Title: The de Rham model for elliptic cohomology from physics

Speakers: Arnav Tripathy

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Abstract: I'll discuss elliptic cohomology from a physical perspective, indicating the importance of the Segal-Stolz-Teichner conjecture and joint work with D. Berwick-Evans on rigorously proving some of these physical predictions.

Pirsa: 20050053 Page 1/18

# The de Rham model for elliptic cohomology

Arnav Tripathy based on joint work with Dan Berwick-Evans

Harvard University

Elliptic Cohomology and Physics Perimeter, May 2020

Pirsa: 20050053

# Table of Contents

What is elliptic cohomology?

2 The Segal-Stolz-Teichner conjecture

Life goes on

Pirsa: 20050053 Page 3/18

# What **is** elliptic cohomology?

Following Grojnowski, Hopkins, Segal, Stolz-Teichner, Witten

- First, the elliptic genus.
- Given a 2d QFT, we have the partition function

$${
m Tr}_{\mathcal H} e^{eta H} = \int e^{i S} d({
m fields} \ {
m on} \ {
m a \ torus}) = Z( au, \overline{ au})$$

satisfying the crucial property<sup>1</sup>

$$Z( au, \overline{ au}) = Z(-1/ au, -1/\overline{ au}).$$

ullet Simplify: suppose at least  $\mathcal{N}=(0,1)$  supersymmetry and insert  $(-1)^F$  to obtain

$$\operatorname{Tr}_{\mathcal{H}}(-1)^F e^{\beta H} = Z_{EG}(\tau).$$

• Witten index argument:  $Z_{EG}(\tau)$  holomorphic, deformation invariant, i.e. a deformation-invariant modular form on  $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ .



<sup>&</sup>lt;sup>1</sup>Assume all anomalies vanish at present.

## What **is** elliptic cohomology?

- So,  $Z_{EG}(\tau)$  a genus (à la Hirzebruch for  $\sigma$ -models + more) valued in modular forms.
- Behavior in families? What is a "family of modular forms" over some base space B? Supersymmetric QM:  $H^*(B; MF_{\mathbb{C}}) =: Ell(B)_{\mathbb{C}}$ .
- Ok, suppose we want a refinement. Natural idea: simply use deformation classes of families of the 2d  $\mathcal{N}=(0,1)$  theories themselves. [Segal, Stolz-Teichner]
- Other obvious idea: quotient the complex cobordism spectrum MU by the formal group law induced from (families of) elliptic curves. [Landweber-Ravenel-Stong, Hopkins-Mahowald-Miller]
- Conjecture: These constructions agree. Applications: Manifold.

Pirsa: 20050053 Page 5/18

### What **is** elliptic cohomology?

• Conjecture: These constructions agree. Applications: Manifold.

#### Topological Vafa-Witten [Gukov-Pei-Putrov-Vafa]

Consider the 6d  $\mathcal{N}=(0,2)$  theory. Compactifying on an elliptic curve yields 4d  $\mathcal{N}=4$ , and a further twisted compactification on a four-manifold M yields the modular form  $VW_M(\tau)$ . Reversing the order of compactification, the twisted compactification of the 6d theory on M yields a 2d theory whose elliptic genus would return  $VW_M(\tau)$ . Segal-Stolz-Teichner predicts a canonical lift of said modular form to a **topological** modular form. If one instead has a family of such manifolds parametrized by some base-space B (for example, G-symmetry), one obtains a class in  $\mathrm{Ell}_G(B)$ .

Pirsa: 20050053 Page 6/18

# What is equivariant elliptic cohomology?

- First, the equivariant elliptic genus.
- Given a 2d QFT with G flavor symmetry, we have the partition function with background gauge fields

$$\operatorname{Tr}_{\mathcal{H}_{\boldsymbol{g}}} \boldsymbol{h} e^{\beta H} = \int_{\mathsf{twisted b.c.s}} e^{iS} d(\mathsf{fields on a torus}) = Z(\tau, \overline{\tau}, \boldsymbol{g}, \boldsymbol{h})$$

satisfying the crucial property

$$Z(\tau, \overline{\tau}, \mathbf{g}, \mathbf{h}) = Z(-1/\tau, -1/\overline{\tau}, \mathbf{h}^{-1}, \mathbf{g}).$$

ullet Simplify: suppose at least  $\mathcal{N}=(0,1)$  supersymmetry and insert  $(-1)^F$  to obtain

$$\operatorname{Tr}_{\mathcal{H}_{\mathbf{g}}}(-1)^F h e^{\beta H} = Z_{EG}(\tau, \mathbf{g}, \mathbf{h}).$$

• Witten index argument:  $Z_{EG}(\tau, g, h)$  holomorphic, deformation invariant, i.e. a deformation-invariant equivariant modular form on  $\mathrm{SL}_2(\mathbb{Z}) \times G \backslash \mathbb{H} \times C^2(G) =: \mathrm{Bun}_G(\mathcal{E})$ .

Arnav Tripathy (Harvard University)

The de Rham model for elliptic cohomology

Perimeter, May 2020

7 / 17

## What **is** equivariant elliptic cohomology?

- So,  $Z_{EG}(\tau, \mathbf{g}, \mathbf{h})$  a twisted, twined genus (à la Hirzebruch for  $\sigma$ -models + more) valued in equivariant modular forms.
- Behavior in families? What is a "family of modular forms" over some base space B? Supersymmetric QM:  $H^*(B; MF_{G,\mathbb{C}}) =: \mathrm{Ell}(B)_{G,\mathbb{C}}$ .
- Ok, suppose we want a refinement. Natural idea: simply use deformation classes of families of the 2d  $\mathcal{N}=(0,1)$  theories with G flavor symmetry. [Segal, Stolz-Teichner]
- Other obvious idea: build a moduli space of derived algebro-geometric objects, oriented elliptic curves with equivariant structure. [Lurie, Gepner-Meier]
- Even more directly: build a family of algebras directly over  $Bun_G(\mathcal{E})$ , at least over  $\mathbb{C}$ . [Grojnowski]
- Conjecture: These constructions all agree.

Pirsa: 20050053 Page 8/18

# Table of Contents

What is elliptic cohomology?

2 The Segal-Stolz-Teichner conjecture

3 Life goes on

## The equivariant Segal-Stolz-Teichner conjecture

- Conjecture: Phases of 2d  $\mathcal{N} = (0,1)$  theories with G flavor symmetry with some worldsheet torus E and parametrized by a base-space M yield a model for equivariant elliptic cohomology  $\mathrm{Ell}_G(M)$  (as defined in topology).
- Too hard to start with. Let's try a 0-categorical, 0-chromatic height version first.
- Conjecture: The algebra of supersymmetric observables of the 2d  $\mathcal{N}=(0,1)$   $\sigma$ -model to a G-manifold M, with background gauge fields turned on, yields  $\widehat{\mathrm{Ell}}_G(M)_{\mathbb{C}}$  (as defined in Grojnowski, BE-T).
- Additional structure one could ask for:
  - ▶ Universal Euler classes in  $\mathrm{Ell}_{U(n)}(\mathrm{pt}), \mathrm{Ell}_{\mathrm{Spin}(2\mathrm{n})}(\mathrm{pt})$  arising from  $\mathcal{N}=(0,1)$  free fermions in a (complex or real) representation. (Similar statement for Thom classes.) [Ando-Hopkins-Rezk]
  - Specializing the above to U(1) intertwines the natural monoidal structures on both sides. [Ando-Hopkins-Strickland]

Pirsa: 20050053 Page 10/18

#### A theorem!

#### Theorem [Berwick-Evans-T]

For G any compact Lie group and M any compact G-manifold,

#### Theorem [Berwick-Evans-T]

The function in the above model induced by n gauged free fermions agrees with the universal elliptic Euler class of  $\mathrm{Ell}_{U(n)}(\mathrm{pt})$ .

#### Theorem [Berwick-Evans-T]

The multiplicative structure on the universal elliptic Euler class  $\sigma(\tau,z) \in \mathrm{Ell}_{U(1)}(\mathrm{pt})$  induced from multiplying U(1) gauge fields agrees with the (formal) elliptic group law defining elliptic cohomology.

Pirsa: 20050053 Page 11/18

# Idea of proof

#### Definition [Grojnowski]

For G a compact Lie group and M a G-manifold, the fiber of  $\mathrm{Ell}_G(M)_\mathbb{C}$  at  $(g_1,g_2)\in C^2(G)$  is

$$\left(\mathrm{Ell}_G(M)_{\mathbb{C}}\right)_{(g_1,g_2)}:=H^*(M^{(g_1,g_2)};\mathbb{C})[\beta,\beta^{-1}].$$

#### Example

Consider U(1) acting on  $S^2$  by rotation, with fixed points the north and south poles. Then  $\mathrm{Ell}_{U(1)}(S^2)$  is a rank-two vector bundle over  $\mathrm{Bun}_{U(1)}(\mathcal{E})$ .

• So, proof strategy: (i) understand local (super)geometry of  $Bun_G(\mathcal{E})$ , (ii) perform local calculation of the supersymmetric observables as de Rham cohomology, (iii) successfully glue together.

Pirsa: 20050053 Page 12/18

# Table of Contents

What is elliptic cohomology?

2 The Segal-Stolz-Teichner conjecture

3 Life goes on

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Pirsa: 20050053 Page 13/18

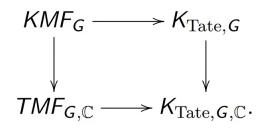
## What **is** elliptic cohomology, physically?

- **EII**(M), a 2-category of boundary conditions of some 3d  $\sigma$ -model with target M. (A priori, needs 3d  $\mathcal{N}=4$  supersymmetry.)
- Ell(M), phases of 2d  $\mathcal{N}=(0,1)$  theories parametrized by M.
- $\mathrm{Ell}(M)_{\mathbb{C}}$ , the BPS Hilbert space of a 3d  $\mathcal{N}=1$   $\sigma$ -model to M on a torus.
- $\mathrm{Ell}(M)_{\mathbb{C}}$ , the algebra of BPS observables for the 2d  $\mathcal{N}=(0,1)$   $\sigma$ -model with torus worldsheet and target M. (And behavior of extended observables?)
- Compare to K-theory: boundary conditions for the B-model, phases of SQMs, BPS Hilbert space of 2d  $\sigma$ -model, algebra of BPS observables for SQMs. ( $K_{\rm top}$  from a category?)
- Why  $K(M)_{\mathbb{C}}$  rather than  $H^*(M;\mathbb{C})$  as the Hilbert space above? Discrete torsion. Functoriality?

Pirsa: 20050053 Page 14/18

#### So, what to attack next?

• Most obviously, return to the Segal-Stolz-Teichner conjecture but with increased chromatic height. We believe we have a model (cf. [Luecke]) for  $KMF_G$  (cf. [Bunke-Naumann]), which fits in the square



- What about the  $\sigma$ -models above where the torus is replaced by a higher-genus surface? Enter  $gll_G(M)$ , which exists over  $\mathbb C$  with necessarily poor integral properties but, for example, should contain the information of  $Z_g(M)$  [Alvarez-Singer].
- M2-branes can end on M5-branes.

Pirsa: 20050053 Page 15/18

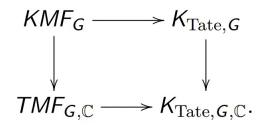
### M-theory and TMF

- The D-brane charge lattice in type II string theories is most naturally K-theory (and type I, KO) as they represent boundary conditions for the fundamental string.
- Consider an F1-ending-on-D4 configuration in IIA and lift to M-theory to obtain an M2-ending-on-M5 configuration.
- Should the charge lattice of M5 branes most naturally be topological modular forms?
- Freed-Moore-Segal suggests TMF should then have some self-Pontryagin duality.
- Indeed,  $Tmf_{\mathbb{C}}$  is self-dual with shift 21 by Serre duality and  $\Delta(\tau)d\tau$  exhibiting  $K_{\overline{\mathcal{M}_1}} \simeq \omega^{-10}$ . [Stojanoska]

Pirsa: 20050053 Page 16/18

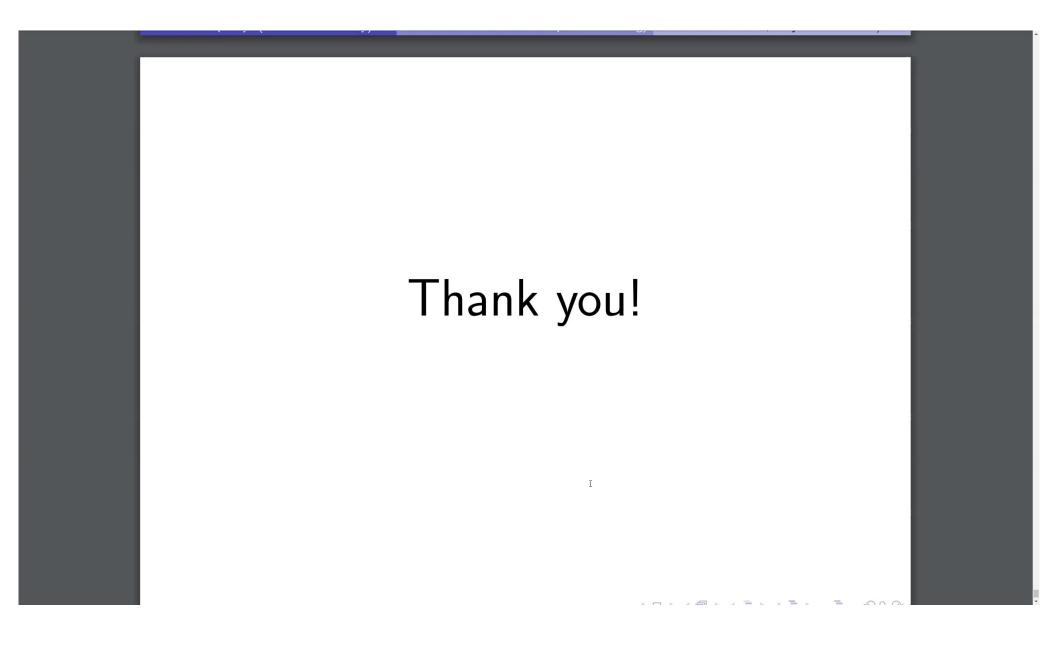
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Pirsa: 20050053 Page 17/18



Pirsa: 20050053 Page 18/18