Title: Quasisymmetric characteristic numbers for Hamiltonian toric manifolds

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Collection: Elliptic Cohomology and Physics

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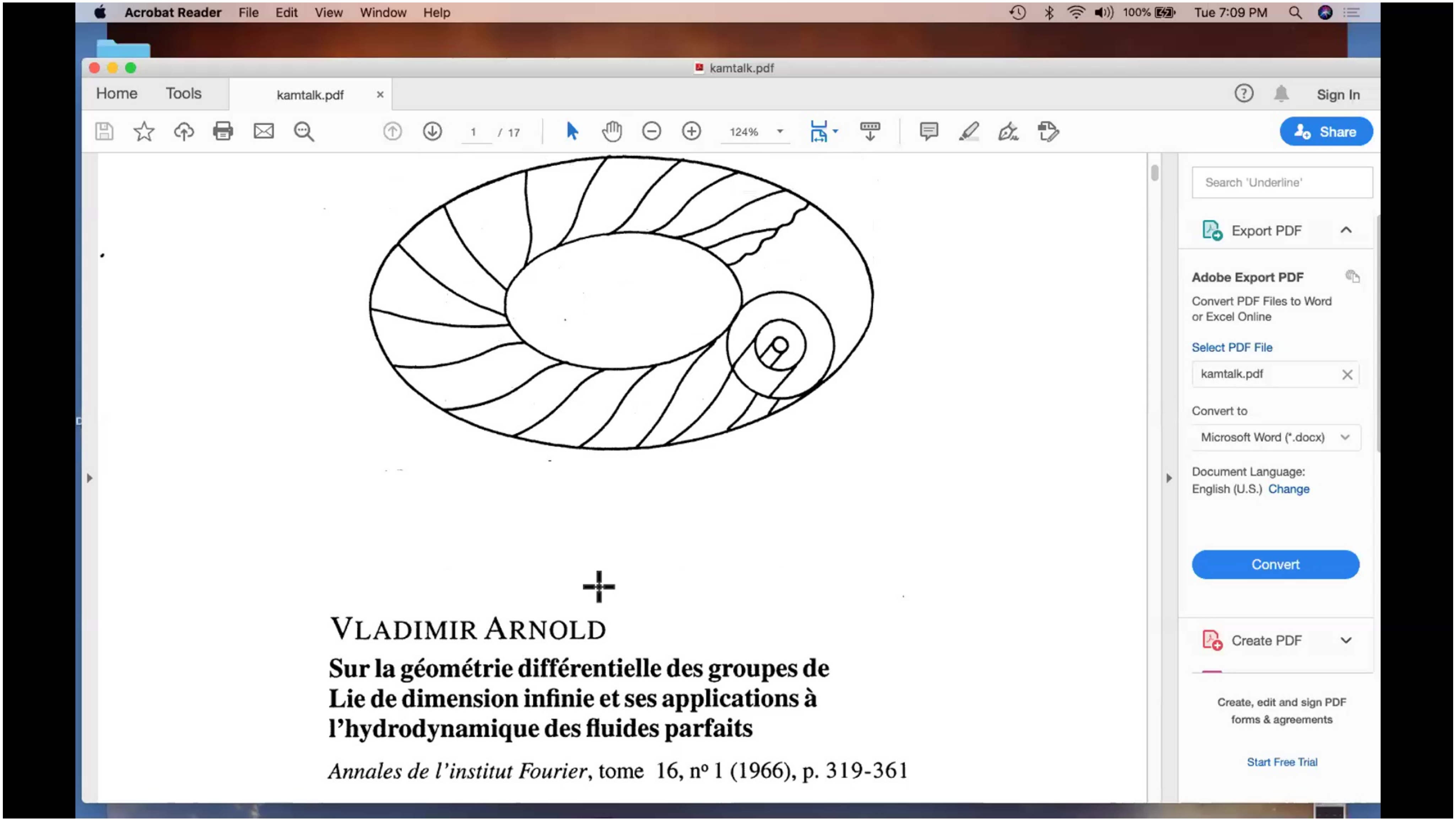
Abstract: Baker and Richter's A_∞ analog of the complex cobordism spectrum provides characteristic numbers for complex-oriented toric manifolds, which generalize to define similar invariants for Hamiltonian toric dynamical systems: roughly, the `completely integrable' systems of classical mechanics which (by KAM theory) possess remarkable stability properties. arXiv:1910.12609



VLADIMIR ARNOLD

Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits

Annales de l'institut Fourier, tome 16, nº 1 (1966), p. 319-361

Cobordism of Hamiltonian toric manifolds, & applications

§I Examples of Hamiltonian toric manifolds

- rigid bodies (Euler . . .)
- fluid mechanics (Kolmogorov, Arnol'd, Moser) (1960s)
- chemical reaction networks (Gatermann *et al* ∼ 2000)

§II Characteristic numbers and cobordism, TBA

Takeaways:

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Takeaways:

- Hamiltonian toric manifolds have interesting applications to physical chemistry, analogous to their applications to hydrodynamics; in particular, to quasi-periodic phenomena, perhaps such as DNA replication; and

- The spectrum $M\xi \wedge \mathbb{C}P^{\infty}$ is a plausible repository for the statistical mechanics of such systems.

§I Introduction: Hamiltonian toric manifolds generalize classical Euler-Liouville-Hamiltonian symplectic 'completely integrable' systems.

They hybridize

- the toric manifolds of Davis and Januszkiewicz,

= compact smooth even ($= 2n$)-dim'l M with effective $T = T^n$-action and a polyhedral quotient M/T, tient M/T,

together with

- equivariant symplectic manifolds with compatible stably-almost-complex structures (Quillen's complex orientations).

They are, roughly, compactifications of classical harmonic oscillators, much as toric varieties are compactifications of classical toric actions.

Examples:

- smooth projective toric varieties, with the Kähler symplectic structure inherited from their projective embeddings

- mechanics of rigid bodies (Euler), generalized in the modern theory of geometric quantization (Kirillov, Kostant, Souriau, *cf.* [GGK] = Guillemin-Ginzburg-Karshon):

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- mechanics of rigid bodies (Euler), generalized in the modern theory of geometric quantization (Kirillov, Kostant, Souriau, *cf.* [GGK] = Guillemin-Ginzburg-Karshon):

G = Lie group, with its canonical coadjoint action on $(\text{Lie } G)^*$. Its orbits

$$G \cdot \lambda \cong G/\text{iso}(\lambda) \to (\text{Lie } G)^*$$

have a natural G-equivariant symplectic structure, which restricts nicely to the left action by a subtorus of G, yielding manifolds foliated by toruses

- Arnol'd argued that the geodesic flow on the group $G = \mathrm{SDiff}(M)$ of volume-preserving diffeomorphisms of a compact Riemannian manifold recovers Euler's equation for incompressible fluid flow on M.

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Earlier work by Kolmogorov, Arnol'd, Moser, and others implies that large classes of integrable systems are **structurally stable** (in a sense close to Thom's); for example D'Arcy Thompson's vortex rings **generically** reproduce themselves. Toric dynamical systems in physical chemistry have remarkably similar formal properties.

- Delzant's classification of symplectic toric manifolds has further (Hamiltonian) structure: a 'moment map'

$$\Phi : M \to M/T \to (\text{Lie}\, T)^*$$

with a polyhedral quotient, which (roughly) parametrizes Noether's associated conserved quantities.

There are alternative cohomological formulations [GGK]:

$$\bar{\omega} = \omega - \Phi \in \Omega^2_T(M),\ d_T\bar{\omega} = 0,$$

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$\exists T$-equivariant cx line bundle on M with connection lifting $[\bar{\omega}] \in H^2_T(M, \mathbb{Z})$ as in geometric quantization.

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Note however that what moment maps are on **odd**-dim'l manifolds make questions of cobordism tricky.

⋆ **Corollary** *Hamiltonian toric manifolds have stable equivariant decompositions*

$$T_M \sim \oplus L_i$$

of their tangent bundles into sums of complex line bundles, indexed by faces of P.

This is a **very strong** kind of splitting principle! It provides an abundant supply of cohomological data

$$c_1(L_i) \in H^2(M, \mathbb{Z})$$

used below to define characteristic numbers.

Remark For general groups $G, H, \ldots,$ Hamiltonian manifolds (*i.e.* with moment maps) have products

$$(M, G),\ (N, H) \Rightarrow (M \times N, G \times H)\ ;$$

in particular

$$(M^{2m}, T^m) \times (N^{2n}, T^n) \Rightarrow (M \times N, T^{m+n})\ .$$

Similarly, if $T_M \sim \oplus L_i,\ T_N \sim \oplus L'_k$ then

$$T_{M \times N} \sim \oplus (L_i \otimes_{\mathbb{C}} L'_k)\ .$$

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Such decompositions make good sense on odd-dimensional manifolds, and suggest the interest of a cobordism theory for manifolds with stable tangent bundle splittings.

Example Gatermann and others (∼ 2000-2010) study chemical reaction networks defined by systems

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \sum F_{k,I}(\kappa) \cdot x^I \in \mathbb{Q}[\boldsymbol{\kappa}][\mathbf{x}]$$

of autonomous ODEs, where $\mathbf{x} = \mathbf{x}(t)$ is a vector of chemical concentrations, and $\boldsymbol{\kappa}$ is a matrix of (non-negative) reaction rates, suitably **sparse** (*i.e.* indexed by a collection of (strongly) connected graphs).

More precisely

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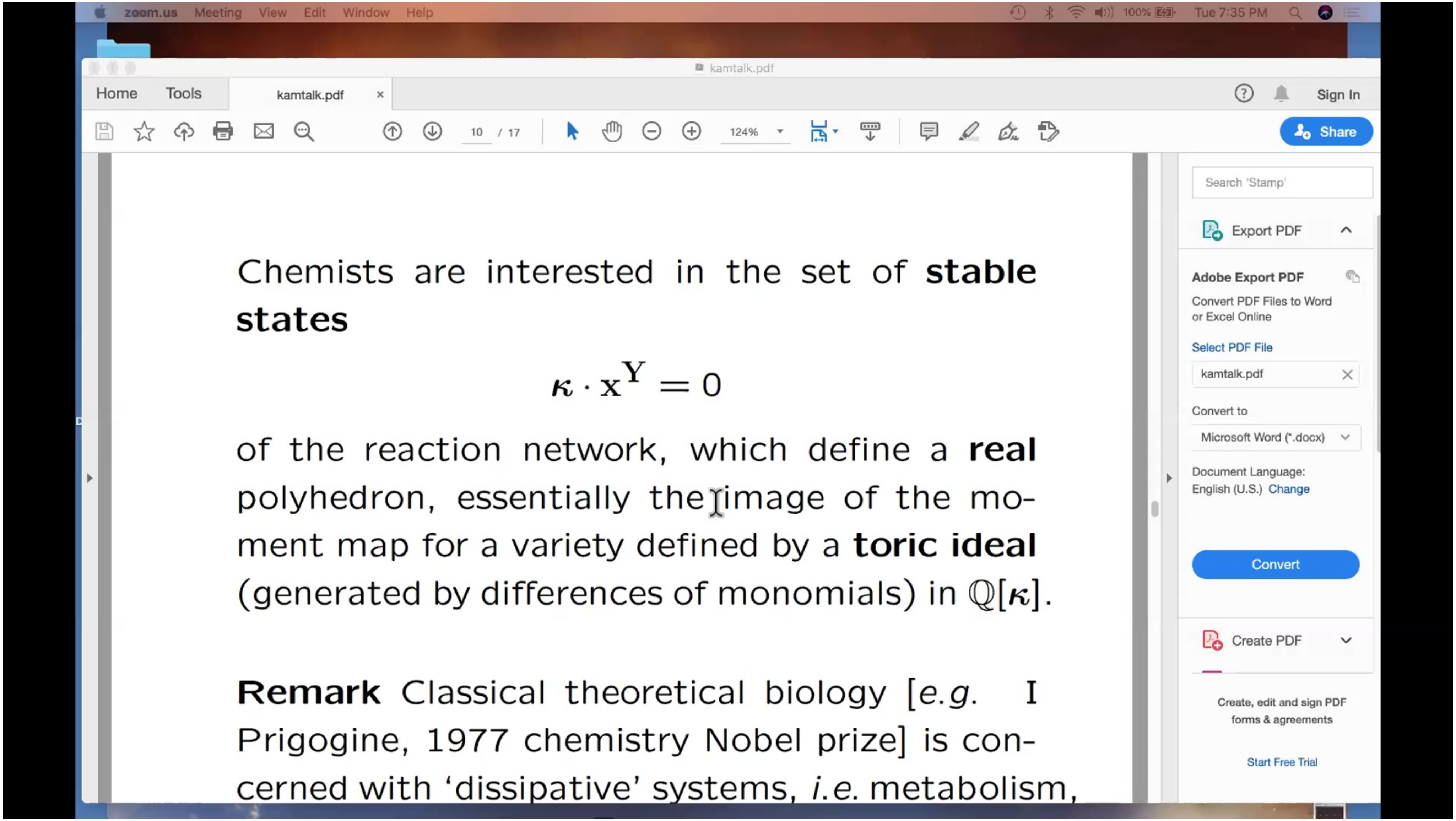
Chemists are interested in the set of **stable states**
$$\boldsymbol{\kappa} \cdot \mathbf{x}^{\mathbf{Y}} = 0$$
of the reaction network, which define a **real** polyhedron, essentially the image of the moment map for a variety defined by a **toric ideal** (generated by differences of monomials) in $\mathbb{Q}[\boldsymbol{\kappa}]$.
Remark Classical theoretical biology [*e.g.* I Prigogine, 1977 chemistry Nobel prize] is concerned with 'dissipative' systems, *i.e.* metabolism,
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Remark Classical theoretical biology [*e.g.* I Prigogine, 1977 chemistry Nobel prize] is concerned with 'dissipative' systems, *i.e.* metabolism, as opposed to (quasi-)periodic phenomena such as DNA replication. The stable states of reaction networks are asymptotic limits of such dissipative systems, while the complex points of the toric variety are their periodic shadows.

§II Characteristic numbers and cobordism

- symmetric and quasisymmetric functions
- . . . in terms of (Spectra)
- ? statistical mechanics

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Thom proved that oriented cobordism classes are charaxterized by their characteristic numbers, which can be defined, via Chern-Weil theory, in terms of curvature invariants. To topologists the remarkable integrality properties of these invariants are algebraic consequences of the structure of the cohomology of certain universal examples, some of which are summarized below.

- The graded commutative and cocommutative Hopf algebra

$$S_* = \mathbb{Z}[e_*] \subset \mathbb{Z}[x_*], \ \sum_{k \geq 0} e_k t^k = \prod_{i \geq 0} (1 + x_i t) \ ,$$

of classical symmetric functions

$$\Delta e_j = \sum_{j+k=l} e_k \otimes e_l \ ,$$

is mysteriously self-dual

$$H_*(B\mathsf{U}, \mathbb{Z}) \cong S_* \cong S^* \cong H^*(B\mathsf{U}, \mathbb{Z})$$

$(e_k \leftrightarrow h_k)$, *cf.* MacDonald.

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Complex cobordism is represented by the Thom spectrum $M\mathrm{U}$ defined by the classifying bundle $\xi \to B\mathrm{U}$, and the map

$$M\mathrm{U} \simeq M\mathrm{U} \wedge S^0 \to M\mathrm{U} \wedge H\mathbb{Z}$$

defines the Hurewicz homomorphism

$$M\mathrm{U}_* \to H_*(M\mathrm{U},\mathbb{Z}) \cong H_*(B\mathrm{U},\mathbb{Z})$$

which sends a manifold to such (normal) characteristic numbers.

- The free graded associative cocommutative Hopf algebra

$$\mathsf{N}_* = \mathbb{Z}\langle Z_* \rangle, \ \Delta Z_j = \sum_{j+k=l} Z_k \otimes Z_l$$

[Baker-Richter, Cartier, Hazewinkel] is dual to the commutative Hopf algebra $\mathsf{Q}_* \subset \mathbb{Z}[x_*]$ of **quasi**-symmetric functions, with basis elements

$$\langle \mathbf{n} \rangle (x_*) = \sum_{i_1 < \cdots < i_k} x_{i_1}^{n_1} \cdots x_{i_k}^{n_k} \in \mathsf{Q}^{2n}$$

indexed by **ordered** partitions

$$\mathbf{n} = n_1 + \cdots + n_k$$

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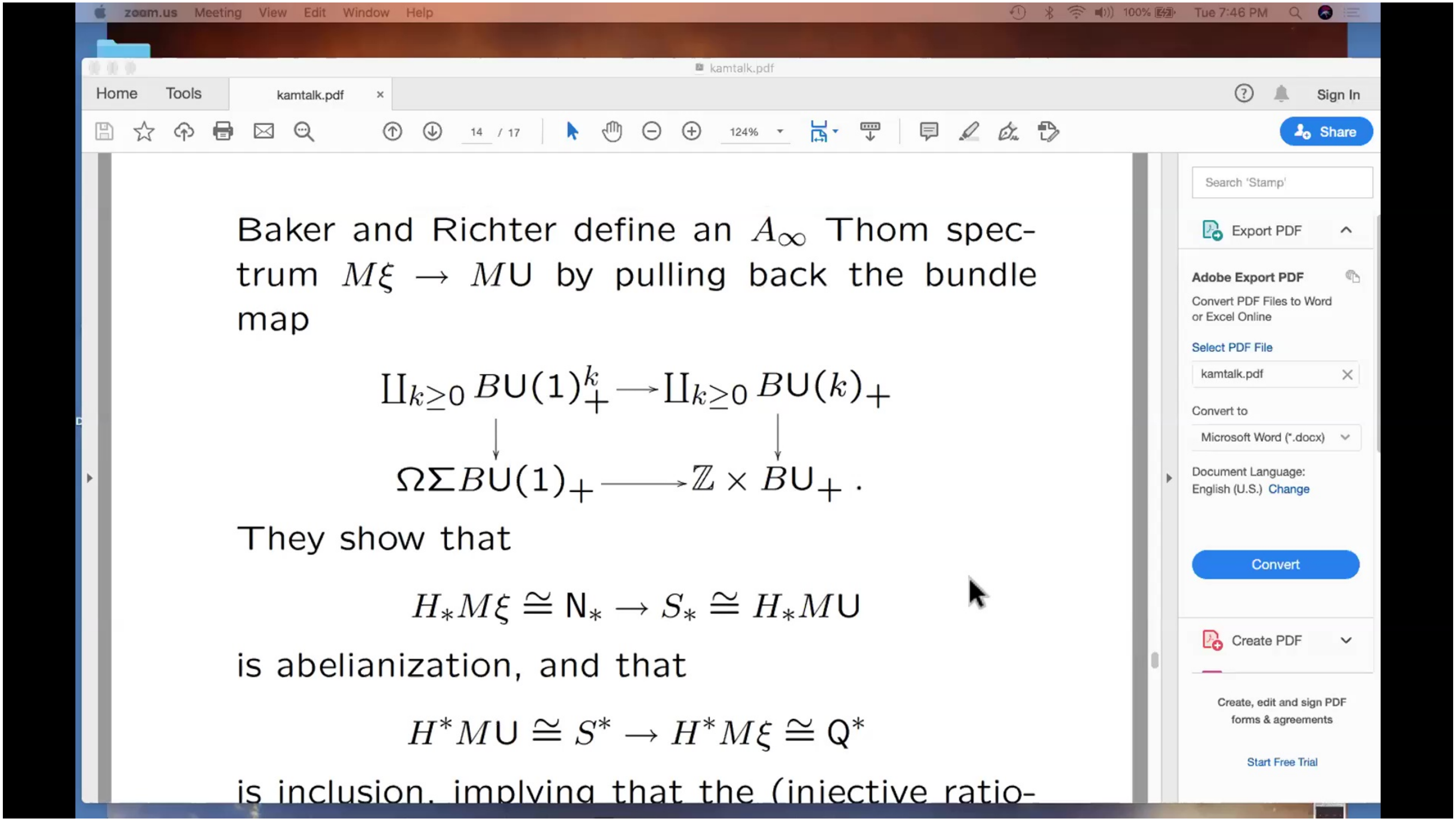
of $n = \sum n_i$.

Manifolds M^{2n} with split tangent bundles $T_M \sim \oplus L_*$ have characteristic classes

$$\langle \mathbf{n} \rangle (\nu_M) = \langle \mathbf{n} \rangle (c_1(-L_*)) \in H^{2n}(M, \mathbb{Z})$$

and thus characteristic numbers

$$[\mathsf{Q}^{2n} \ni \langle \mathbf{n} \rangle \mapsto \langle \mathbf{n} \rangle (\nu_M) \in H^{2n}(M, \mathbb{Z})] \in \mathsf{N}_{2n}$$



Baker and Richter define an A_∞ Thom spectrum $M\xi \to M\mathsf{U}$ by pulling back the bundle map

$$\begin{array}{ccc} \coprod_{k\geq 0} B\mathsf{U}(1)^k_+ & \longrightarrow & \coprod_{k\geq 0} B\mathsf{U}(k)_+ \\ \downarrow & & \downarrow \\ \Omega\Sigma B\mathsf{U}(1)_+ & \longrightarrow & \mathbb{Z} \times B\mathsf{U}_+ \,. \end{array}$$

They show that

$$H_* M\xi \cong \mathsf{N}_* \to S_* \cong H_* M\mathsf{U}$$

is abelianization, and that

$$H^* M\mathsf{U} \cong S^* \to H^* M\xi \cong \mathsf{Q}^*$$

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is inclusion, implying that the (injective rational isomorphism) nal isomorphism)

$$M\xi_* \to H_* M\xi \cong \mathsf{N}_*$$

can be identified with the characteristic number map for (cx-oriented) manifolds with totally split tangent bundles. Note that toric **varieties** are global quotients of toric **manifolds** by finite groups, and thus have rational characteristic numbers.

The spectrum $M\xi \wedge \mathbb{C}P^\infty$ is a plausible repository for Hamiltonian toric cobordism, with $\mathbb{C}P^\infty$ classifying a line bundle with connection encoding the symplectic structure. The homotopy groups $[B\mathbb{T}^n, M\mathrm{U} \wedge B\mathbb{T}]_{2n}$ are an alternative (*cf.* [GGK]) but its grading is more complicated.

Note that [BR] $M\xi_*$ is p-locally a free BP$_*$-module, even though it is not (locally) a BP-algebra spectrum. This provides some justification for regarding $M\xi$ as a noncommutative analog of MU

Recall $MO_*MO = MO_*\otimes\mathbb{Z}[t_*]$ is a tensor product with the Hopf algebra with diagonal $(\Delta t)(T) = (t\otimes 1)((1\otimes t)(T))$, where

$$t(T) = \sum_{k\geq 0} t_k T^{k+1} \; ,$$

represents the group of formal diffeomorphisms of the line.

Conjecture *$M\xi_* M\xi$ injects into $\mathsf{N}_* \otimes \mathsf{N}_*$, with the Brouder-Frabetti-Krattenthaler(-Cauchy) diagonal $(\Delta Z)(T) =$*

$$\mathrm{res}_{U=0}(U^{-1}Z(U) \otimes (1 - U^{-1}Z(T))^{-1})$$

defining a Hopf algebra of formal diffeomorphisms of the noncommutative line.

Speculation: The analog of Miscenko's logarithm for this Hopf algebra defines a version of the cumulant generating function in Voiculescu's theory of free probability, analogous to the Helmholtz Gibbs free energy

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$$\log \mathcal{E}(\exp(tX))$$

in physical chemistry, or to the Cramér risk function in the statistical mechanics of large deviations.