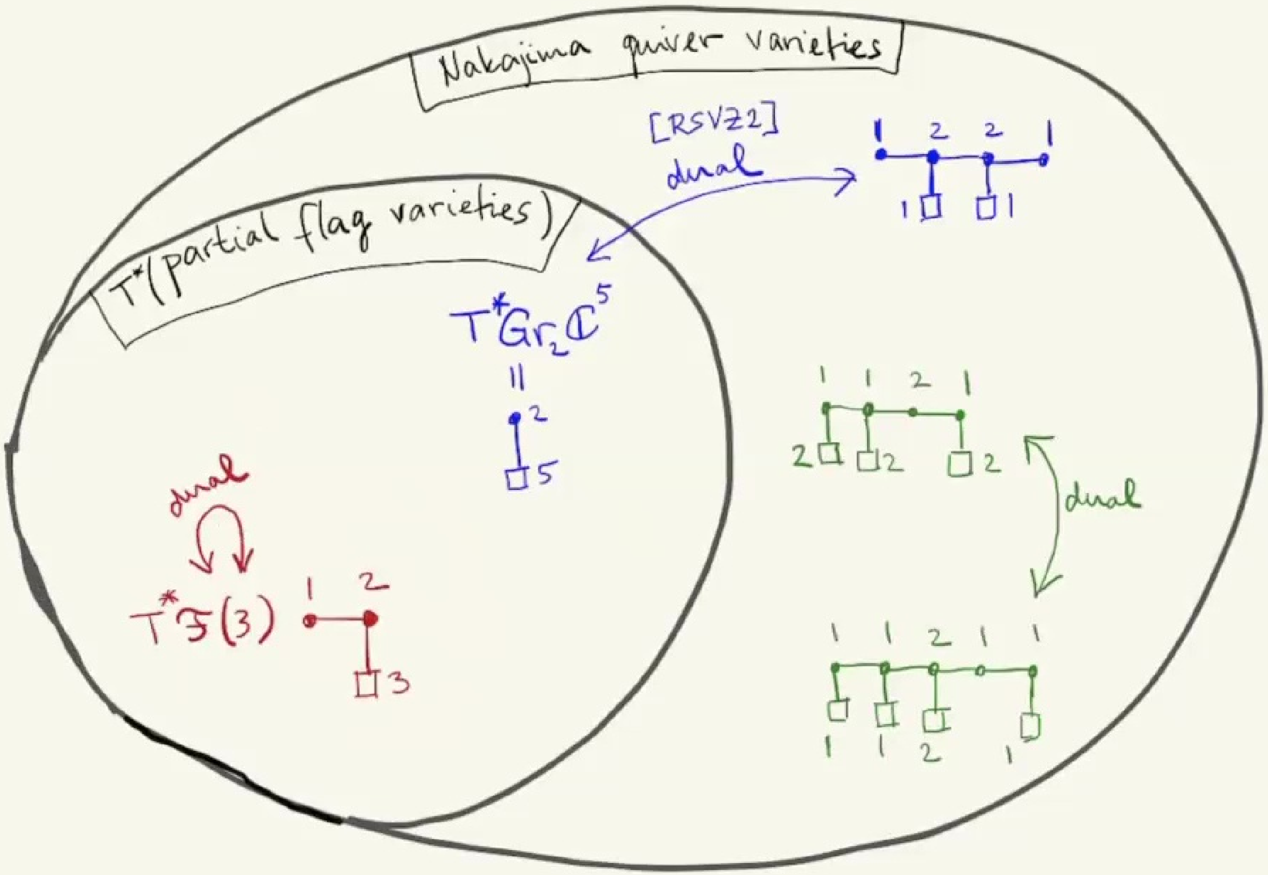
$$X \overset{\text{dual}}{\longleftrightarrow} X'$$

$$Ell(\Sigma \subset X) \text{ “}=\text{” } Ell(\Sigma' \subset X')$$

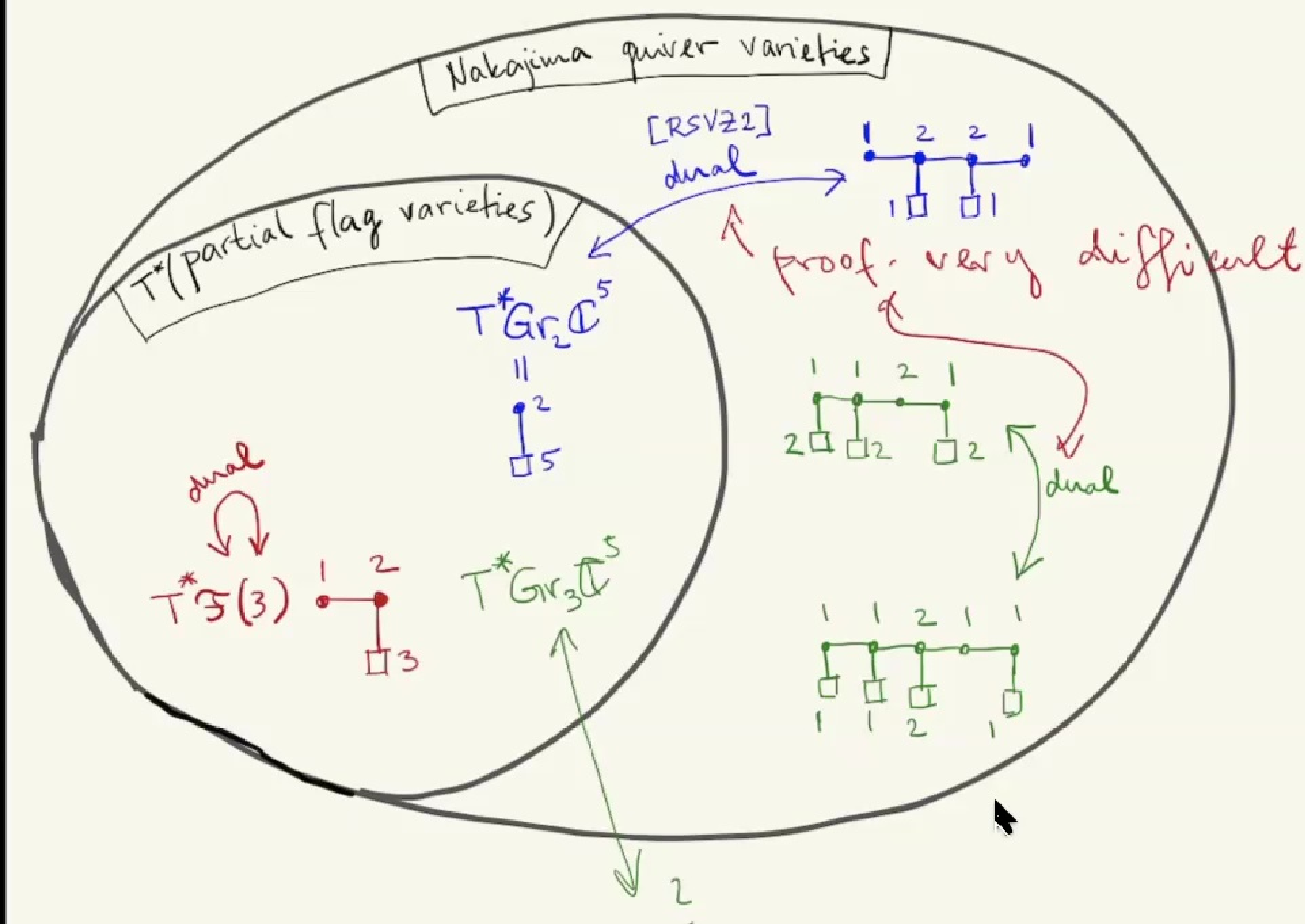
switch

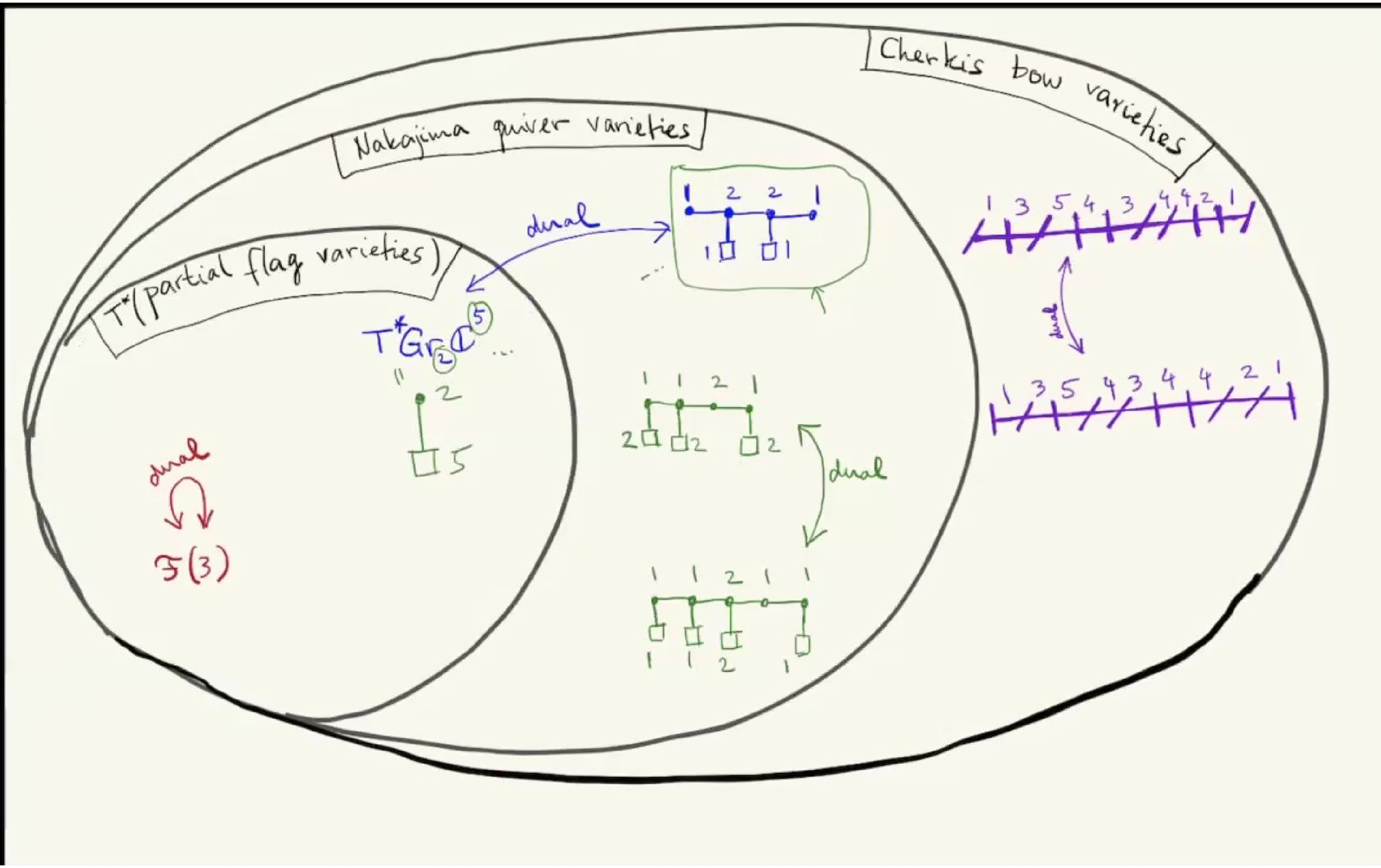
equivariant $\Leftrightarrow$ dynamical

$\hbar \Leftrightarrow \hbar^{-1}$



Nakajima quiver varieties
T*(partial flag varieties)
[RSVZ22]
dual
T*Gr₂ℂ⁵
dual
T*F(3)
dual


Nakajima quiver varieties
T*(partial flag varieties)
[RSVZ22]
dual
proof: very difficult
$T^*Gr_2\mathbb{C}^5$
$T^*Gr_3\mathbb{C}^5$
$T^*\mathcal{F}(3)$
dual
dual



Cherkis bow varieties
Nakajima quiver varieties
T*(partial flag varieties)
dual
T*Gr(2, ℂ^5)
dual
𝔉(3)
dual
dual

\end{walkthrough}

→ sketch of elliptic char. classes (4 slides)

→ bow varieties (18 slides)

Elliptic classes, stable envelopes

(sketch through an example)

<u>Example</u> $\mathbb{P}^2 = pt \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^2$ (or $T^*\mathbb{P}^2$) $\circlearrowleft T^3 \times \mathbb{C}^*_\hbar$

3 fix pts $1 = (1:0:0)$ $2 = (0:1:0)$ $3 = (0:0:1)$

$E(pt) = \mathrm{Stab}(1)$

$E(\mathbb{A}^1) = \mathrm{Stab}(2)$ $\in Ell_{T\times\mathbb{C}^*}(T^*\mathbb{P}^2)$

$E(\mathbb{A}^2) = \mathrm{Stab}(3)$

$$\begin{array}{c} Ell_{T\times\mathbb{C}^*}(T^*\mathbb{P}^2) \\ \Big\downarrow \text{restriction map} \\ Ell_{T\times\mathbb{C}^*}(1) \oplus Ell_{T\times\mathbb{C}^*}(2) \oplus Ell_{T\times\mathbb{C}^*}(3) \end{array}$$

<u>2 definitions</u>

- geometry (R-Weber): "twisted Borisov-Libgober classes"
- repr. theory (Okounkov, Maulik, Aganagic)

axioms on these $\xrightarrow{\text{restrictions}}$

• repr. theory (Okounkov, Maulik, Aganagic) $\longrightarrow \mathrm{Ell}_{T\times\mathbb{C}^*}(1) \oplus \mathrm{Ell}_{T\times\mathbb{C}^*}(2) \oplus \mathrm{Ell}_{T\times\mathbb{C}^*}(3)$

axioms on these — restrictions

axioms are motivated by this philosophy

Okounkov + Maulik, Aganagic, ... (see also R-Tarasov-Varchenko-...):

$$\boxed{H^*_{\mathbb{T}}(\mathcal{M}), K_{\mathbb{T}}(\mathcal{M}), \mathrm{Ell}_{\mathbb{T}}(\mathcal{M})} \overset{1:1}{\longleftrightarrow} \boxed{\text{Bethe algebra of quantum integrable systems}}$$

$V^{\otimes \text{tan}}$

fixed point basis (easy) — spin basis (easy)

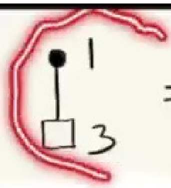
geometric basis (hard) (named: "stable envelopes") — Bethe (eigen-) basis (hard)

$T^3 \circlearrowright T^*\mathbb{P}^2$ 3 fixed pts $1 = (1:0:0)$ $2 = (0:1:0)$ $3 = (0:0:1)$

equivariant parameters u_1, u_2, u_3

dynamical parameters μ_1, μ_2, μ_3

$$
\begin{array}{cc}
 & \begin{array}{ccc} @1 & @2 & @3 \end{array} \\
\begin{array}{c} \mathrm{Stab}(1) = \\ \mathrm{Stab}(2) = \\ \mathrm{Stab}(3) = \end{array} &
\begin{bmatrix}
\theta\left(\frac{u_2}{u_1}\right)\theta\left(\frac{u_3}{u_1}\right) & 0 & 0 \\
\dfrac{\theta(h)\,\theta\left(\frac{u_2 h \mu_2}{u_1 \mu_1}\right)\theta\left(\frac{u_3}{u_1}\right)}{\theta\left(\frac{h \mu_2}{\mu_1}\right)} & \theta\left(\frac{h u_1}{u_2}\right)\theta\left(\frac{u_3}{u_2}\right) & 0 \\
\dfrac{\theta(h)\,\theta\left(\frac{h u_2}{u_1}\right)\theta\left(\frac{u_3 \mu_2}{u_1 \mu_1}\right)}{\theta\left(\frac{\mu_2}{\mu_1}\right)} & \dfrac{\theta\left(\frac{h u_1}{u_2}\right)\theta(h)\,\theta\left(\frac{u_3 \mu_2}{u_2 \mu_1}\right)}{\theta\left(\frac{\mu_2}{\mu_1}\right)} & \theta\left(\frac{h u_1}{u_3}\right)\theta\left(\frac{h u_2}{u_3}\right)
\end{bmatrix}
\end{array}
$$



$= T^*\mathbb{P}^2$

$$W_1 := \theta\left(\frac{u_1 h \mu_2}{t \mu_1}\right)\theta\left(\frac{u_2}{t h}\right)\theta\left(\frac{u_3}{t h}\right)$$

$$W_2 := \theta\left(\frac{u_1}{t}\right)\theta\left(\frac{u_2 \mu_2}{t \mu_1}\right)\theta\left(\frac{u_3}{t h}\right)$$

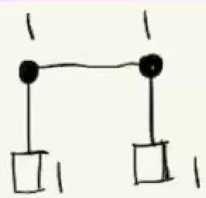
$$W_3 := \theta\left(\frac{u_1}{t}\right)\theta\left(\frac{u_2}{t}\right)\theta\left(\frac{u_3 \mu_2}{t h \mu_1}\right)$$

fixpt1 $t = \frac{u_1}{\hbar}$ fixpt 2 $t = \frac{u_2}{\hbar}$ fixpt3 $t = \frac{u_3}{\hbar}$

$\boxed{\text{substitute } u_1 = z_1 \hbar^2 \quad u_2 = z_2 \hbar \quad u_3 = z_3 \text{ (reparametrize)}}$

$$\begin{pmatrix} \theta\left(\frac{h^2\mu_2}{\mu_1}\right)\theta\left(\frac{z_2}{h z_1}\right)\theta\left(\frac{z_3}{z_1 h^2}\right) & 0 & 0 \\ \theta(h)\,\theta\left(\frac{z_2\mu_2}{z_1\mu_1}\right)\theta\left(\frac{z_3}{z_1 h^2}\right) & \theta\left(\frac{h^2 z_1}{z_2}\right)\theta\left(\frac{h\mu_2}{\mu_1}\right)\theta\left(\frac{z_3}{z_2 h}\right) & 0 \\ \theta(h)\,\theta\left(\frac{z_2}{z_1}\right)\theta\left(\frac{z_3\mu_2}{z_1 h^2 \mu_1}\right) & \theta\left(\frac{h^2 z_1}{z_2}\right)\theta(h)\,\theta\left(\frac{z_3\mu_2}{z_2 h \mu_1}\right) & \theta\left(\frac{h^3 z_1}{z_3}\right)\theta\left(\frac{h^2 z_2}{z_3}\right)\theta\left(\frac{\mu_2}{\mu_1}\right) \end{pmatrix}$$

transpose, $\hbar \leftrightarrow \hbar^{-1}$, $z^{-1} \leftrightarrow \mu$



fixpt1 $t_1 = \frac{u_1}{\hbar}$ $t_2 = u_2$ fixpt2 $t_1 = u_1$ $t_2 = u_2$ fixpt3 $t_1 = u_1$ $t_3 = u_1$

$\boxed{\text{substitute: } u_1 = z_1 \hbar,\ u_2 = z_2 \hbar \text{ (reparametrization)}}$

$$F_1 := \theta\left(\frac{t_1}{u_1 h}\right)\theta\left(\frac{u_2 h^2 \mu_3}{t_2 \mu_1}\right)\theta\left(\frac{t_2 \mu_2}{t_1 \mu_1}\right)$$

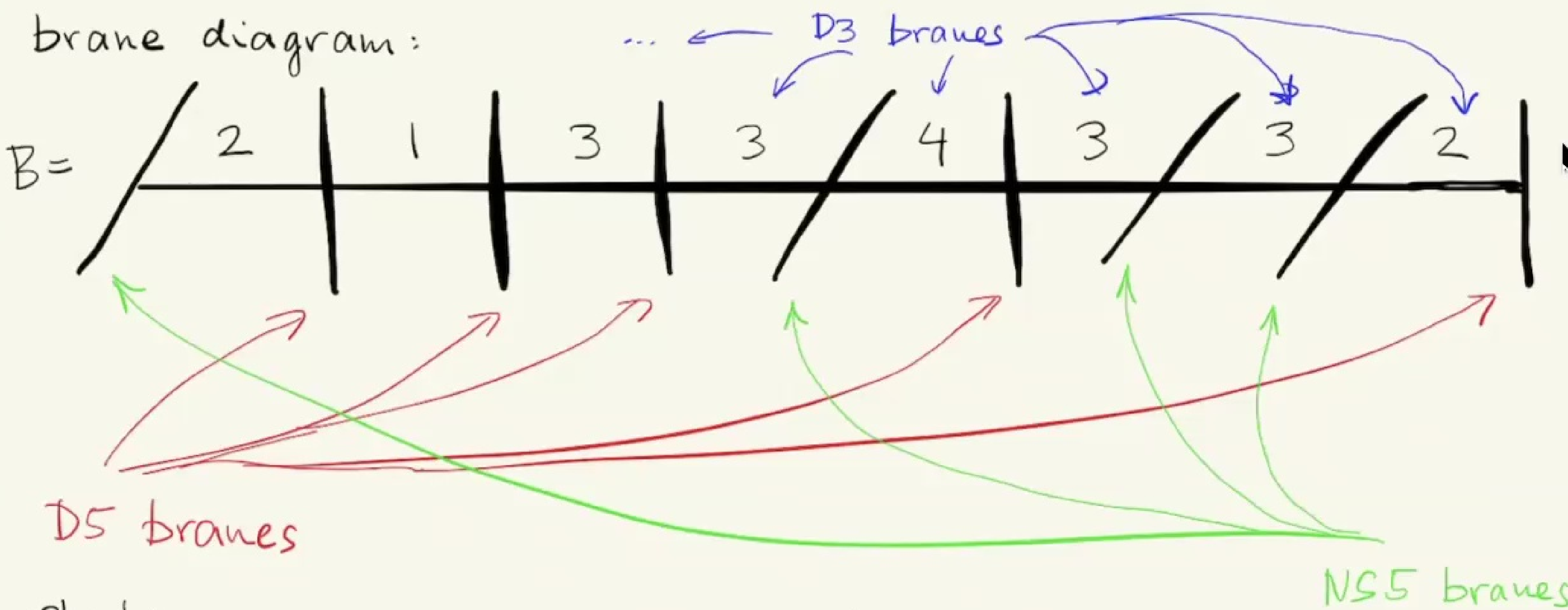
$$F_2 := \theta\left(\frac{t_1 \mu_1}{u_1 h^2 \mu_2}\right)\theta\left(\frac{u_2 h \mu_3}{t_2 \mu_2}\right)\theta\left(\frac{t_2}{t_1 h}\right)$$

$$F_3 := \theta\left(\frac{t_1 \mu_1}{u_1 h^3 \mu_3}\right)\theta\left(\frac{u_2}{t_2}\right)\theta\left(\frac{t_2 \mu_2}{t_1 h^2 \mu_3}\right)$$

$$\begin{pmatrix} \theta\left(\frac{z_2}{h^2 z_1}\right)\theta\left(\frac{h^2\mu_3}{\mu_1}\right)\theta\left(\frac{h\mu_2}{\mu_1}\right) & \theta\left(\frac{1}{h}\right)\theta\left(\frac{h^2\mu_3}{\mu_1}\right)\theta\left(\frac{z_2\mu_2}{z_1\mu_1}\right) & \theta\left(\frac{1}{h}\right)\theta\left(\frac{z_2 h^2\mu_3}{z_1\mu_1}\right)\theta\left(\frac{\mu_2}{\mu_1}\right) \\ 0 & \theta\left(\frac{\mu_1}{h^2\mu_2}\right)\theta\left(\frac{h\mu_3}{\mu_2}\right)\theta\left(\frac{z_2}{h z_1}\right) & \theta\left(\frac{\mu_1}{h^2\mu_2}\right)\theta\left(\frac{z_2 h\mu_3}{z_1\mu_2}\right)\theta\left(\frac{1}{h}\right) \\ 0 & 0 & \theta\left(\frac{\mu_1}{h^3\mu_3}\right)\theta\left(\frac{z_2}{z_1}\right)\theta\left(\frac{\mu_2}{h^2\mu_3}\right) \end{pmatrix}$$

Bow varieties

brane diagram:



$$\xrightarrow[\text{Nakajima-Takayama}]{\text{Cherkis}} \quad X(B) \circlearrowleft T^{\#\text{D5-branes}} \times \mathbb{C}^*_\hbar$$

(well... stability parameters...)

variety with holomorphic symplectic str.

$\xi_1^2, \xi_2^1, \xi_3^3, \ldots$ tautological bundles ($\xleftrightarrow{1:1}$ D3 branes)

$X(B) = ?$

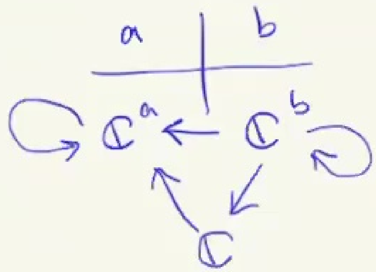
- [Cherkis 09, 10, 11]

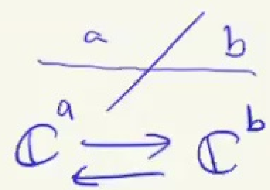
$$X(B) = \mathcal{M}\left(U(n)\text{-instantons on multi-Taub-NUT spaces} \right)$$

$\Updownarrow$

gauge equiv. class of bow solutions (⇔ lin maps, connection satisfying constraints (eg Nahm's eq'n))

- [Nakajima – Takayama '17] : "quiver" description





- some "commuting" eq'ns [moment $\mathbb{R}, \mathbb{C}$]
- open conditions (stability)
- then $/ \times GL_a$

How to study cohomology classes on $X(B)$?

localization. eg $\quad T^5 \circlearrowright T^*Gr_2\mathbb{C}^5 \leftarrow \xi^2$ need

- elements in H^* : Chern classes of tautological b's; equivariant parameters $u_1 \dots u_5$
  $H^*_{T^5}(pt) = \mathbb{C}[u_1, -, u_5]$
- torus fixed pts $\leftrightarrow$ two element subsets of $\{1, 2, \dots, 5\}$
- specializations of taut. bundles at fixed pts
  e.g. fixpt $\leftrightarrow \{1, 3\}$ $\qquad H^*_{T^5}(T^*Gr_2\mathbb{C}^5) \longrightarrow H^*_{T^5}(\text{fixpt})$
  $$\gamma_1 \longmapsto u_1$$
  $$\gamma_2 \longmapsto u_3$$
- tangent bundle $= Hom(\xi, \mathbb{C}^5 - \xi) \oplus \hbar\, Hom(\mathbb{C}^5 - \xi, \xi)$

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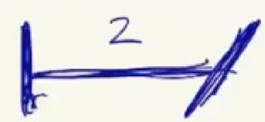
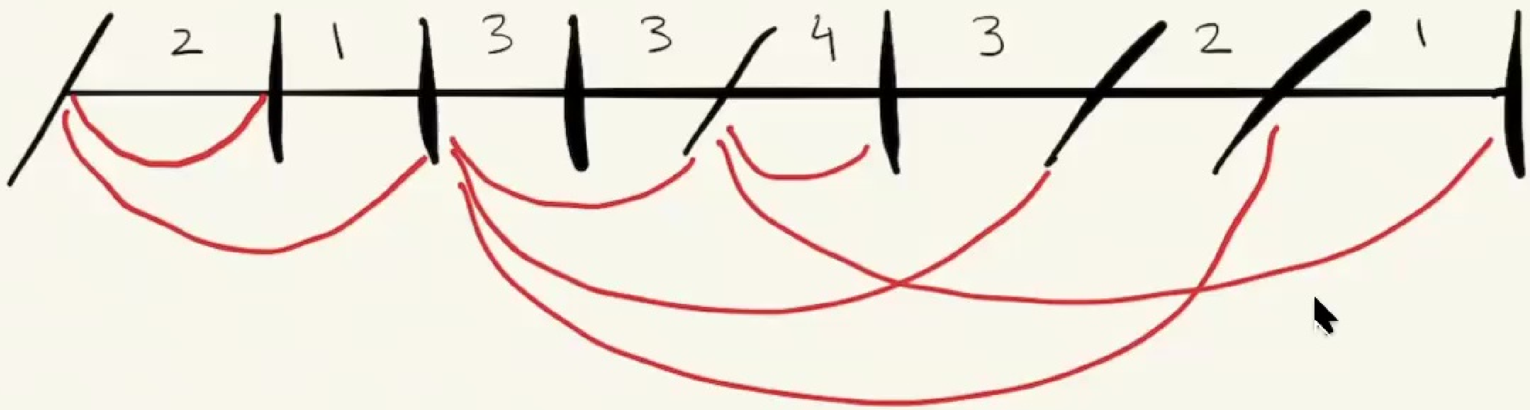
torus fixed pts $\longleftrightarrow$ sets of "ties"



$\nexists$ fix pt



- tie : connects 2 different type 5-branes
- each D3 brane position is <u>covered</u> as many times as its multiplicity

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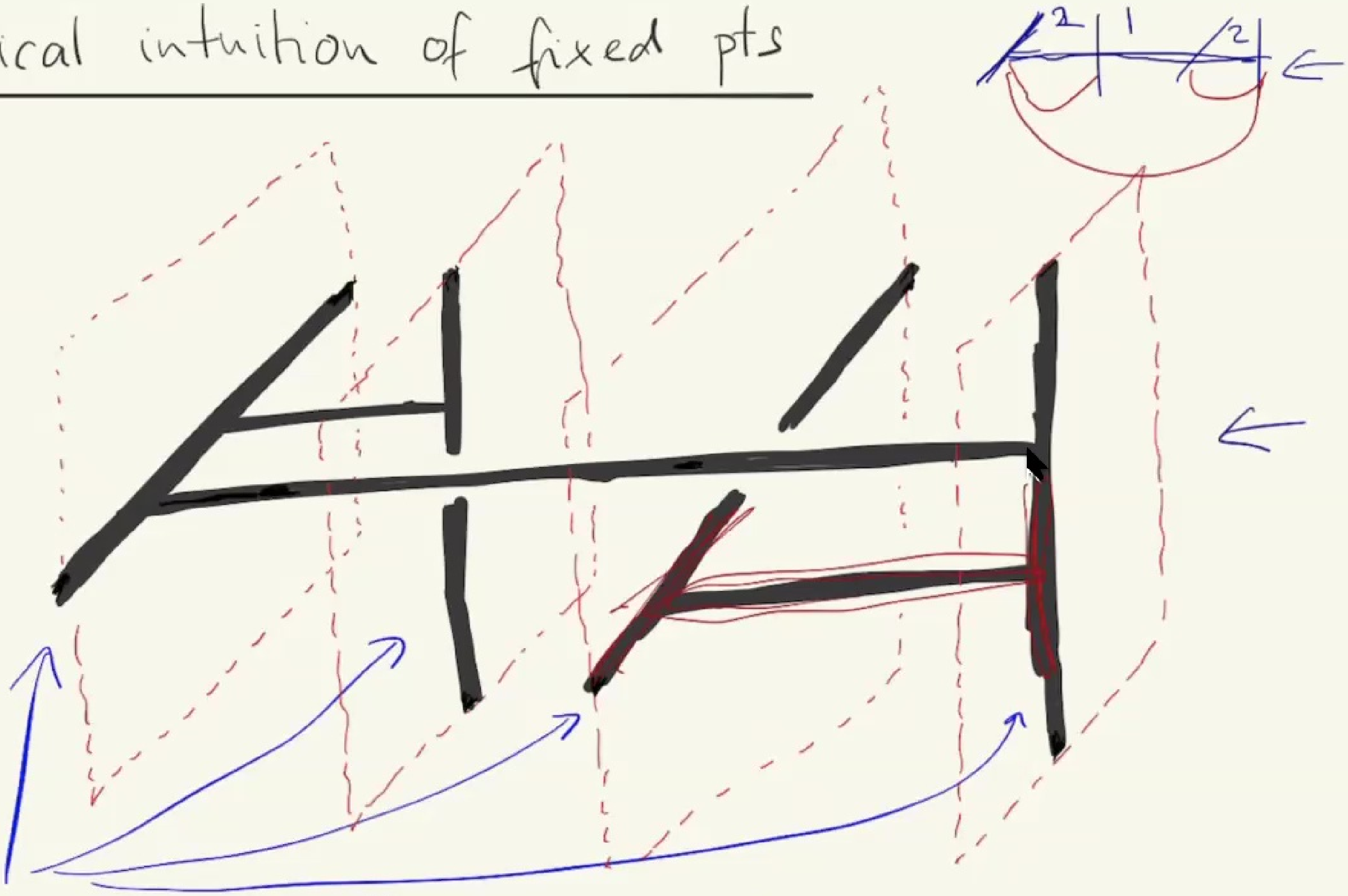
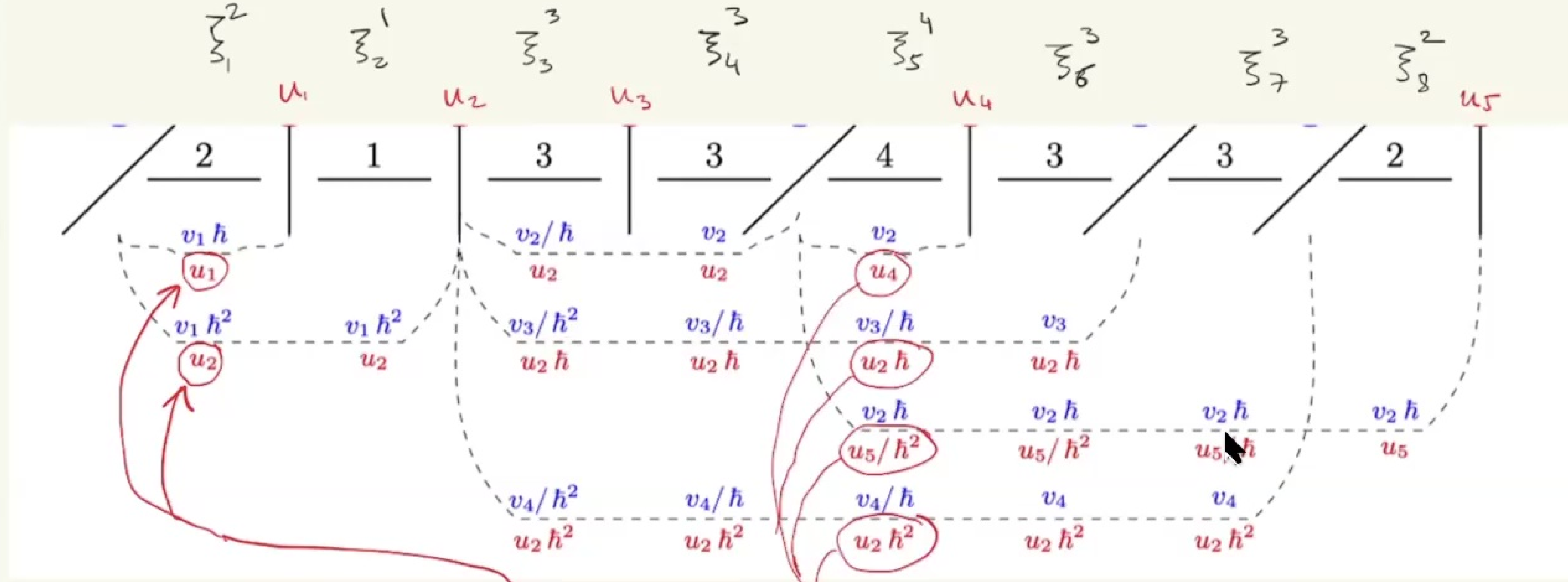
From now on: assume fix pt <u>exists</u>. <u>Thm</u> $X(B)$ smooth.

Physical intuition of fixed pts



5 branes: stuck in their vertical planes



Chern roots of ξ_1^2 $\mapsto$

Chern roots of ξ_5^4 $\mapsto$

$K_T(pt) = \mathbb{C}[u_1^{\pm}, \dots, u_5^{\pm}, \hbar^{\pm}]$

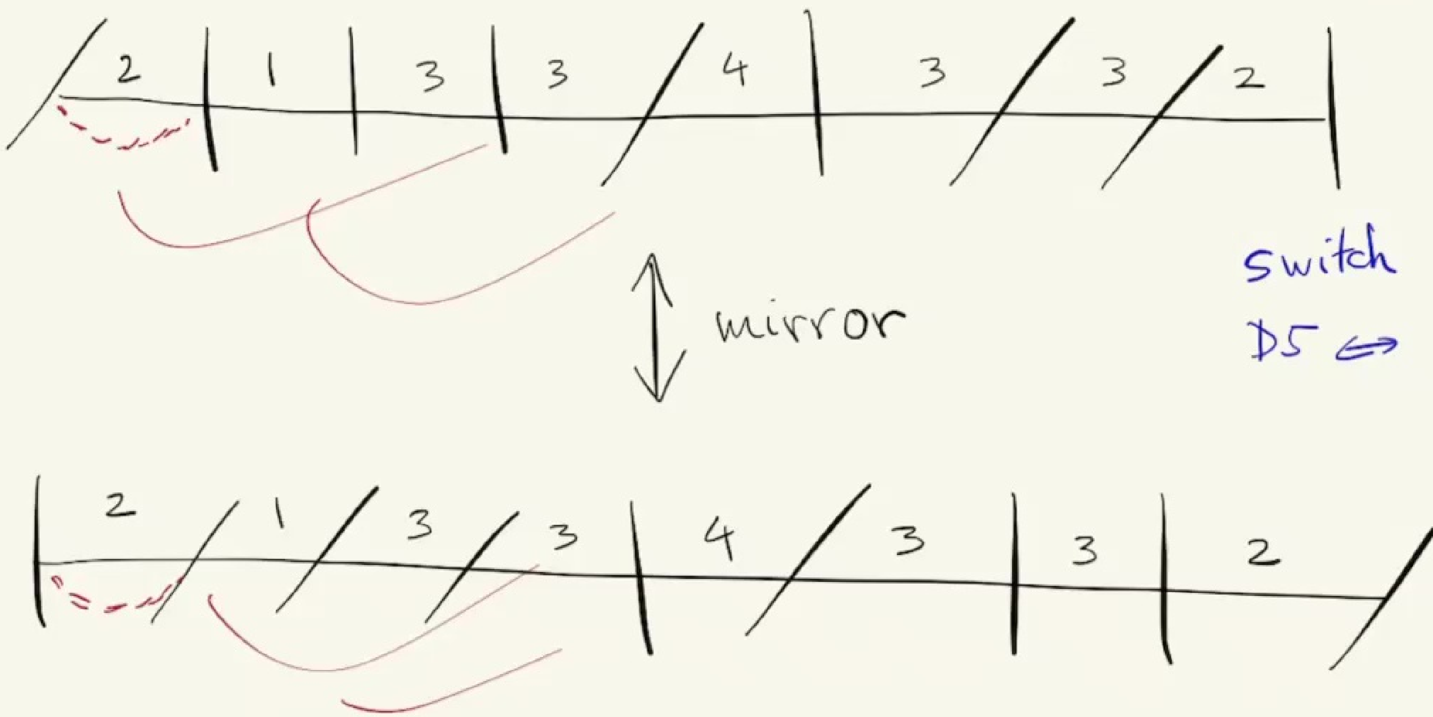
[ignore everything in <u>blue</u> on this slide] tangent bundle

$$TX(\mathcal{B}) = \left(\bigoplus_{J \text{ is D5}} T_J \right) \oplus \left(\bigoplus_{J \text{ is NS5}} T_J \right) \ominus$$

$$\left(\begin{array}{l} \bigoplus_{J \text{ is D5}} \hbar \, \mathrm{Hom}\left(\xi_J^{\text{right}}, \xi_J^{\text{left}}\right) \\ \bigoplus_{J \text{ is D3}} (1+\hbar)\, \mathrm{End}(\xi_J) \end{array} \right)$$

Two important operations on brane diagrams

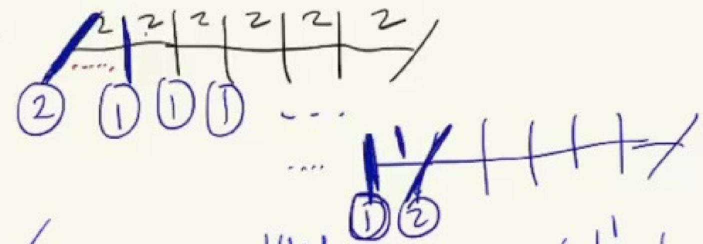
① Mirror:



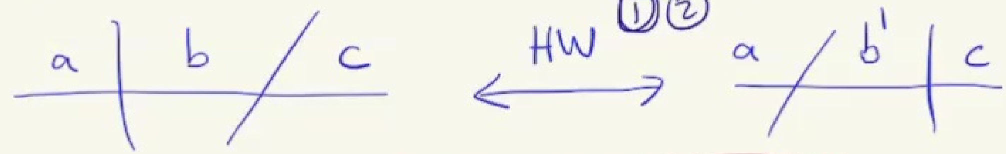
switch
D5 ⇔ NS5

- observe: natural bijection on fixed pts

Other important operation on B.



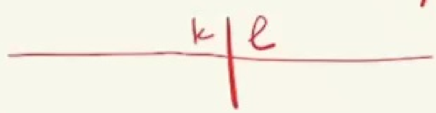
2 Hanany-Witten move :



$$\boxed{b + b' = a + c + 1}$$

forced by conservation of brane charge

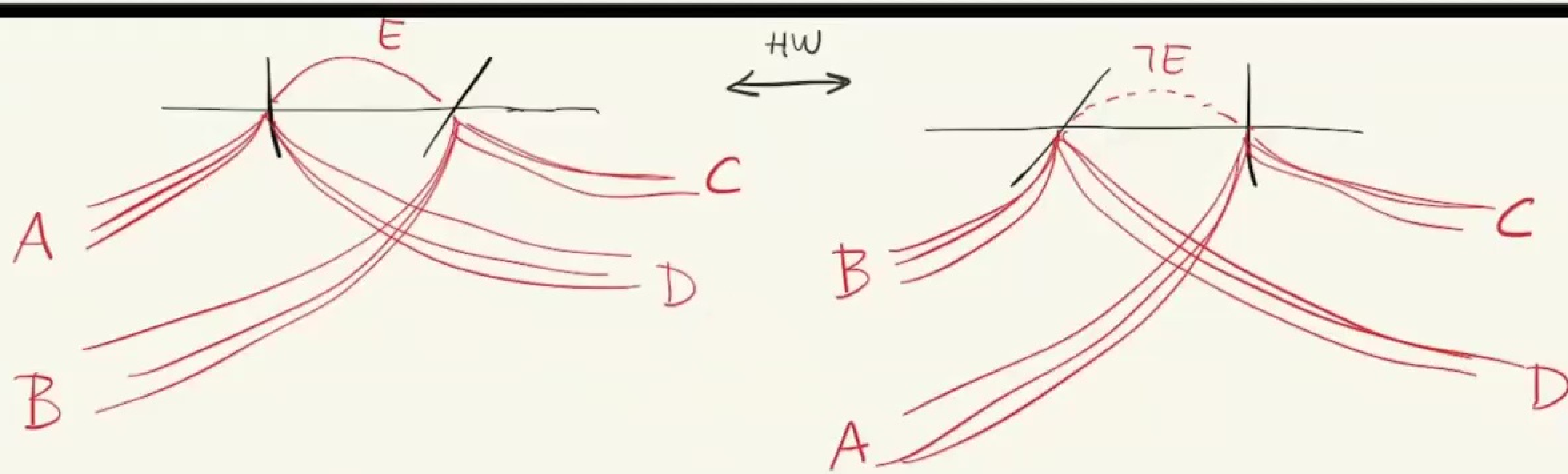
charge(5-brane) = #(opposite type 5-branes to the left) + $\ell - k$



- Thm $\boxed{X(B) \cong X(HW(B))}$ with torus reparametrized
- Thm natural bijection of torus fixed pts

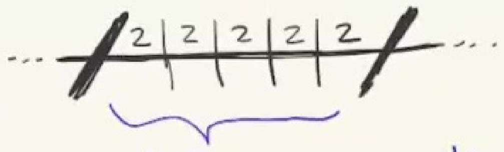
• Thm natural bijection of torus fixed pts

reparametrized

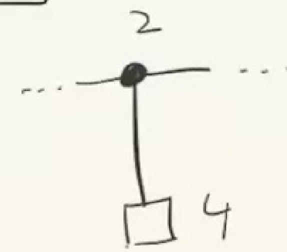


[natural bijection on fixed pts]

Nakajima quiver var's vs bow varieties

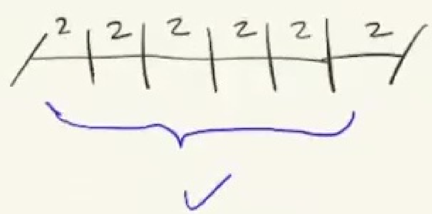


$=$

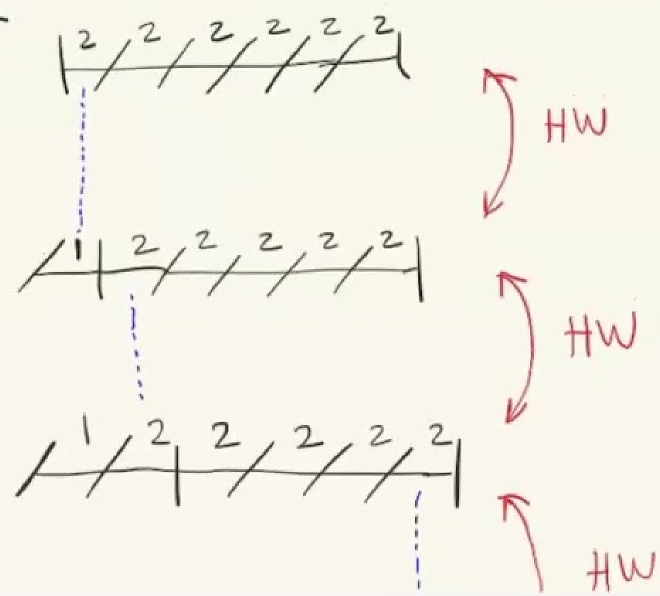


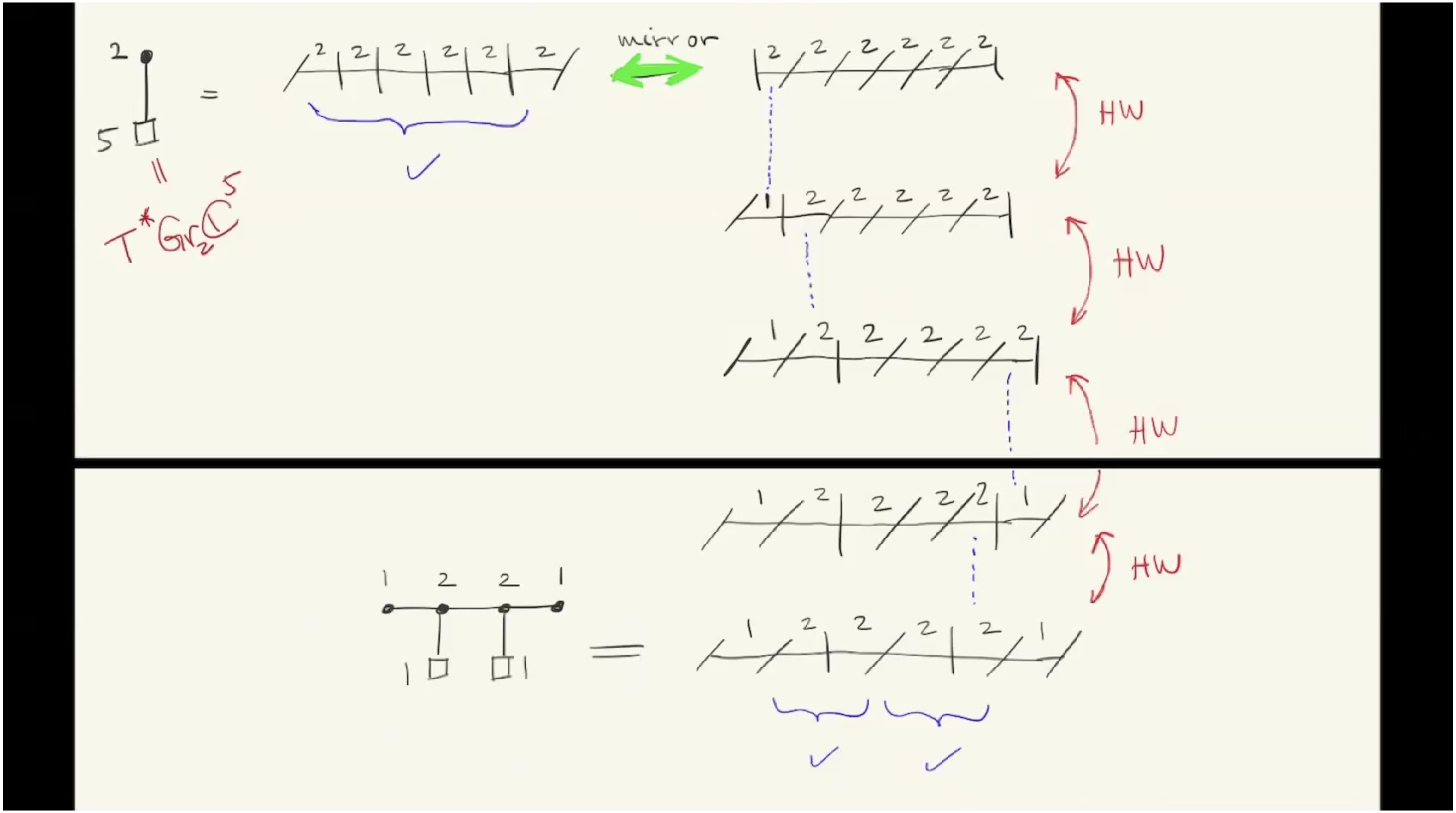
between consecutive NS5 branes all D3 multiplicities are the same ("cobalanced")

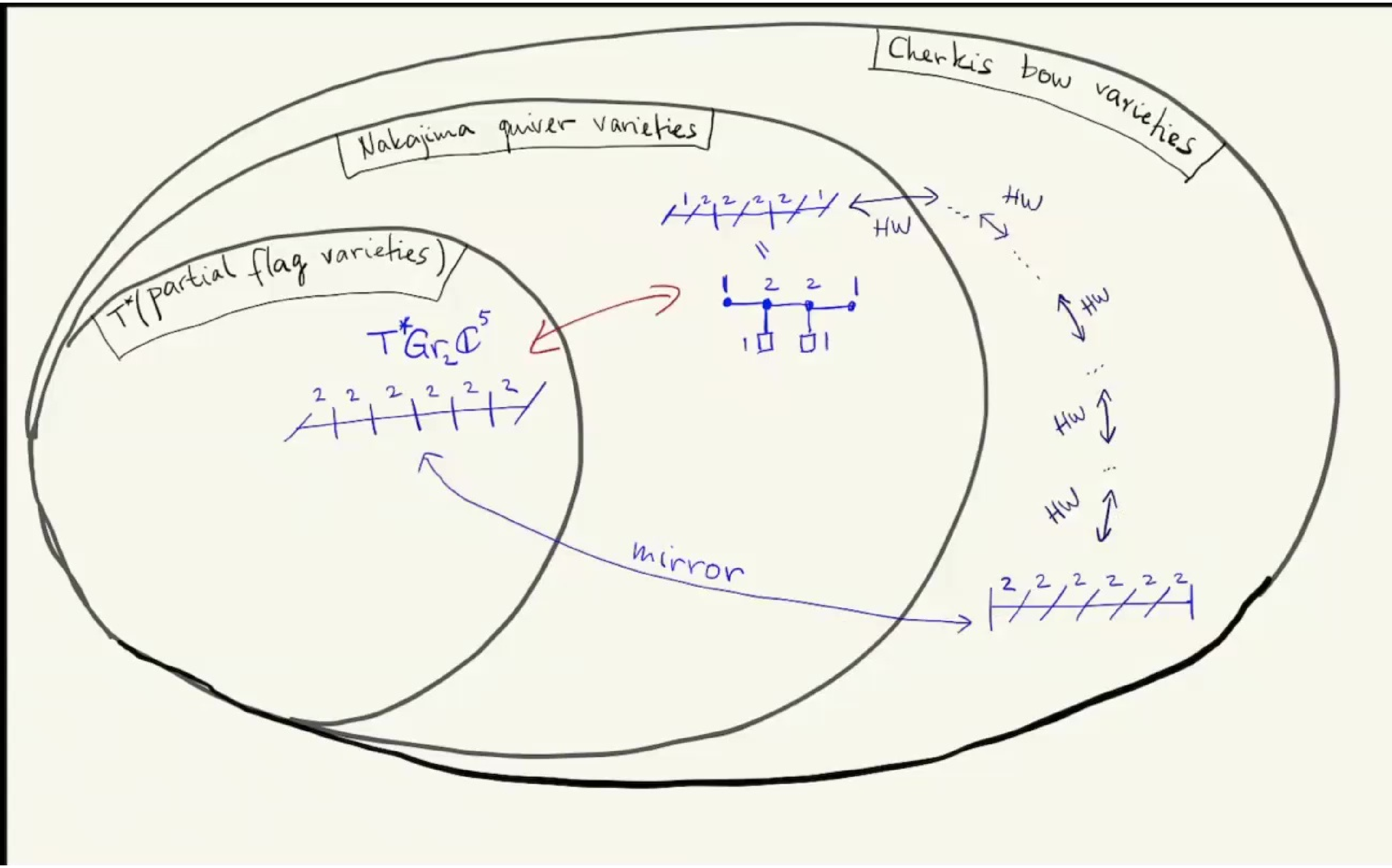
$=$

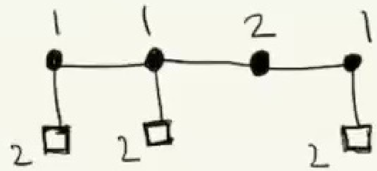


mirror




2
5
$T^*Gr_2\mathbb{C}^5$
mirror
HW
HW
HW
HW
1 2 2 1


Cherkis bow varieties
Nakajima quiver varieties
T*(partial flag varieties)
T*Gr2C5
mirror
HW
HW
HW
HW
HW
HW



smooth
holomorphic
symplectic

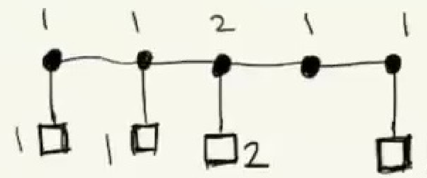
dim = 8

with $GL_2 \times GL_2 \times GL_2$ action

tautological line bundles:

$\xi_1^1 \quad \xi_2^1 \quad \xi_3^2 \quad \xi_4^1$

torus fixed pts = 16



smooth
holomorphic
symplectic

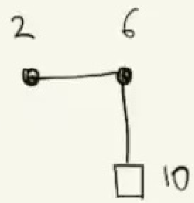
dim = 10

with $GL_1 \times GL_1 \times GL_2 \times GL_1$ action

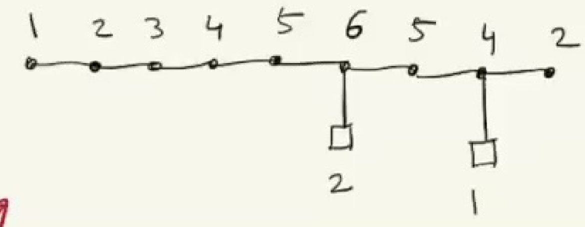
tautological line bundles

$\xi_1^1 \quad \xi_2^1 \quad \xi_3^2 \quad \xi_4^1 \quad \xi_5^1$

torus fixed pts = 16



dual

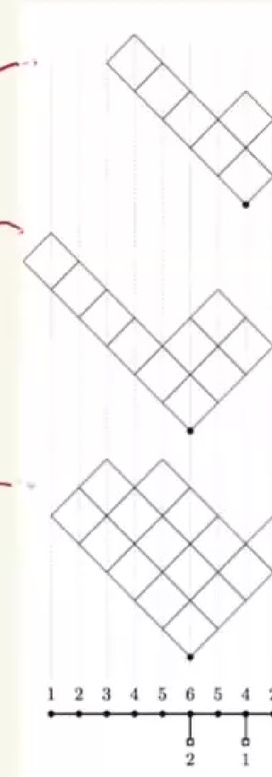


$T^*\mathcal{F}_{2,4,4}$ =

$\#$ of fix pts = $\dfrac{10!}{2!\,4!\,4!}$

=

$\#$ of fix pts = $\#\{(\lambda_1, \lambda_2, \lambda_3)$: partitions like these $\}$



λ_1

λ_2

λ_3

Expectation (work in progress)

- $\exists$ notion of elliptic stable envelope for bow varieties

- Stab (B) " $=$ " Stab (mirror (B))

equivar $\Leftrightarrow$ Kähler

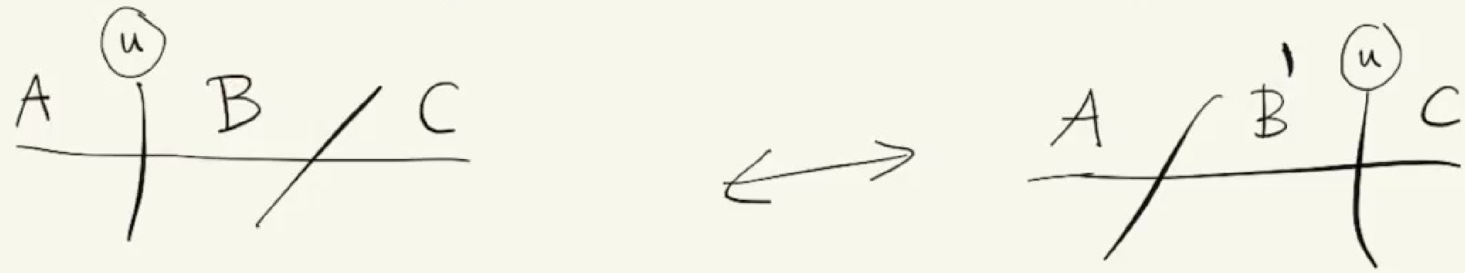
$\hbar \Leftrightarrow \hbar^{-1}$ [as in example above]

(and this should be easy (?) ...)

But then why are known cases [SRVZ1] [SRVZ2] difficult?

$T^*\mathcal{F}(n)$ $\circlearrowleft$

$T^*Gr_k\mathbb{C}^n \Leftrightarrow$ its dual because the algebraic combinatorics of HW moves:



$$B' = A \oplus C \oplus u \ominus B$$

(+ reparametrization)

symmetric function of Chern roots of B'

symmetric function of "difference of alphabets"

$$s_{22}(B-A) = s_{22}(B) + s_{21}(B)\,s_1(A) + s_2(B)\,s_2(A) + s_{11}(B)\,s_{11}(A) + s_1(B)\,s_{21}(A) + s_{22}(A)$$

Expectation (work in progress)

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$T^* \mathcal{F}(n)$ $\circlearrowleft$

$T^* Gr_k \mathbb{C}^n \leftrightarrow$ its dual

	[ ]	ħ-deformed	
H^*_T			CSM
K_T			mCL(
Ell_T	$\nexists$		$E(\ , \mu)$

$\nexists$

pay price!

mirror

$d=3 \quad N=4$