Title: The de Rham model for elliptic cohomology from physics

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Abstract: I'll discuss elliptic cohomology from a physical perspective, indicating the importance of the Segal-Stolz-Teichner conjecture and joint work with D. Berwick-Evans on rigorously proving some of these physical predictions.

The de Rham model for elliptic cohomology

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based on joint work with Dan Berwick-Evans

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What **is** elliptic cohomology?

Following Grojnowski, Hopkins, Segal, Stolz-Teichner, Witten

- First, the elliptic genus.
- Given a 2d QFT, we have the partition function

$$\mathrm{Tr}_{\mathcal{H}} e^{\beta H} = \int e^{iS} d(\text{fields on a torus}) = Z(\tau, \overline{\tau})$$

satisfying the crucial property[1]

$$Z(\tau, \overline{\tau}) = Z(-1/\tau, -1/\overline{\tau}).$$

- Simplify: *suppose at least* $\mathcal{N} = (0,1)$ *supersymmetry* and insert $(-1)^F$ to obtain

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F e^{\beta H} = Z_{EG}(\tau).$$

- Witten index argument: $Z_{EG}(\tau)$ holomorphic, deformation invariant, i.e. a deformation-invariant modular form on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

[1] Assume all anomalies vanish at present.

What **is** elliptic cohomology?

- So, $Z_{EG}(\tau)$ a genus (à la Hirzebruch for σ-models + more) valued in modular forms.
- Behavior in families? What is a "family of modular forms" over some base space B? Supersymmetric QM: $H^*(B; MF_{\mathbb{C}}) =: \mathrm{Ell}(B)_{\mathbb{C}}$.
- Ok, suppose we want a refinement. Natural idea: simply use deformation classes of families of the 2d $\mathcal{N} = (0,1)$ theories themselves. [Segal, Stolz-Teichner]
- Other obvious idea: quotient the complex cobordism spectrum MU by the formal group law induced from (families of) elliptic curves. [Landweber-Ravenel-Stong, Hopkins-Mahowald-Miller]
- **Conjecture:** These constructions agree. **Applications:** Manifold.

What **is** elliptic cohomology?

- **Conjecture:** These constructions agree. **Applications:** Manifold.

Topological Vafa-Witten [Gukov-Pei-Putrov-Vafa]

Consider the 6d $\mathcal{N} = (0, 2)$ theory. Compactifying on an elliptic curve yields 4d $\mathcal{N} = 4$, and a further twisted compactification on a four-manifold M yields the modular form $VW_M(\tau)$. Reversing the order of compactification, the twisted compactification of the 6d theory on M yields a 2d theory whose elliptic genus would return $VW_M(\tau)$. Segal-Stolz-Teichner predicts a canonical lift of said modular form to a **topological** modular form. If one instead has a family of such manifolds parametrized by some base-space B (for example, G-symmetry), one obtains a class in $\mathrm{Ell}_G(B)$.

What **is** equivariant elliptic cohomology?

- First, the equivariant elliptic genus.
- Given a 2d QFT with G flavor symmetry, we have the partition function with background gauge fields

$$\mathrm{Tr}_{\mathcal{H}_g} h e^{\beta H} = \int_{\text{twisted b.c.s}} e^{iS} d(\text{fields on a torus}) = Z(\tau, \overline{\tau}, g, h)$$

satisfying the crucial property

$$Z(\tau, \overline{\tau}, g, h) = Z(-1/\tau, -1/\overline{\tau}, h^{-1}, g).$$

- Simplify: *suppose at least* $\mathcal{N} = (0,1)$ *supersymmetry* and insert $(-1)^F$ to obtain

$$\mathrm{Tr}_{\mathcal{H}_g} (-1)^F h e^{\beta H} = Z_{EG}(\tau, g, h).$$

- Witten index argument: $Z_{EG}(\tau, g, h)$ holomorphic, deformation invariant, i.e. a deformation-invariant equivariant modular form on $\mathrm{SL}_2(\mathbb{Z}) \times G \backslash \mathbb{H} \times C^2(G) =: \mathrm{Bun}_G(\mathcal{E})$.

What **is** equivariant elliptic cohomology?

- So, $Z_{EG}(\tau, g, h)$ a twisted, twined genus (à la Hirzebruch for σ-models + more) valued in equivariant modular forms.
- Behavior in families? What is a "family of modular forms" over some base space B? Supersymmetric QM: $H^*(B; MF_{G,\mathbb{C}}) =: \mathrm{Ell}(B)_{G,\mathbb{C}}$.
- Ok, suppose we want a refinement. Natural idea: simply use deformation classes of families of the 2d $\mathcal{N} = (0,1)$ theories with G flavor symmetry. [Segal, Stolz-Teichner]
- Other obvious idea: build a moduli space of derived algebro-geometric objects, oriented elliptic curves with equivariant structure. [Lurie, Gepner-Meier]
- Even more directly: build a family of algebras directly over $\mathrm{Bun}_G(\mathcal{E})$, at least over $\mathbb{C}$. [Grojnowski]
- **Conjecture:** These constructions all agree.

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The equivariant Segal-Stolz-Teichner conjecture

- **Conjecture:** Phases of 2d $\mathcal{N} = (0,1)$ theories with G flavor symmetry with some worldsheet torus E and parametrized by a base-space M yield a model for equivariant elliptic cohomology $\mathrm{Ell}_G(M)$ (as defined in topology).
- Too hard to start with. Let's try a 0-categorical, 0-chromatic height version first.
- **Conjecture:** The algebra of supersymmetric observables of the 2d $\mathcal{N} = (0,1)$ σ-model to a G-manifold M, with background gauge fields turned on, yields $\widehat{\mathrm{Ell}}_G(M)_{\mathbb{C}}$ (as defined in Grojnowski, BE-T).
- Additional structure one could ask for:
  - Universal Euler classes in $\mathrm{Ell}_{U(n)}(\mathrm{pt}), \mathrm{Ell}_{\mathrm{Spin(2n)}}(\mathrm{pt})$ arising from $\mathcal{N} = (0,1)$ free fermions in a (complex or real) representation. (Similar statement for Thom classes.) [Ando-Hopkins-Rezk]
  - Specializing the above to $U(1)$ intertwines the natural monoidal structures on both sides. [Ando-Hopkins-Strickland]

A theorem!

Theorem [Berwick-Evans–T]

For G any compact Lie group and M any compact G-manifold,

$$\mathcal{O}\Big(\mathrm{Maps}_0((\Pi\mathbb{S})E, [M/\!\!/^{\nabla}G])\Big) \simeq \widehat{\mathrm{Ell}}_G(M).$$

Theorem [Berwick-Evans–T]

The function in the above model induced by n gauged free fermions agrees with the universal elliptic Euler class of $\mathrm{Ell}_{U(n)}(\mathrm{pt})$.

Theorem [Berwick-Evans–T]

The multiplicative structure on the universal elliptic Euler class $\sigma(\tau, z) \in \mathrm{Ell}_{U(1)}(\mathrm{pt})$ induced from multiplying $U(1)$ gauge fields agrees with the (formal) elliptic group law defining elliptic cohomology.

Idea of proof

Definition [Grojnowski]

For G a compact Lie group and M a G-manifold, the fiber of $\mathrm{Ell}_G(M)_{\mathbb{C}}$ at $(g_1, g_2) \in C^2(G)$ is

$$\left(\mathrm{Ell}_G(M)_{\mathbb{C}}\right)_{(g_1,g_2)} := H^*(M^{(g_1,g_2)}; \mathbb{C})[\beta, \beta^{-1}].$$

Example

Consider $U(1)$ acting on S^2 by rotation, with fixed points the north and south poles. Then $\mathrm{Ell}_{U(1)}(S^2)$ is a rank-two vector bundle over $\mathrm{Bun}_{U(1)}(\mathcal{E})$.

- So, proof strategy: (i) understand local (super)geometry of $\mathrm{Bun}_G(\mathcal{E})$, (ii) perform local calculation of the supersymmetric observables as de Rham cohomology, (iii) successfully glue together.

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What **is** elliptic cohomology, physically?

- **Ell**(M), a 2-category of boundary conditions of some 3d σ-model with target M. (*A priori*, needs 3d $\mathcal{N} = 4$ supersymmetry.)
- $\mathrm{Ell}(M)$, phases of 2d $\mathcal{N} = (0,1)$ theories parametrized by M.
- $\mathrm{Ell}(M)_{\mathbb{C}}$, the BPS Hilbert space of a 3d $\mathcal{N} = 1$ σ-model to M on a torus.
- $\mathrm{Ell}(M)_{\mathbb{C}}$, the algebra of BPS observables for the 2d $\mathcal{N} = (0,1)$ σ-model with torus worldsheet and target M. (And behavior of extended observables?)
- Compare to K-theory: boundary conditions for the B-model, phases of SQMs, BPS Hilbert space of 2d σ-model, algebra of BPS observables for SQMs. (K_{top} from a category?)
- Why $K(M)_{\mathbb{C}}$ rather than $H^*(M;\mathbb{C})$ as the Hilbert space above? Discrete torsion. Functoriality?

So, what to attack next?

- Most obviously, return to the Segal-Stolz-Teichner conjecture but with increased chromatic height. We believe we have a model (cf. [Luecke]) for KMF_G (cf. [Bunke-Naumann]), which fits in the square

$$\begin{array}{ccc} KMF_G & \longrightarrow & K_{\mathrm{Tate},G} \\ \downarrow & & \downarrow \\ TMF_{G,\mathbb{C}} & \longrightarrow & K_{\mathrm{Tate},G,\mathbb{C}}. \end{array}$$

- What about the σ-models above where the torus is replaced by a higher-genus surface? Enter $g\mathrm{ll}_G(M)$, which exists over $\mathbb{C}$ with necessarily poor integral properties but, for example, should contain the information of $Z_g(M)$ [Alvarez-Singer].
- M2-branes can end on M5-branes.

M-theory and TMF

- The D-brane charge lattice in type II string theories is most naturally K-theory (and type I, KO) as they represent boundary conditions for the fundamental string.
- Consider an F1-ending-on-D4 configuration in IIA and lift to M-theory to obtain an M2-ending-on-M5 configuration.
- Should the charge lattice of M5 branes most naturally be topological modular forms?
- Freed-Moore-Segal suggests TMF should then have some self-Pontryagin duality.
- Indeed, $Tmf_{\mathbb{C}}$ is self-dual with shift 21 by Serre duality and $\Delta(\tau)d\tau$ exhibiting $K_{\overline{\mathcal{M}_1}} \simeq \omega^{-10}$. [Stojanoska]

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Thank you!