Title: Projective elliptic genera and applications

Speakers: Fei Han

Collection: Elliptic Cohomology and Physics

Date: May 25, 2020 - 9:00 AM

URL: http://pirsa.org/20050050

Abstract: Projective vector bundles (or gerbe modules) are generalizations of vector bundles in the presence of a gerbe on manifolds. Given a projective vector bundle, we will first show how to use it to twist the Witten genus to get modular invariants, which we call projective elliptic genera. Then we will give two applications: (1) given any pseudodifferential operator, we will construct modular invariants generalizing the Witten genus, which corresponds to the Dirac operator; (2) we will enhance the Hori map in T-duality to the graded Hori map and show that it sends Jacobi forms to Jacobi forms. This represents our joint works with Mathai.

下午9:01 5月25日周一 93%

Proj ell and app

Projective elliptic genera and applications

HAN Fei

National University of Singapore

Based on the following papers :

1. Fei Han and Varghese Mathai, *Projective Elliptic Genera and Elliptic Pseudodifferential Genera*, Advances in Mathematics, 358 (2019).

2.Fei Han and Varghese Mathai, *T-duality, Jacobi forms and Witten Gerbe Modules*, arXiv: 2001.00322.

Topological Side

By Hirzebruch, a genus is a ring homomorphism

$$\varphi : \Omega^*_{SO} \otimes \mathbb{Q} \to R,$$

where R is an integral domain over $\mathbb{Q}$.

$$g(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} \varphi(\mathbb{C}P^n) z^{n+1}$$

is called the logarithm of φ.
Let $f = g^{-1}$. Then following Ochanine, φ is called an elliptic genus if f satisfies the ODE

$$(f')^2 = 1 - 2\delta f + \epsilon f^4,$$

where ϵ, δ are two paramaters. Or in other words, (f, f') parametrize the elliptic curve

$$y^2 = 1 - 2\delta x + \epsilon x^4.$$

Topological Side

Equivalently, f satisfies

$$f(u+v) = \frac{f(u)f'(v) + f'(u)f(v)}{1 - \epsilon f(u)^2 f(v)^2}.$$

Denotes $y_1 = f(u)$, $y_2 = f(v)$, $F(y_1, y_2) = f(f^{-1}(y_1) + f^{-1}(y_2))$, then F has the formal group law

$$F(y_1 + y_2) = \frac{y_1\sqrt{P(y_2)} + y_2\sqrt{P(y_1)}}{1 - \epsilon y_1^2 y_2^2} = y_1 + y_2 + \cdots,$$

where $P(x) = 1 - 2\delta x^2 + \epsilon x^4$.

Topological Side

The Jacobi theta functions are defined as follows :

$$\theta(v,\tau) = 2q^{1/8}\sin(\pi v)\prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^j)(1-e^{-2\pi\sqrt{-1}v}q^j)\right],$$

$$\theta_1(v,\tau) = 2q^{1/8}\cos(\pi v)\prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^j)(1+e^{-2\pi\sqrt{-1}v}q^j)\right],$$

$$\theta_2(v,\tau) = \prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^{j-1/2})(1-e^{-2\pi\sqrt{-1}v}q^{j-1/2})\right],$$

$$\theta_3(v,\tau) = \prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^{j-1/2})(1+e^{-2\pi\sqrt{-1}v}q^{j-1/2})\right].$$

They are all holomorphic functions for $(v,\tau)\in\mathbb{C}\times\mathbb{H}$, where $\mathbb{C}$ is the complex plane and $\mathbb{H}$ is the upper half plane. $q=e^{2\pi i\tau}$.

Topological Side

The Jacobi theta functions are defined as follows :

$$\theta(v,\tau)=2q^{1/8}\sin(\pi v)\prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^j)(1-e^{-2\pi\sqrt{-1}v}q^j)\right],$$

$$\theta_1(v,\tau)=2q^{1/8}\cos(\pi v)\prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^j)(1+e^{-2\pi\sqrt{-1}v}q^j)\right],$$

$$\theta_2(v,\tau)=\prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^{j-1/2})(1-e^{-2\pi\sqrt{-1}v}q^{j-1/2})\right],$$

$$\theta_3(v,\tau)=\prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^{j-1/2})(1+e^{-2\pi\sqrt{-1}v}q^{j-1/2})\right].$$

They are all holomorphic functions for $(v,\tau)\in\mathbb{C}\times\mathbb{H}$, where $\mathbb{C}$ is the complex plane and $\mathbb{H}$ is the upper half plane. $q=e^{2\pi i\tau}$.

Topological Side

Then we have

$$f_1(u,\tau) = 2u\frac{\theta'(0,\tau)}{\theta(u,\tau)}\frac{\theta_1(u,\tau)}{\theta_1(0,\tau)},$$

$$f_2(u,\tau) = u\frac{\theta'(0,\tau)}{\theta(u,\tau)}\frac{\theta_2(u,\tau)}{\theta_2(0,\tau)},$$

Let M be a $4i$-dimensional closed smooth oriented manifold. Let $\{\pm 2\pi\sqrt{-1}x_i\}$ be the formal Chern roots for $TM \otimes \mathbb{C}$. Define

$$\Phi_L(M,\tau) = \left\langle \prod_{i=1}^{2k} f_1(\frac{x_i}{2\pi i},\tau),[M]\right\rangle,$$

$$\Phi_W(M,\tau) = \left\langle \prod_{i=1}^{2k} f_2(\frac{x_i}{2\pi i},\tau),[M]\right\rangle.$$

$\Phi_L(M,\tau), \Phi_W(M,\tau)$ are called the elliptic genera of M.

Topological Side

Let

$$f(u,\tau) = u\frac{\theta'(0,\tau)}{\theta(u,\tau)}.$$

Define

$$\Psi_W(M,\tau) = \left\langle \prod_{i=1}^{2k} f\!\left(\frac{x_i}{2\pi i},\tau\right), [M] \right\rangle.$$

$\Psi_W(M,\tau)$ is called the Witten genus of M.

Topological Side

Proposition (Zagier)

1) $\Phi_L(M,\tau)$ *is a modular form of weight 2i over* $\Gamma_0(2)$ *;*
$\Phi_W(M,\tau)$ *is a modular form of weight 2i over* $\Gamma^0(2)$ *;*
if the first rational Pontrjagin class $p_1(M)=0$*, then*
$\Psi_W(M,\tau)$ *is a modular form of weight 2i over* $SL_2(\mathbb{Z})$*.*
2) the following equality hold,

$$\Phi_L(M,-1/\tau)=(2\tau)^{2i}\Phi_W(M,\tau).$$

Topological Side

A spin manifold M is called string if $\frac{1}{2}p_1(M) = 0$. This means the free loop space LM is spin. To make sense out of this, important works by Killingback, McLaughlin, Stolz-Teichner, Waldorf, Kottke-Melrose.

The homotopy lifting of Witten genus lifts leads to the theory *tmf*(topological modular forms) developed by Hopkins-Miller and the σ-orientation :

$$\sigma : MString \to tmf.$$

Geometric Side

Let M be a $4k$-dimensional oriented Riemannian manifold. Let ∇^{TM} be the associated Levi-Civita connection on TM and $R^{TM} = (\nabla^{TM})^2$ be the curvature of ∇^{TM}. Define

$$\Phi_L(M,\tau) = \int_M \det{}^{\frac{1}{2}}\left(\frac{R^{TM}}{2\pi^2}\frac{\theta'(0,\tau)}{\theta(\frac{R^{TM}}{4\pi^2},\tau)}\frac{\theta_1(\frac{R^{TM}}{4\pi^2},\tau)}{\theta_1(0,\tau)}\right) \in \mathbb{Q}[[q^{1/2}]],$$

$$\Phi_W(M,\tau) = \int_M \det{}^{\frac{1}{2}}\left(\frac{R^{TM}}{4\pi^2}\frac{\theta'(0,\tau)}{\theta(\frac{R^{TM}}{4\pi^2},\tau)}\frac{\theta_2(\frac{R^{TM}}{4\pi^2},\tau)}{\theta_2(0,\tau)}\right) \in \mathbb{Q}[[q^{1/2}]],$$

$$\Psi_W(M,\tau) = \int_M \det{}^{\frac{1}{2}}\left(\frac{R^{TM}}{4\pi^2}\frac{\theta'(0,\tau)}{\theta(\frac{R^{TM}}{4\pi^2},\tau)}\right) \in \mathbb{Q}[[q]].$$

$\Phi_L(M,\tau), \Phi_W(M,\tau)$ are called the elliptic genera of M, and $\Psi_W(M,\tau)$ is called the Witten genus of M.

Geometric Side

By the Chern-Weil theory of realizing characteristic classes via curvatures, we see that this topological definition coincide with the geometric definition.

The advantage of the geometric definition is that the definition is local, namely we get the elliptic density and Witten density, and therefore more flexible. For example, we can perform Chern-Simons transgression on them.

Geometric Side

Twisted by a vector bundle

Let (V, g^V) be a Euclidean vector bundle on M. Let ∇^V be an Euclideian connection on V and $R^V = (\nabla^V)^2$ be the curvature of ∇^V. Define

$$\Phi_L(M, V, \tau) = \int_M \det{}^{\frac{1}{2}} \left(\frac{R^{TM}}{2\pi^2} \frac{\theta'(0,\tau)}{\theta(\frac{R^{TM}}{4\pi^2}, \tau)} \frac{\theta_1(\frac{R^V}{4\pi^2}, \tau)}{\theta_1(0,\tau)} \right) \in \mathbb{Q}[[q^{1/2}]],$$

$$\Phi_W(M, V, \tau) = \int_M \det{}^{\frac{1}{2}} \left(\frac{R^{TM}}{4\pi^2} \frac{\theta'(0,\tau)}{\theta(\frac{R^{TM}}{4\pi^2}, \tau)} \frac{\theta_2(\frac{R^V}{4\pi^2}, \tau)}{\theta_2(0,\tau)} \right) \in \mathbb{Q}[[q^{1/2}]],$$

$\Phi_L(M, V, \tau), \Phi_W(M, V, \tau)$ are called the V-twisted elliptic genera. When $V = TM$, we return to the elliptic genera of M.

Analytic Side

Let E, F be two Hermitian vector bundles over M carrying Hermitian connections ∇^E, ∇^F respectively. Let $R^E = (\nabla^E)^2$ (resp. $R^F = (\nabla^F)^2$) be the curvature of ∇^E (resp. ∇^F).

Let $\widehat{A}(TM, \nabla^{TM})$ and $\widehat{L}(TM, \nabla^{TM})$ be the Hirzebruch characteristic forms defined respectively by

$$\widehat{A}(TM, \nabla^{TM}) = \det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^{TM}}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^{TM}\right)}\right),$$

$$\widehat{L}(TM, \nabla^{TM}) = \det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{2\pi}R^{TM}}{\tanh\left(\frac{\sqrt{-1}}{4\pi}R^{TM}\right)}\right).$$

If we set the formal difference $G = E - F$, then G carries an induced Hermitian connection ∇^G in an obvious sense. We define the associated Chern character form

$$\mathrm{ch}(G, \nabla^G) = \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right] - \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^F\right)\right].$$

Analytic Side

For any complex number t, let

$$\Lambda_t(E) = \mathbb{C}|_M + tE + t^2\Lambda^2(E) + \cdots, S_t(E) = \mathbb{C}|_M + tE + t^2 S^2(E) + \cdots$$

denote respectively the total exterior and symmetric powers of E, which live in $K(M)[[t]]$.

Analytic Side

If W is a real Euclidean vector bundle over M carrying a Euclidean connection ∇^W, then its complexification $W_{\mathbb{C}} = W \otimes \mathbb{C}$ is a complex vector bundle over M carrying a canonically induced Hermitian metric from that of W, as well as a Hermitian connection $\nabla^{W_{\mathbb{C}}}$ induced from ∇^W. If E is a vector bundle (complex or real) over M, set $\widetilde{E} = E - \dim E$ in $K(M)$ or $KO(M)$.
Set

$$\Theta(TM) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}).$$

$$\Theta_1(TM) = \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{T_{\mathbb{C}}M}),$$

$$\Theta_2(TM) = \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{T_{\mathbb{C}}M}),$$

Analytic Side

Suppose M is a spin manifold. Denote by D the Dirac operator and by B_M the signature operator. Define

$$\Psi_W(M,\tau) = Ind(D\otimes\Theta(TM)) = \int_M \widehat{A}(M)\mathrm{ch}(\Theta(TM)) \in \mathbb{Z}[[q]].$$

$$\begin{aligned}\Phi_L(M,\tau) =& Ind(B_M\otimes\Theta(TM)\otimes\Theta_1(TM))\\ =& \int_M \widehat{L}(M)\mathrm{ch}(\Theta(TM)\otimes\Theta_1(TM)) \in \mathbb{Z}[[q^{1/2}]],\end{aligned}$$

$$\begin{aligned}\Phi_W(M,\tau) =& Ind(D\otimes\Theta(TM)\otimes\Theta_2(TM))\\ =& \int_M \widehat{A}(M)\mathrm{ch}(\Theta(TM)\otimes\Theta_2(TM)) \in \mathbb{Z}[[q^{1/2}]]\end{aligned}$$

Analytic Side

This analytic definition coincide with the geometric definition by the Atiyah-Singer index theorem. Moreover, the elliptic genera and Witten genus are refined to be modular forms with integral Fourier development on spin manifolds.

We call the operators $D \otimes \Theta(TM), B_M \otimes \Theta_1(TM), D \otimes \Theta_2(TM)$ the Witten operators. They can be formally viewed as the Signature, Dirac operaors on LM, the free loop space of M.

Analytic Side

Let V be a spin vector bundle over M.

Define V-twisted elliptic genera

$$\Phi_L(M, V, \tau) = Ind(D \otimes \Theta(TM) \otimes (S^+(V) + S^-(V)) \otimes \Theta_1(V))$$
$$= \int_M \widehat{A}(M)\mathrm{ch}(\Theta(TM))\mathrm{ch}((S^+(V) + S^-(V)) \otimes \Theta_1(V)) \in \mathbb{Z}[[q^{1/2}]],$$

$$\Phi_W(M, V, \tau) = Ind(D \otimes \Theta(TM) \otimes \Theta_2(V))$$
$$= \int_M \widehat{A}(M)\mathrm{ch}(\Theta(TM))\mathrm{ch}(\Theta_2(V)) \in \mathbb{Z}[[q^{1/2}]].$$

Analytic Side

$$(S^+(V) + S^-(V)) \otimes \Theta_1(V) = (S^+(V) + S^-(V)) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{V_{\mathbb{C}}}),$$

$$\Theta_2(V) = \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{V_{\mathbb{C}}})$$

formally are vector bundles over loop space LM by Witten.

The Mathai-Melrose-Singer index theory in the nonspin or nonspin^c settings leads us to construct projective elliptic genera.

The Mathai-Melrose-Singer Projective Index Theory : Projective Dirac operators fractional analytic index theorem

V. Mathai, R.B. Melrose and I.M. Singer,
Fractional Analytic Index,
J. Differential Geometry, 74, (2006), no. 2, 265-292.
[math.DG/0402329]

The Index Theorem for Dirac operators

Atiyah and Singer defined the Dirac operator, $\partial\!\!\!/^{+}$ on any compact spin manifold Z of even dimension, and computed the analytic index,

$$\begin{aligned}\mathrm{index}_a(\partial\!\!\!/^{+}) &= \dim(\mathrm{nullspace}\,\partial\!\!\!/^{+}) - \dim(\mathrm{nullspace}\,\partial\!\!\!/^{-}) \\ &= \int_Z \widehat{A}(Z) \quad \in \mathbb{Z}\end{aligned}$$

Question Since $\int_Z \widehat{A}(Z) \notin \mathbb{Z}$ continues to make sense for non-spin manifolds Z, what corresponds to the analytic index in this situation, since the usual Dirac operator does not exist ?

MMS generalize the notion of "ψdo", to "projective ψdo".
In particular, on an oriented even dimensional Riemannian manifold, MMS define the notion of projective spin Dirac operator. They define its fractional analytic index, and prove an index theorem showing that it equals the $\widehat{A}$-genus.

Bundle gerbes

A degree 3 analogue of complex line bundles are "bundles gerbes" due to Murray.

A special case consists of a principal $PU(n)$-bundle,

$$PU(n) \longrightarrow Y \xrightarrow{\phi} Z$$

and a central extension of the structure group

$$U(1) \longrightarrow U(n) \longrightarrow PU(n)$$

Bundle gerbes

The Dixmier-Douady invariant of Y,

$$DD(Y) = \delta(Y) \in \mathrm{Torsion}(H^3(Z, \mathbb{Z}))$$

is the obstruction to lifting the principal $PU(n)$-bundle Y to a principal $U(n)$-bundle.

The associated algebra bundle

$$\mathcal{A} = Y \times_{PU(n)} M_n(\mathbb{C})$$

is called the associated Azumaya bundle.

Projective vector bundles

A projective vector bundle on a manifold Z is not a global bundle on Z, but rather it is a vector bundle $\mathcal{E} \to Y$, where $\mathcal{E}$ also satisfies

$$\mathcal{L}_g \otimes \mathcal{E}_y \cong \mathcal{E}_{g.y}, \qquad g \in PU(n),\ y \in Y \qquad (1)$$

where $\mathcal{L} = U(n) \times_{U(1)} \mathbb{C} \to PU(n)$ is the primitive line bundle,

$$\mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2} \cong \mathcal{L}_{g_1.g_2}, \qquad g_i \in PU(n).$$

The identification (1) gives a projective action of $PU(n)$ on $\mathcal{E}$, i.e. an action of $U(n)$ on $\mathcal{E}$ s.t. the center $U(1)$ acts as scalars.

For a projective vector bundle, there is a twisted Chern character $\mathrm{Ch}_B\mathcal{E}$, where B is the B-field of the bundle gerbe Y, defined by Bouwknegt-Carey-Mathai-Murray-Stevenson.

Projective vector bundle of spinors

Let $E \to Z$ be a real oriented Riemannian vector bundle,

$$SO(n) \longrightarrow SO(E) \xrightarrow{\psi} Z$$

the principal bundle of oriented orthonormal frames on E.

Let N denote the (co)normal bundle to the fibres. Then it is easy to see that $w_2(N) = 0$, so that N always has a bundle of spinors $\mathcal{S}$, which is a projective vector bundle over Z.

Also $End(\mathcal{S}) \cong \psi^* Cl(E)$.

Projective spin Dirac operator

There is always a projective bundle of spinors $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ on any even dimensional oriented manifold Z, having the property that $\hom(\mathcal{S}, \mathcal{S}) \cong Cl(Z)$.

There are natural spin connections on the Clifford algebra bundle $Cl(Z)$ and $\mathcal{S}^\pm$ induced from the Levi-Civita connection on T^*Z.

Recall also that $\hom(\mathcal{S}, \mathcal{S}) \cong Cl(Z)$, has an extension to $\tilde{Cl}(Z)$ in a tubular neighbourhood of the diagonal Δ, with an induced connection ∇, called the projective spin connection.

The projective spin Dirac operator is defined as the distributional section

$$\not{\partial} = cl \cdot \nabla_L(\kappa_{Id}), \qquad \kappa_{Id} = \delta(z - z')Id_S$$

Here ∇_L is the projective spin connection ∇ restricted to the left variables with cl the contraction given by the Clifford action of T^*Z on the left.

As in the usual case, the projective spin Dirac operator $\not{\partial}$ is elliptic and odd wrt $\mathbb{Z}_2$ grading of $\mathcal{S}$.

Index of projective spin Dirac operators

Theorem (Mathai-Melrose-Singer)

The projective spin Dirac operator on an even-dimensional compact oriented manifold Z, has fractional analytic index,

$$\mathrm{index}_a(\partial\!\!\!/^+) = \int_Z \widehat{A}(Z) \in \mathbb{Q}.$$

Recall that $Z = \mathbb{C}P^{2n}$ is an oriented but non-spin manifold such that $\int_Z \widehat{A}(Z) \notin \mathbb{Z}$, justifying the title of the talk. e.g.

$$Z = \mathbb{C}P^2 \quad \Longrightarrow \quad \mathrm{index}_a(\partial\!\!\!/^+) = -1/8.$$
$$Z = \mathbb{C}P^4 \quad \Longrightarrow \quad \mathrm{index}_a(\partial\!\!\!/^+) = 3/128.$$

Index of projective spin Dirac operators

Twisted by a projective vector bundle

Let $\mathcal{E}$ be a projective vector bundle over Z.

Theorem (Mathai-Melrose-Singer)

The projective spin Dirac operator on an even-dimensional compact oriented manifold Z, has fractional analytic index,

$$\mathrm{index}_a(\partial\!\!\!/^+ \otimes \mathcal{E}) = \int_Z \widehat{A}(Z)\mathrm{Ch}_B(\mathcal{E}) \in \mathbb{Q}.$$

Projective Witten bundles and graded twisted Chern character

Let $\mathcal{E}$ be a projective vector bundle over Z. Construct the Witten projective bundle

$$\Theta_2(\mathcal{E}) = \bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-\frac{1}{2}}}(\mathcal{E}) \otimes \bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-\frac{1}{2}}}(\bar{\mathcal{E}})$$

Formally the projective vector bundle over loop space LZ.

Define the projective elliptic genus in an analytic way

$$\Phi_W(M, \mathcal{E}, \tau) = \mathrm{index}_a(\not{\partial}^+ \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}} M}) \otimes \Theta_2(\mathcal{E})),$$

where ∂^+ is the projective elliptic operator.

Projective Witten bundles and graded twisted Chern character

To get a geometric formula for this via the Mathai-Melrose-Singer index theorem, we need to introduce a graded twisted Chern character

$$\mathrm{GCh}_B(\Theta_2(\mathcal{E})),$$

a loop version of the twisted BCMMS Chern character, counting in various levels in the Fourier expansion of $\Theta_2(\mathcal{E})$.

Projective Witten bundles and graded twisted Chern character

Then by the Mathai-Melrose-Singer projective index theorem, we have

$$
\begin{aligned}
&\Phi_W(M,\mathcal{E},\tau)\\
=&\mathrm{Index}_a\left(\partial\!\!\!/^+\otimes\bigotimes_{n=1}^{\infty}S_{q^n}(\widetilde{T_{\mathbb{C}}M})\otimes\Theta_2(\mathcal{E})\right)\\
=&\int_Z\widehat{A}(Z)\mathrm{Ch}\left(\bigotimes_{n=1}^{\infty}S_{q^n}(\widetilde{T_{\mathbb{C}}M})\right)\mathrm{GCh}_B(\Theta_2(\mathcal{E}))\in\mathbb{Q}[[q^{1/2}]].
\end{aligned}
$$

Projective Witten bundles and graded twisted Chern character

We can show that the geometric formula

$$\Phi_W(M,\mathcal{E},\tau) = \int_Z \det{}^{\frac{1}{2}}\left(\frac{R^{TZ}}{4\pi^2}\frac{\theta'(0,\tau)}{\theta(\frac{R^{TZ}}{4\pi^2},\tau)}\frac{\theta_2(\frac{B+R^{\mathcal{E}}}{4\pi^2},\tau)}{\theta_2(0,\tau)}\right),$$

where the B-field is inserted.

Theorem (H.-Mathai)

If $p_1(TZ) = \mathfrak{p}_1(\mathcal{E})$, then $\Phi_W(M,\mathcal{E},\tau)$ is a rational modular form.

The first application of the projective elliptic genera is constructing projective elliptic pseudodifferential genera for any projective elliptic pseudodifferential operator.

Our construction of elliptic pseudodifferential genera suggests the existence of putative S^1-equivariant elliptic pseudodifferential operators on loop space that localises to the elliptic pseudodifferential genera, by a formal application of the Atiyah-Segal-Singer localisation theorem.

We also compute the elliptic pseudodifferential genera for some concrete elliptic pseudodifferential operators.

Let Z be an $4r$-dimensional compact oriented smooth manifold.

Let $W_3(Z) \in H^3(M, \mathbb{Z})$ be the third integral Stiefel-Whitney class.

Fix a projective spin^c structure on Z and let $S_c^{\pm}(TZ)$ be the complex projective spin^c bundle of TZ with twist $W_3(Z)$.

Denote by $S_c(TZ)$ the bundle $S_c^+(TZ) \oplus S_c^-(TZ)$.

Let $TZ^{\perp}$ be the stable complement of the tangent bundle TZ. Let $S_c^{\pm}(TZ^{\perp})$ be the projective spin^c bundle of $TZ^{\perp}$ with twist $-W_3(Z)$.

Denote by $S_c(TZ^{\perp})$ the bundle $S_c^+(TZ^{\perp}) \oplus S_c^-(TZ^{\perp})$.

Let $\nabla^{S_c^{\pm}(TZ^{\perp})}$ be projective Hermitian connections on $S_c^{\pm}(TZ^{\perp})$. Denote by $\nabla^{S_c(TZ^{\perp})}$ the $\mathbb{Z}_2$-graded projective Hermitian connection on $S_c(TZ^{\perp})$.

Let $P : C^\infty(F_0) \to C^\infty(F_1)$ be a projective pseudodifferential elliptic operator with F_0 and F_1 being projective Hermitian vector bundles over Z with twist H.

Let ∇_0 and ∇_1 be projective Hermitian connections on F_0, F_1 respectively.

Denote by ∇^F the $\mathbb{Z}_2$-graded projective Hermitian connection on the bundle $F = F_0 \oplus F_1$.

There exists $m \in \mathbb{Z}^+$ and projective complex vector bundle E on Z with twist $H - W_3(Z)$ such that $TM \oplus TZ^\perp \cong M \times \mathbb{R}^m$ and $F \otimes S_c(TZ^\perp) = E^{\oplus 2^m}$.

Suppose the rank of E is l. Define *the first rational Pontryagin class of* P by

$$\mathfrak{p}_1(P) := \mathfrak{p}_1(E) \in H^4(Z, \mathbb{Q}).$$

It is clear that the it is well defined.

Let $\mathbb{S}_\lambda$ be the *Schur functor* (c.f. Fulton-Harris) They are indexed by Young diagram λ and are functors from the category of vector spaces to itself . The Schur functor is a continuous functor and therefore if (E, ∇^E) is a vector bundle with connection, then applying the Schur functor gives us a vector bundle with connection $(\mathbb{S}_\lambda(E), \mathbb{S}_\lambda(\nabla^E))$. If U, V be two vector spaces, the exterior power of a tensor product has the following nice expression via the Schur functors :

$$\Lambda^n(U \otimes V) = \bigoplus \mathbb{S}_\lambda(U) \otimes \mathbb{S}_{\lambda'}(V), \tag{1}$$

where $\mathbb{S}_\lambda$ is the Schur functor with λ running over all the Young diagram with n cells, at most $\dim(U)$ rows, $\dim(V)$ columns, and λ' being the transposed Young diagram. Hence on the projective bundle $\Lambda^n(F \otimes S_c(TZ^\perp))$, there is a projective Hermitian connection

$$\bigoplus \left(\mathbb{S}_\lambda(\nabla^F) \otimes 1 + 1 \otimes \mathbb{S}_{\lambda'}(\nabla^{S_c(TZ^\perp)}) \right), \tag{2}$$

Denote this projective bundle with connection by $\Lambda^n(\nabla^F, \nabla^{S_c(TZ^\perp)})$.

In the following, when we write $\psi(\nabla^F, S_c(TZ^\perp))$, where ψ is certain operations on vector bundles constructed from exterior power, it always means the connections constructed in this way. For instance,

$$\Theta(\nabla^F, \nabla^{S_c(TZ^\perp)}) = \bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(\nabla^F, \nabla^{S_c(TZ^\perp)}) \otimes \bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(\overline{\nabla^F}, \overline{\nabla^{S_c(TZ^\perp)}})$$

is a Hermitian connection on the q-series with virtual projective bundle coefficients,

$$\Theta(F \otimes S_c(TZ^\perp)) = \bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(F \otimes S_c(TZ^\perp)) \otimes \bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(\overline{F \otimes S_c(TZ^\perp)}).$$

Definition

For the (projective) elliptic pseudodifferential operator P, define the (projective) elliptic pseudodifferential genera $Ell(P,\tau)$

$$Ell(P,\tau)=\left(\prod_{j=1}^{\infty}(1-q^j)\right)^{4r-2l}\cdot\int_Z\widehat{A}(Z)\mathrm{Ch}(\Theta(TZ))\mathbb{H}(\nabla^F,\nabla^{S_c(TZ^\perp)}),$$

where

$$\mathbb{H}(\nabla^F,\nabla^{S_c(TZ^\perp)})=\mathrm{Ch}^{\frac{1}{2^{m+1}}}_{-l(H\text{-}W_3(Z))}\det(\overline{\nabla^F},\overline{\nabla^{S_c(TZ^\perp)}})\cdot\mathrm{Ch}^{\frac{1}{2^m}}_{H\text{-}W_3(Z)}$$
$$\left((\wedge^{even}(\nabla^F,\nabla^{S_c(TZ^\perp)})-\wedge^{odd}(\nabla^F,\nabla^{S_c(TZ^\perp)}))\otimes\Theta(\nabla^F,\nabla^{S_c(TZ^\perp)})\right),$$

Theorem (H.-Mathai)

If $p_1(TZ) = \mathfrak{p}_1(P)$, then $Ell(P, \tau)$ is a rational modular form.

The second application of the construction of projective elliptic genera is in T-duality.

T-duality in an H-flux

[Bouwknegt-Evslin-Mathai]

$$
\begin{tikzcd}
& (Z \times_X \widehat{Z}, [p^* H] = [\widehat{p}^* \widehat{H}]) \arrow[dl, "p"] \arrow[dr, "\widehat{p}"'] & \\
(Z, A, H) \arrow[dr, "\pi"] & & (\widehat{Z}, \widehat{A}, \widehat{H}) \arrow[dl, "\widehat{\pi}"'] \\
& X &
\end{tikzcd}
$$

$$\pi_!(H) = R^{\widehat{A}}, \quad \widehat{\pi}_!(\widehat{H}) = R^{A}$$

T-duality in an H-flux

Hori's formula

$$T_H G = \int_{\mathbb{T}} e^{-A\wedge\widehat{A}}\ G, \tag{3}$$

where $G \in \Omega^\bullet(Z)^{\mathbb{T}}$ is the total RR fieldstrength,

$$\begin{array}{lll} G \in \Omega^{even}(Z)^{\mathbb{T}} & \text{for} & \text{Type IIA}; \\ G \in \Omega^{odd}(Z)^{\mathbb{T}} & \text{for} & \text{Type IIB}. \end{array}$$

$$T_H \colon \Omega^{\bar{k}}(Z)^{\mathbb{T}} \to \Omega^{\overline{k+1}}(\widehat{Z})^{\widehat{\mathbb{T}}},$$

for $k = 0, 1$, (where $\bar{k}$ denotes the parity of k) is isomorphism, inducing isomorphism on twisted cohomology groups,

$$T_H : H^\bullet(Z, H) \xrightarrow{\cong} H^{\bullet+1}(\widehat{Z}, \widehat{H}).$$

Gerbe and gerbe modules

a gerbe $\mathcal{G}$ on M is a collection of line bundles $\{L_{\alpha\beta}\}$ on double overlaps, $L_{\alpha\beta} \to U_{\alpha\beta} = U_\alpha \cap U_\beta$ such that on triple overlaps $U_{\alpha\beta\gamma}$ there is a trivialization

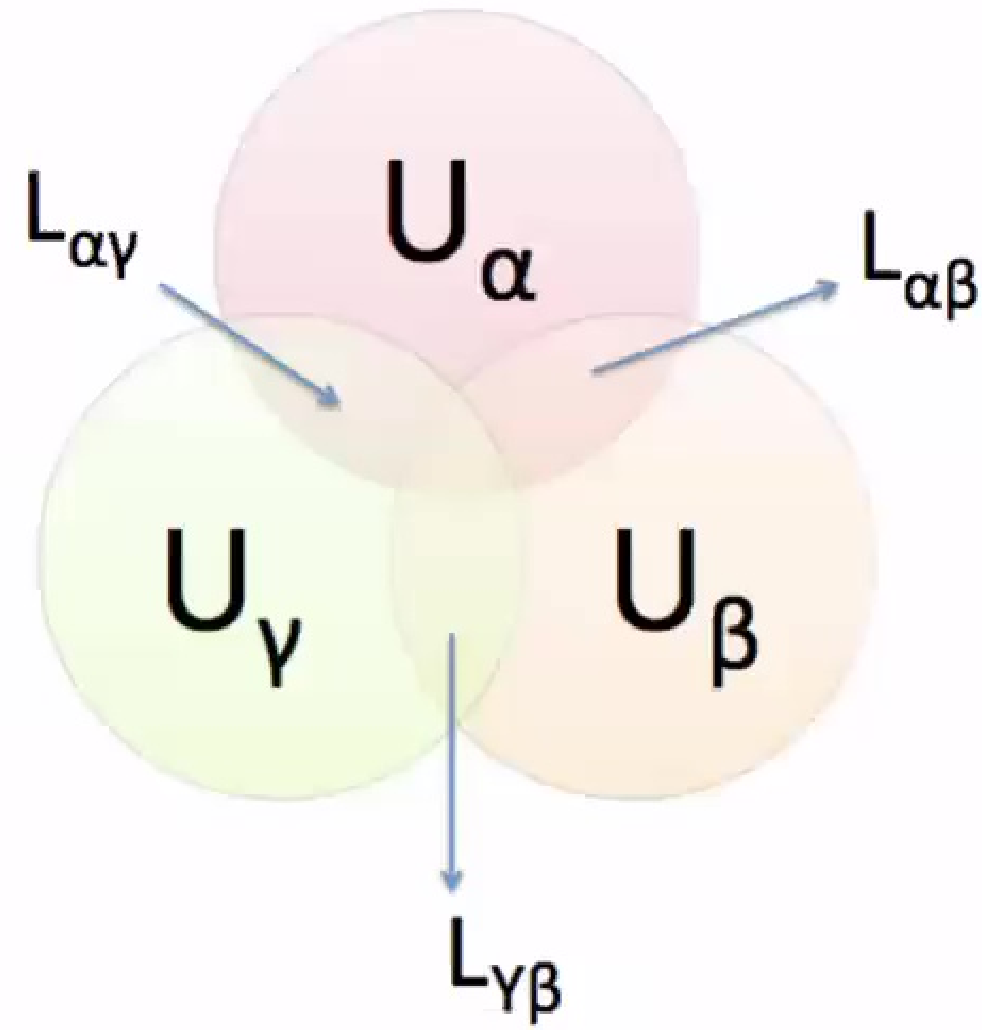
$$\phi_{\alpha\beta\gamma} : L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha} \xrightarrow{\cong} \mathbb{C}$$

Then $\{\phi_{\alpha\beta\gamma}\}$ is a $U(1)$-valued Cech 2-cocycle gives the Dixmier-Douady class of the gerbe in $H^3(M, \mathbb{Z})$.

Upto equivalence, gerbes on Z are classified by $H^3(M, \mathbb{Z})$.

A trivial gerbe $\{L_{\alpha\beta}\}$ is of the form $L_{\alpha\beta} = L_\alpha \otimes L^*_\beta$, where $\{L_\alpha \to U_\alpha\}$ is a collection of line bundles.

Gerbe and gerbe modules



Gerbe and gerbe modules

A connection on the gerbe $\mathcal{G}$ is $\{(L_{\alpha\beta}, \nabla^L_{\alpha\beta})\}$, a collection of line bundles $L_{\alpha\beta} \to U_{\alpha\beta}$ such that there is an isomorphism $L_{\alpha\beta} \otimes L_{\beta\gamma} \cong L_{\alpha\gamma}$ on $U_{\alpha\beta\gamma}$ and collection of connections $\{\nabla^L_{\alpha\beta}\}$ such that $\nabla^L_{\alpha\beta} = d + A_{\alpha\beta}$ (note that as $H^2(U_\alpha \cap U_\beta) = 0$, the bundle $L_{\alpha\beta}$ is trivial). Then we have

$$(\nabla^L_{\alpha\beta})^2 = F^L_{\alpha\beta} = B_\beta - B_\alpha.$$

H such that $H|_{U_\alpha} = dB_\alpha$ is called the curvature of the gerbe.

A principal $PU(n)$-bundle is a model for a torsion gerbe ($H = 0$).

Gerbe and gerbe modules

[Bouwknegt-Carey-Mathai-Murray-Stevenson]

Let $\{U_\alpha\}$ be an open cover of Z and $E = \{E_\alpha\}$ be a collection of (infinite dimensional) Hilbert bundles $E_\alpha \to U_\alpha$ whose structure group is reduced to U_{tr}, which are unitary operators on the model Hilbert space $\mathcal{H}$ of the form
(identity + trace class operator).

Here tr denotes the Lie algebra of trace class operators on $\mathcal{H}$. In addition, assume that on the overlaps $U_{\alpha\beta}$ that there are isomorphisms $\phi_{\alpha\beta} : L_{\alpha\beta} \otimes E_\beta \cong E_\alpha$, which are consistently defined on triple overlaps because of the gerbe property. Then $\{E_\alpha\}$ is said to be a gerbe module for the gerbe $\{L_{\alpha\beta}\}$.

Gerbe and gerbe modules

A gerbe module connection ∇^E is a collection of connections $\{\nabla^E_\alpha\}$ is of the form $\nabla^E_\alpha = d + A^E_\alpha$ where $A^E_\alpha \in \Omega^1(U_\alpha) \otimes tr$ whose curvature F^E_α on the overlaps $U_{\alpha\beta}$ satisfies

$$\phi^{-1}_{\alpha\beta}(F^E_\alpha)\phi_{\alpha\beta} = F^L_{\alpha\beta} I + F^E_\beta.$$

Using $F^L_{\alpha\beta} = B_\beta - B_\alpha$, this becomes

$$\phi^{-1}_{\alpha\beta}(B_\alpha I + F^E_\alpha)\phi_{\alpha\beta} = B_\beta I + F^E_\beta.$$

It follows that $\exp(-B)\,\mathrm{Tr}\left(\exp(-F^E) - I\right)$ is a globally well defined differential form on Z of even degree. Notice that $\mathrm{Tr}(I) = \infty$ which is why we need to consider the subtraction.

Gerbe and gerbe modules

Let $E = \{E_\alpha\}$ and $E' = \{E'_\alpha\}$ be be a gerbe modules for the gerbe $\{L_{\alpha\beta}\}$. Then an element of twisted K-theory $K^0(Z, H)$ is represented by the pair (E, E').

Two such pairs (E, E') and (G, G') are equivalent if $E \oplus G' \oplus K \cong E' \oplus G \oplus K$ as gerbe modules for some gerbe module K for the gerbe $\{L_{\alpha\beta}\}$. We can assume without loss of generality that these gerbe modules E, E' are modeled on the same Hilbert space.

Gerbe and gerbe modules

Suppose that $\nabla^E, \nabla^{E'}$ are gerbe module connections on the gerbe modules E, E' respectively. Then we can define the twisted Chern character as

$$Ch_H : K^0(Z, H) \to H^{even}(Z, H)$$

$$Ch_H(E, E') = \exp(-B)\, \mathrm{Tr}\left(\exp(-F^E) - \exp(-F^{E'})\right)$$

When the gerbe is torsion gerbe, $H = 0$, we see that the twisted Chern character lands in $H^{even}(Z, \mathbb{C})$.

Witten gerbe modules

Given a gerbe (non torsion) and a gerbe module pair (E, E'), analogous to the Witten projective bundle for the torsion case, one can construct the Witten gerbe module

$$\Theta_2(E, E') \in \oplus_{m\in\mathbb{Z}} K^0(Z, mH).$$

To take the Chern character of the Witten gerbe modules, similar to the projective Witten bundle case, we have to take into account of the levels of twists and apply the twisted Chern character of twist mH when the module has twist mH and mix them. So we take graded twisted Chern character GCh_B.

Unlike the projective bundles case, which are of finite rank, the gerbe modules E, E' are infinite dimensional, and therefore there are some analytic difficulties to overcome for the convergence of the graded twisted Chern characters. We use the Holomorphic functional calculus and Fredholm determinant to deal with this.

Witten gerbe modules

The target space of the graded twisted Chern character are q-series with coefficients being differential forms on Z, who are sums of $(d+mH)$-closed differential forms for various m. To distinguish these $(d+mH)$-closed forms for various level m, we introduce a formal variable y such that if a form ω is $(d+mH)$-closed, we write $\omega \cdot y^m$. Then the graded twisted Character of the Witten gerbe modules take values in $\Omega^*(Z)[[y, y^{-1}, q]]$ or $\Omega^*(Z)[[y, y^{-1}, q^{1/2}]]$. One can formally view them as spaces of twisted differential forms on free double loop space, and a model for the configuration space for Ramond-Ramond fields on this space. So we have

$$\mathrm{GCh}_B(\Theta_2(E, E')) \in \Omega^*(Z)[[y, y^{-1}, q^{1/2}]].$$

T-duality and Jacobi forms

Let M be a manifold with H-flux. Let $\mathcal{A}^{\bar{k}}(M)_{(d+mH)-cl}$ denote the space of holomorphic functions on $\mathbb{H}$ except for a set of isolated points, which take values in $\Omega^{\bar{k}}(M)_{(d+mH)-cl}$, the $(d+mH)$-closed forms on M with degree parity $\bar{k}$. Let $\mathcal{H}^{\bar{k}}(M, mH)$ denote the space of holomorphic functions on $\mathbb{H}$ except for a set of isolated points, which take values in $H^{\bar{k}}(M, mH)$.

T-duality and Jacobi forms

Denote $q = e^{2\pi\sqrt{-1}\tau}, \tau \in \mathbb{H}$ and $y = e^{-2\pi\sqrt{-1}z}$, $z \in \mathbb{C}$. On M, consider the 2-variable series

$$\omega(z,\tau) \in \bigoplus_{m \in \mathbb{Z}} \mathcal{H}^{\bar{k}}(M, mH) \cdot y^m$$

with the following properties : $\omega(z,\tau)$ is represented by

$$\sum_{m\in\mathbb{Z}} \omega_m(\tau) y^m, \tag{4}$$

with $\omega_m(\tau) \in \mathcal{A}^{\bar{k}}(M)_{(d+mH)-cl}$, $m \in \mathbb{Z}$ such that the degree p (with $\bar{p} = \bar{k}$) component

$$\sum_{m\in\mathbb{Z}} \omega_m(\tau)^{[p]} y^m \tag{5}$$

is the expansion at $y = 0$ of a Jacobi form of weight $\frac{p+\bar{k}}{2}$ and index 0 over $L \rtimes \Gamma$. Denote the abelian group of all such $\omega(z,\tau)$ by $\mathcal{J}_0^{\bar{k}}(M, H; L, \Gamma)$.

T-duality and Jacobi forms

Theorem (H.-Mathai)

If $Ch_H^{[2]}(E, E') = 0$ *and* $Ch_H^{[4]}(E, E') = 0$, *then*

$$\mathrm{GCh}_B(\Theta_2(E, E')) \in \mathcal{J}_0^{\bar{0}}(Z, H; \mathbb{Z}^2, \Gamma^0(2)).$$

T-duality and Jacobi forms

This motivates us to enhance to T-duality Hori map to graded Hori map to handle this two-variable situation.

T-duality and Jacobi forms

Consider the situation of T-duality with pair $(Z, H), (\hat{Z}, \hat{H})$. For $m \in \mathbb{Z}$, define the level m Hori map by

$$T_{*,m}(G) = \int_{\mathbb{T}} e^{-mA\wedge\hat{A}} G,$$

for G is an $\mathbb{T}$-invariant form on Z and $(d + mH)G = 0$. As we have

$$m\hat{H} = mH + d(mA \wedge \hat{A}),$$

it is not hard to see that $T_{*,m}G$ is a $\hat{\mathbb{T}}$-invariant form on $\hat{Z}$ and

$$(d + m\hat{H})(T_{*,m}(G)) = 0,$$

similar to the $m = 1$ case.

T-duality and Jacobi forms

Denote $\mathcal{A}^{\bar{k}}(Z)^{\mathbb{T}}_{(d+mH)-cl}$ the space of holomorphic functions on $\mathbb{H}$ except for a set of isolated points, which take values in $\Omega^{\bar{k}}(Z)^{\mathbb{T}}_{(d+mH)-cl}$, the $\mathbb{T}$-invariant $(d+mH)$-closed forms on Z with degree parity $\bar{k}$. Denote $\mathcal{A}^{\bar{k}}(\hat{Z})^{\hat{\mathbb{T}}}_{(d+m\hat{H})-cl}$ the similar stuff on the dual side. Define the graded Hori map

$$LT_*: \bigoplus_{m\in\mathbb{Z}} \mathcal{A}^{\bar{k}}(Z)^{\mathbb{T}}_{(d+mH)-cl}\cdot y^m \to \bigoplus_{m\in\mathbb{Z}} \mathcal{A}^{\overline{k+1}}(\hat{Z})^{\hat{\mathbb{T}}}_{(d+m\hat{H})-cl}\cdot y^m$$

by

$$LT_*\left(\sum_{m\in\mathbb{Z}}\omega_m(\tau)y^m\right) = \sum_{m\in\mathbb{Z}} T_{*,m}(\omega_m(\tau))y^m.$$

Passing to cohomology, we have the graded Hori map

$$LT: \bigoplus_{m\in\mathbb{Z}} \mathcal{H}^{\bar{k}}(Z, mH)\cdot y^m \to \bigoplus_{m\in\mathbb{Z}} \mathcal{H}^{\overline{k+1}}(\hat{Z}, m\hat{H})\cdot y^m.$$

T-duality and Jacobi forms

One can similarly define on the dual side,

$$\widehat{LT}_* : \bigoplus_{m\in\mathbb{Z}} \mathcal{A}^{\bar{k}}(\hat{Z})^{\hat{\mathbb{T}}}_{(d+m\hat{H})-cl} \cdot y^m \to \bigoplus_{m\in\mathbb{Z}} \mathcal{A}^{\overline{k+1}}(Z)^{\mathbb{T}}_{(d+mH)-cl} \cdot y^m$$

and

$$\widehat{LT} : \bigoplus_{m\in\mathbb{Z}} \mathcal{H}^{\bar{k}}(\hat{Z}, m\hat{H}) \cdot y^m \to \bigoplus_{m\in\mathbb{Z}} \mathcal{H}^{\overline{k+1}}(Z, mH) \cdot y^m.$$

T-duality and Jacobi forms

Theorem (H.-Mathai)

Let $H(\mathbb{H})$ denote the space of holomorphic functions on $\mathbb{H}$. The following identities hold :
(i)

$$\widehat{LT} \circ LT = -y\frac{\partial}{\partial y} = -\frac{\sqrt{-1}}{2\pi}\frac{\partial}{\partial z}, \quad LT \circ \widehat{LT} = -y\frac{\partial}{\partial y} = -\frac{\sqrt{-1}}{2\pi}\frac{\partial}{\partial z};$$

in particular when restricting to $m \neq 0$ parts, we get isomorphisms of $H(\mathbb{H})$ modules,

$$LT_* : \bigoplus_{m\in\mathbb{Z}, m\neq 0} \mathcal{A}^{\bar{k}}(Z)^{\mathbb{T}}_{(d+mH)-cl} \cdot y^m \to \bigoplus_{m\in\mathbb{Z}, m\neq 0} \mathcal{A}^{\overline{k+1}}(\hat{Z})^{\hat{\mathbb{T}}}_{(d+m\hat{H})-cl} \cdot y^m;$$

$$\widehat{LT}_* : \bigoplus_{m\in\mathbb{Z}, m\neq 0} \mathcal{A}^{\bar{k}}(\hat{Z})^{\hat{\mathbb{T}}}_{(d+m\hat{H})-cl} \cdot y^m \to \bigoplus_{m\in\mathbb{Z}, m\neq 0} \mathcal{A}^{\overline{k+1}}(Z)^{\mathbb{T}}_{(d+mH)-cl} \cdot y^m.$$

T-duality and Jacobi forms

Theorem (H.-Mathai)

(ii) Restrcting to the Jacobi forms, we have

$$LT\Big(\mathcal{J}_0^{\bar{k}}(Z,H;L,\Gamma))\Big) \subseteq \mathcal{J}_0^{\overline{k+1}}(\hat{Z},\hat{H};L,\Gamma);$$

dually, we have

$$\widehat{LT}\Big(\mathcal{J}_0^{\bar{k}}(\hat{Z},\hat{H};L,\Gamma)\Big) \subseteq \mathcal{J}_0^{\overline{k+1}}(Z,H;L,\Gamma).$$

Thank you very much!