Title: Codes, vertex algebras and topological modular forms

Speakers: Gerd Laures

Collection: Elliptic Cohomology and Physics

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Abstract: The talk illuminates the role of codes and lattice vertex algebras in algebraic topology. These objects come up naturally in connection with string structures or topological modular forms. The talk tries to unify these different concepts in an introductory manner.

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Codes, vertex algebras and topological modular forms

Gerd Laures

Ruhr-University Bochum

Elliptic Cohomology and Physics workshop, May 2020

jt with Nora Ganter

Codes

A *monomial transformation* is a linear map of the form

$$f : K^n \to K^n$$
$$e_i \mapsto c_i e_{\sigma^{-1}(i)}$$

for some $c_i \in K^\times$, $\sigma \in \Sigma_n$.

The (linear) *code* of length n is a subset (subspace)

$$C \subset K^n$$

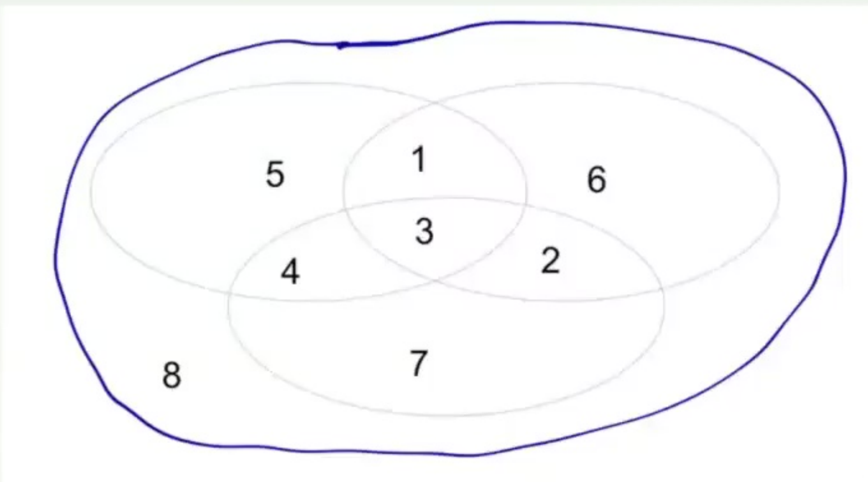
A *morphism* $f : C \to C'$ is monomial transformation with

$$f(C) \subset C'.$$

The *weight* of a word $c \in C$ is $wt(c) = \#\{i | \, c_i \neq 0\}$.

Codes

Example (Hamming code, $K = \mathbb{F}_2$, $n = 8$)



$$C = \{x \in \mathbb{F}_2^8 | \sum_{i \in circle} x_i = 0\}$$

$$\cong \mathbb{F}_2^4$$

$$C = \left\{ \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ \dots & & & & & & & \end{array} \right\}$$

self dual: $C = C^{\perp}$

doubly even: $\min_{c \neq 0}\{wt(c)\} = 4$

Codes

Example (Ternary code, $K = \mathbb{F}_3$, $n = 4$)

$$C = \{(s, t, s+t, -s+t) \mid s, t \in \mathbb{F}_3\} = \text{graph}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

- $\min_{c \neq 0}\{wt(c)\} = 3$
- self dual
- $\text{Aut}(C) = GL_2(\mathbb{F}_3) = \text{Aut}_{\mathbb{F}_4}(y^2 + y = x^3)$

Example (Ternary Golay code, $K = \mathbb{F}_3$, $n = 12$)

- $3^6 = 729$ words
- self dual
- $\text{Aut}(C) = 2.M_{12}$ Mathieu group

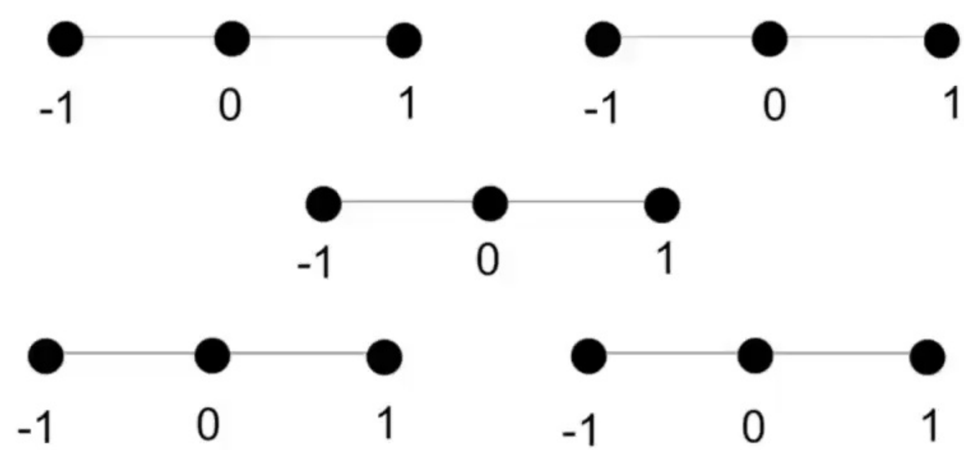
Lattices

$$
\begin{array}{lcll}
K & = & \mathbb{Q}[\zeta] & \zeta \text{ primitive } p \text{ root} \\
\mathbb{O} & = & \mathbb{Z}[\zeta] & \text{ring of integers} \\
\mathbb{P} & = & (1-\zeta)\mathbb{O} & \text{sublattice of } \mathbb{O}
\end{array}
$$

exact sequence:

$$
\begin{array}{ccccc}
\mathbb{P} & \to & \mathbb{O} & \xrightarrow{\rho} & \mathbb{F}_p \\
 & & \sum_{i=0}^{p-2} a_i \zeta^i & \mapsto & \sum_i a_i
\end{array}
$$

$p = 3$



Lattices

$$\begin{array}{lcll} K & = & \mathbb{Q}[\zeta] & \zeta \text{ primitive } p \text{ root} \\ \mathbb{O} & = & \mathbb{Z}[\zeta] & \text{ring of integers} \\ \mathbb{P} & = & (1-\zeta)\mathbb{O} & \text{sublattice of } \mathbb{O} \end{array}$$

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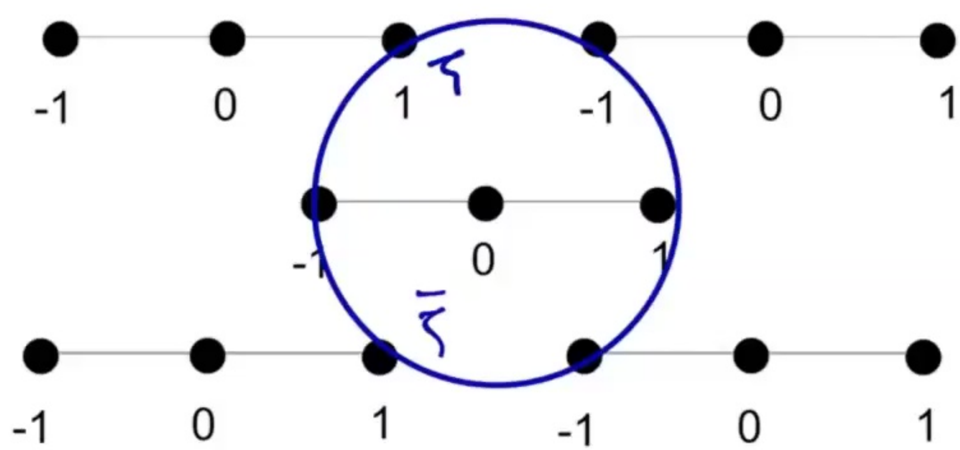

Lattices

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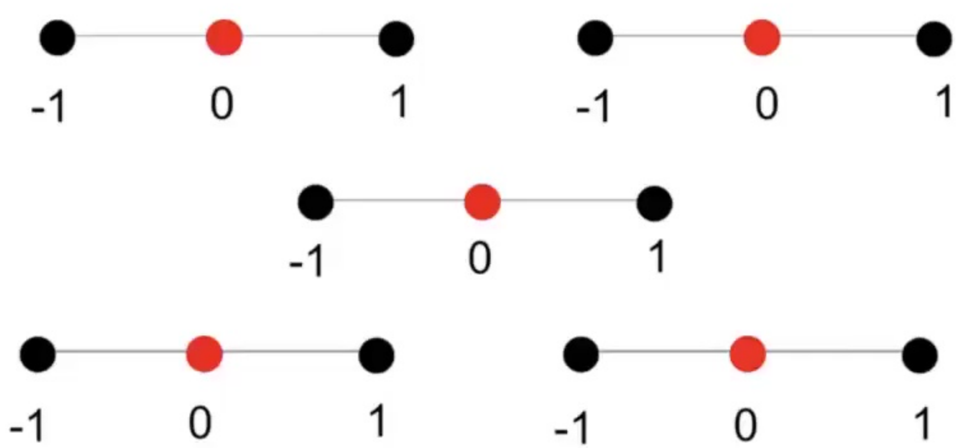


Lattices

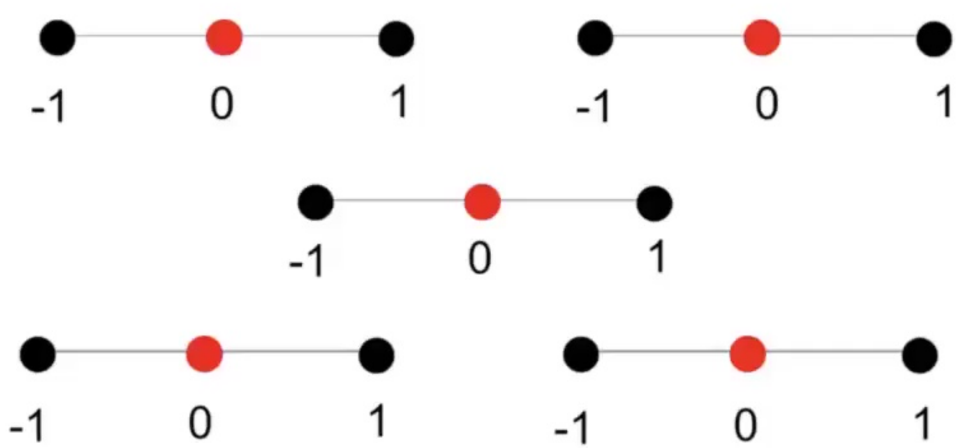
K	$=$	$\mathbb{Q}[\zeta]$	ζ primitive p root
$\mathbb{O}$	$=$	$\mathbb{Z}[\zeta]$	ring of integers
$\mathbb{P}$	$=$	$(1-\zeta)\mathbb{O}$	sublattice of $\mathbb{O}$

exact sequence:

$$\mathbb{P}^n \to \mathbb{O}^n \xrightarrow{\rho} \mathbb{F}_p^n$$

$$\sum_{i=0}^{p-2} a_i\zeta^i \mapsto \sum_i a_i$$

$p = 3$



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Lattices

A linear code $C \subset \mathbb{F}_p^n$ defines a lattice:

$$\Gamma_C = \rho^{-1}(C) \subset \mathbb{O}^n$$

Define symmetric bilinear form:

$$\langle x, y\rangle = \sum_{i=1}^{n} Tr(\frac{x_i \bar{y}_i}{p}), \quad Tr(\alpha) = \sum_{i=1}^{p-1} \sigma_r(\alpha) \in \mathbb{Q} \quad (\sigma_r(\zeta) = \zeta^r)$$

If $C \subset C^\perp$ it holds:

- Γ_C is an even integral lattice of rank $n(p-1)$

$$\langle x, y\rangle \in \mathbb{Z}; \ \langle x, x\rangle \in 2\mathbb{Z}$$

- $\Gamma_{C^\perp}$ is the dual lattice

$$\Gamma_C^\vee = \{x \in \mathbb{O}^n | \ \langle x, y\rangle \in \mathbb{Z} \text{ for all } y \in \Gamma_C\}$$

$C = C^\perp \Rightarrow \Gamma_C^\vee = \Gamma_C$

Theta series ($p = 3$)

For $\tau \in \mathbb{H} = \{\tau | \, \mathrm{im}(\tau > 0\}$ set $\theta_\Gamma(\tau) = \sum_{x\in\Gamma} e^{\pi i \langle x,x\rangle \tau}$.

Theorem (van der Geer-Hirzebruch, Alpbach)

θ_{Γ_C} is a modular form for the group

$$\Gamma(3) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; mod \; 3\},$$

that is, it is a holomorphic function on $\mathbb{H}$ with

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^n f(\tau)$$

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n; \; q = e^{2\pi i \tau}$$

Self dual codes give $SL_2(\mathbb{Z})$-invariant modular forms.

Arithmetic Whitehead Towers

$$
\begin{array}{ccccc}
F_{n,ev} & \xrightarrow{\subset} & Spin_n(\mathbb{Z}) & \longrightarrow & Spin_n(\mathbb{R}) \\
\downarrow{\scriptstyle \mathbb{Z}/2} & & \downarrow{\scriptstyle \mathbb{Z}/2} & & \downarrow{\scriptstyle \mathbb{Z}/2} \\
(\mathbb{F}_2^n)_{ev} & \xrightarrow{\subset} & SO_n(\mathbb{Z}) & \longrightarrow & SO_n(\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C} \subset \; \mathbb{F}_2^n & \xrightarrow{\subset} & O_n(\mathbb{Z}) & \longrightarrow & O_n(\mathbb{R})
\end{array}
$$

$$
\begin{aligned}
F_n &= \langle e_1, e_2, \ldots, e_n, -1 \rangle / (e_i^2 = -1, e_i e_j = -e_j e_i, (-1)^2 = 1, (-1) \text{ ctl}) \\
Cl_n &= \mathbb{R}[F_n]/((-1)_{\mathbb{R}} = -1) \\
\mathcal{A}lg\mathcal{R}ep(Cl_n) &\cong \{\text{repr. } \rho \text{ of } F_n |\ \rho(-1) = -id\}
\end{aligned}
$$

The spinor representation

H Hamming code of length 8

doubly even

$$\begin{array}{c} \omega \\ \uparrow \\ (-1,-1,\ldots,-1) \end{array} \qquad \begin{array}{ccc} \tilde{H} & \longrightarrow & F_{8,ev} \\ \downarrow & & \downarrow \\ H & \longrightarrow & \mathbb{F}_2^8 \end{array}$$

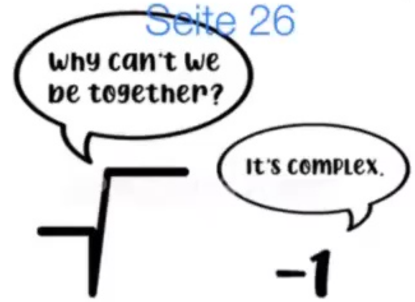
$$\Delta^{\pm} = \mathrm{ind}_{\tilde{H}}^{F_{8,ev}}(\rho^{\pm}), \ \rho(-1) = -id.$$

Arithmetic Whitehead Towers

top group   Stolz
2-group

$$
\begin{array}{ccccc}
? & \xrightarrow{\subset} & String_n(\mathbb{Z}) & \longrightarrow & String_n(\mathbb{R}) \\
\downarrow K(\mathbb{Z},2) & & \downarrow K(\mathbb{Z},2) & & \downarrow K(\mathbb{Z},2) = PU(\mathcal{H}) \\
F_{n,ev} & \xrightarrow{\subset} & Spin_n(\mathbb{Z}) & \longrightarrow & Spin_n(\mathbb{R}) \\
\downarrow \mathbb{Z}/2 & & \downarrow \mathbb{Z}/2 & & \downarrow \mathbb{Z}/2 \\
(\mathbb{F}_2^n)_{ev} & \xrightarrow{\subset} & SO_n(\mathbb{Z}) & \longrightarrow & SO_n(\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C} < \ \mathbb{F}_2^n & \xrightarrow{\subset} & O_n(\mathbb{Z}) & \longrightarrow & O_n(\mathbb{R})
\end{array}
$$

Arithmetic Whitehead Towers (Complex Case)



$$\begin{array}{ccccc}
? & \xrightarrow{\subset} & \widetilde{SU}_n(\mathbb{O}) & \longrightarrow & \widetilde{SU}_n(\mathbb{C}) \\
\downarrow K(\mathbb{Z},2) & & \downarrow K(\mathbb{Z},2) & & \downarrow K(\mathbb{Z},2) \\
(\mathbb{F}_p)^n_{det=1} & \xrightarrow{\subset} & SU_n(\mathbb{O}) & \longrightarrow & SU_n(\mathbb{C}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{F}_p^n & \xrightarrow{\subset} & U_n(\mathbb{O}) & \longrightarrow & U_n(\mathbb{C})
\end{array}$$

$\ell < \{\zeta^i\} = \mathbb{F}_p^n$

$U_n(\mathbb{O})$: $\{\pm\zeta^i\} \rtimes \Sigma_n \to T_0$

$\Omega \widetilde{SU}$

$\downarrow \Omega PU_H = S^1$

ΩSU

$L_0 = P^n_1, L_1 = \mathcal{O}^n$

$L_0 \to L_1 \to \mathbb{F}_p^n \to BL_0 \to BL_1$

$BL_0 = L_0 \otimes S^1 = T_0$, $BL_1 = L_1 \otimes S^1 = T_1$

Look at Ω repr of $\Omega \widetilde{T_0}$ with S^1 acting non trivial

Vertex algebras associated to a lattice L (Borcherds)

Set $\mathfrak{h} = L \otimes \mathbb{C}$. Define the *Heisenberg algebra*

$$\hat{\mathfrak{h}} = \mathfrak{h}[z^{\pm}] \oplus \mathbb{C}K$$

with brackets

$$[v \otimes z^m, w \otimes z^n] = m\langle v, w\rangle \delta_{m,-n} K$$

and K is central. Set

$$\begin{aligned} V_{\hat{\mathfrak{h}}}(1,0) &= U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^+)} \mathbb{C} \cong S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C} \\ V_L &= V_{\hat{\mathfrak{h}}}(1,0) \otimes \mathbb{C}\{\tilde{L}\} \end{aligned}$$

V_L has the structure of a conformal *vertex operator algebra*.

Vertex algebras

A *vertex algebra* consists of the following data:

- a vector space V
- a vacuum vector $\mathbf{1} \in V_0$, $\mathbf{1} \neq 0$
- a linear map $D : V \to V$ of degree one, $D\mathbf{1} = 0$
- a linear map $Y(\cdot, z) : V \to \mathrm{End}(V)[[z, z^{-1}]]$, $Y(v, z) = \sum v_n z^{-n-1}$

These data should satisfy the following properties for all $v, w \in V$:

- $Y(v, z) \in V[[z, z^{-1}]][z^{-1}]$ and there is a $k \geq 0$ with

$$(z_1 - z_2)^k [Y(v_1, z_1), Y(v_2, z_2)] = 0.$$

- $Y(v, z)\mathbf{1} = v + O(z)$
- $[D, Y(v, z)] = \partial Y(u, z)$

A *module* over a vertex operator algebra is a vector space M together with

$$Y_M : V \to \mathrm{End}(M)[[z, z^{-1}]];\ v \to Y_M(v, z) = \sum_n v_n^M z^{-n-1}$$

with the obvious properties.

V_L-modules for even lattices L

The irreducible representation of V_L are indexed by

$$\lambda \in C = L^{\vee}/L$$

and have the form

$$V_{L+\lambda} = S(\hat{\mathfrak{h}}^{-}) \otimes \mathbb{C}\{\tilde{L} + \lambda\}.$$

and they have the partition function

$$Z_{V_{L+\lambda}}(\tau) = tr_{V_{L+\lambda}} q^{\mathcal{L}_0 - d/24} = \eta(q)^{-d} \sum_{\alpha \in L+\lambda} q^{1/2\langle\alpha,\alpha\rangle}.$$

Theorem (Dong)

The V_L-modules $\{ V_{L+\lambda}\}$ for $\lambda \in C$ are all distinct and they provide a complete list for the isomorphism classes of irreducible V_L-modules.

The case $L = \mathbb{P}^d$, $L^\wedge = \mathbb{O}^d$, $L^\wedge/L = \mathbb{F}_p^d = C_d$ (full code)

Set $V_d = V_{\mathbb{P}^d}$ and define

$$\mathcal{R}ep(V_d) = \text{Grothendieck } \mathbb{O}[\tfrac{1}{p}]\text{-module of } V_d\text{-modules}$$

$$\mathcal{R}ep(V) = \bigoplus_{d \geq 0} \mathcal{R}ep(V_d)$$

Theorem (Ganter-L.)

For $p = 3$, the partition function induces a ring isomorphism

$$\mathcal{R}ep(V)/Aut(C) \xrightarrow{\cong} mf(3)$$

$$[M] \mapsto \eta^d Z_M,$$

$e_T \longmapsto E_4$

For a ternary code H set $M_H = \oplus_{\lambda \in H} V_{L+\lambda}$. If $H = H^\perp$ then Z_{M_H} is a $Sl_2(\mathbb{Z})$-modular form, in fact, it is a topological modular form.

Γ(3)-Topological modular forms

There is a complex oriented cohomology theory $TMF(3)$ with coefficients

$$TMF(3)_* = mf(3)_*[\Delta^{-1}]$$

and associated genus $(y = -\zeta)$

$$[M] \mapsto \chi_y(q, \mathcal{L}M)$$

with

$$\begin{aligned}\chi_y(q, \mathcal{L}M) &= \left(\prod_{i=1}^{d} x_i \frac{1 + ye^{-x_i}}{1 - e^{-x_i}} \left(\prod_{n=1}^{\infty} \frac{1 + yq^n e^{-x_i}}{1 - q^n e^{-x_i}} \frac{1 + y^{-1} q^n e^{x_i}}{1 - q^n e^{x_i}} \right) \right) [M] \\ &= \chi_y(M, \bigotimes_{n=1}^{\infty} \Lambda_{yq^n} T^* \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T^* \otimes \bigotimes_{n=1}^{\infty} \Lambda_{y^{-1}q^n} T \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T)\end{aligned}$$

The *real* theory TMF satisfies $TMF[3^{-1}] = TMF(3)^{h\,SL_2(\mathbb{F}_3)}$ (after completion).

The chiral de Rham complex

Corollary (Tamanoi)

The formal S^1-equivariant index of the twisted Dolbeault operator is a virtual module over some vertex operator algebra.

Theorem (Malikov-Schectman-Vaintrob, Cheung)

There is a graded sheaf $MSV(X)$ of differential vertex algebras for any complex manifold X. It possesses a filtration such that the associated graded is isomorphic to

$$\Lambda_{y^{-1}} T^* \otimes \bigotimes_{n=1}^{\infty} \Lambda_{yq^n} T^* \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T^* \otimes \bigotimes_{n=1}^{\infty} \Lambda_{y^{-1}q^n} T \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T.$$

If X is compact

$TMF(3)^* X \xrightarrow{x} K[[q]]^* X$

$e_{TMF(3)} - MSV(X) \longmapsto e_{Tate} = *$

$$char\ H^*(MSV(X)) = \chi_y(q, \mathcal{L}M) \cdot (a\ constant).$$

Chiral complexes

Theorem (Lian-Linshaw)

The cohomology of chiral complexes satisfies a homotopy axiom, has Mayer-Vietoris properties and comes with equivariant versions.

Conjecture/work in progress

The cohomology theory $TMF(3)^*(X)$ and its (not yet defined) equivariant versions have a description in terms of certain module sheaves of vertex algebras for manifolds X.

Attempt of a definition: $L = \mathbb{P}^d$

$$V_{\hat{\mathfrak{h}}}(1,\varphi)(U) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^+)} C^\infty(U) \cong U(\hat{\mathfrak{h}}^-) \otimes C^\infty(U)$$

$$V_d(U) = V_{\hat{\mathfrak{h}}}(1,\varphi)(U) \otimes \mathbb{C}\{\tilde{L}\}$$

Definition

A *sheaf of V_d-modules* on a smooth manifold M is a sheaf which locally is a sheaf of $V_d(U)$-modules.

Note:

- $MSV(X)$ is a sheaf of V_d-modules.
- for $U = *$ they assemble to $mf(3)$
- situation is analogous to K-theory