Title: Quantum homeopathy works: Efficient unitary designs with a system-size independent number of non-Clifford gates

Speakers: Ingo Roth

Series: Perimeter Institute Quantum Discussions

Date: May 27, 2020 - 4:00 PM

URL: http://pirsa.org/20050048

Abstract: Many quantum information protocols require the implementation of random unitaries. Because it takes exponential resources to produce Haar-random unitaries drawn from the full n-qubit group, one often resorts to t-designs. Unitary t-designs mimic the Haar-measure up to t-th moments. It is known that Clifford operations can implement at most 3-designs. In this work, we quantify the non-Clifford resources required to break this barrier. We find that it suffices to inject O(t^4 log^2(t) log(1/Îµ)) many non-Clifford gates into a polynomial-depth random Clifford circuit to obtain an Îµ-approximate t-design. Strikingly, the number of non-Clifford gates required is independent of the system size â€“ asymptotically, the density of non-Clifford gates is allowed to tend to zero. We also derive novel bounds on the convergence time of random Clifford circuits to the t-th moment of the uniform distribution on the Clifford group. Our proofs exploit a recently developed variant of Schur-Weyl duality for the Clifford group, as well as bounds on restricted spectral gaps of averaging operators. Joint work with J. Haferkamp, F. Montealegre-Mora, M. Heinrich, J. Eisert, and D. Gross.

Freie Universität Berlin

arXiv: 2002.09524

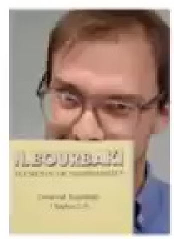
Quantum Homeopathy Works:

Efficient Unitary Designs With A System-Size Independent Number Of Non-Clifford Gates

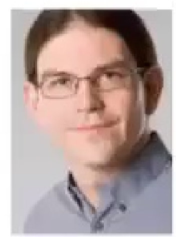
Ingo Roth

joint work with

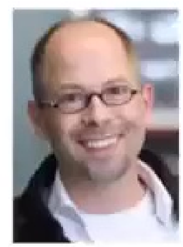
Jonas Haferkamp



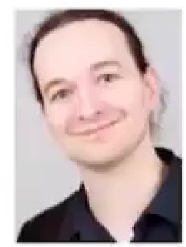
Felipe Montealegre-Mora



Markus Heinrich



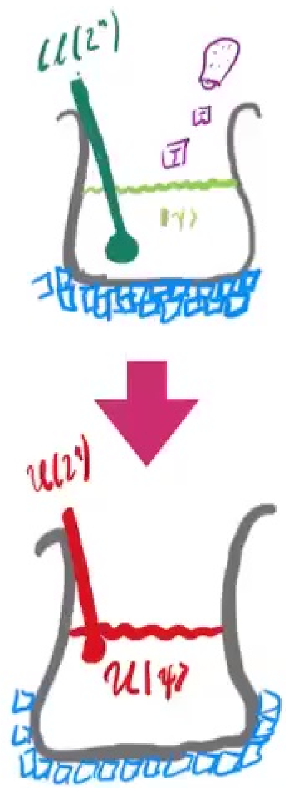
Jens Eisert



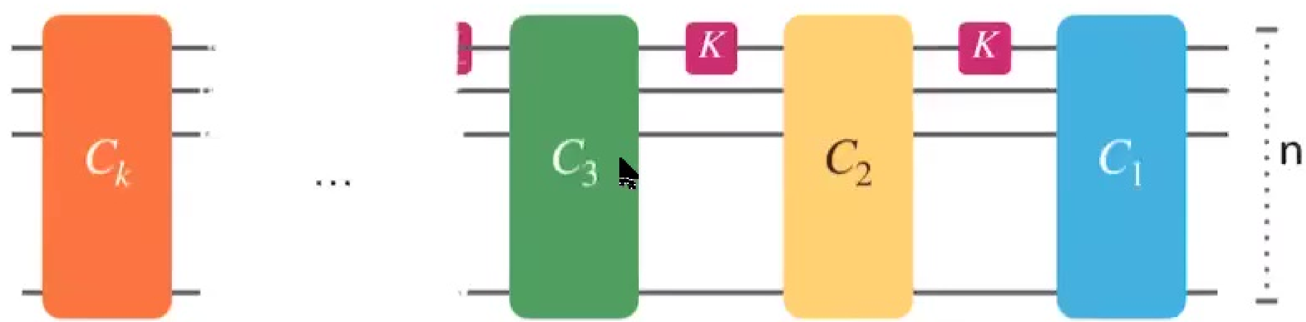
David Gross

Done

What is quantum homeopathy?



$C_i \sim \mu_{\mathrm{Cl}(2^n)}$ $K \notin \mathrm{Cl}(2^n)$



is an ϵ-approximate

unitary t-design

for

$$k \gtrsim \mathrm{poly}(t)\,\log\frac{1}{\epsilon} \qquad n \gtrsim t^2$$

Done

Outline

- Motivation for **'simple' t-designs**
- **Clifford group** 1, 2, 3, and ...?
- *PLAN I: Moving the target*
  - Example low-rank **Randomised Benchmarking tomography**
- *PLAN II: Adding magic (non-Clifford gates)*
- Summary

Applications of Haar-random unitaries

- Demonstration of **Quantum advantages**
- Dynamics of **mixing processes**, models of **black-holes**
- **Quantum system identification**
  - Randomised benchmarking [...]
  - Shadow estimation [Huang, Kueng, Preskill 2020]
  - *Low-rank quantum tomography [... , IR, Kueng, Kimmel, Liu, Gross, Eisert, Kliesch 2018]*

A moment please . . .

Polynomials of degree t $p_t(U, U^\dagger) = \mathrm{Tr}\left[B\, U^{\otimes t} A (U^\dagger)^{\otimes t}\right]$

! Want to calculate $\mathbb{E}_{U \sim \mu_{U(d)}}\left[p_t(U, U^\dagger)\right]$!

$$E_{\mu_{U(d)}}(A) = \mathbb{E}_{U \sim \mu_{U(d)}} U^{\otimes t} A (U^\dagger)^{\otimes t} = \int_{U(d)} U^{\otimes t} A (U^\dagger)^{\otimes t} d\mu_{U(d)}(U)$$

Short calculation shows: $V^{\otimes t} E_{\mu_{U(d)}}(A) = E_{\mu_{U(d)}}(A) V^{\otimes t}$

$E_{\mu_{U(d)}}$ is the projector onto the commutant of $\Delta^t(U(d))$!

Commutant of $\mathscr{A} \subset L(V)$ $\mathrm{comm}(\mathscr{A}) = \{B \in L(V) \mid BA = AB \quad \forall A \in \mathscr{A}\}$

$\Delta^t : U \mapsto U^{\otimes t}$

SCHUR-WEYL DUALITY

$$\mathrm{comm}\, \Delta^t(U(d)) \cong \mathbb{C}[S_t]$$

The same holds for the twirling of any invariant (Haar) measure, e.g. the uniform measure on finite subgroups.

Done

Applications of t-designs

- **Quantum system identification**
  - Simple **Randomised Benchmarking** requires 2-DESIGN
  - Sample-optimal **Shadow estimation** 3-DESIGN
  - low-rank **Quantum tomography** 4-design 4-DESIGN

What are 'simple' t-designs?

The Clifford group

$$\mathrm{Cl}(2^n) := \left\langle \left\{ S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, CZ = \mathrm{diag}(1,1,1,-1) \right\} \right\rangle$$

PHYSICAL REVIEW A **96**, 062336 (2017)

Multiqubit Clifford groups are unitary 3-designs

Huangjun Zhu*

THE CLIFFORD GROUP FORMS A UNITARY 3-DESIGN

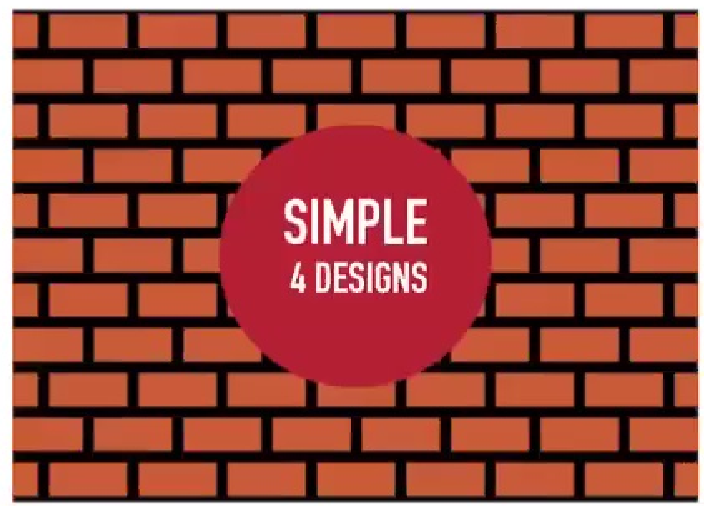
ZAK WEBB[1]

Quantum Information & Computation, 2016

The Clifford group fails gracefully to be a unitary 4-design

Huangjun Zhu[1], Richard Kueng[1], Markus Grassl[2], and David Gross[1]

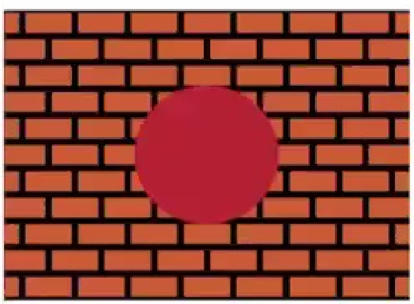
arXiv:1609.08172



MOTIVATION

Unitary t-design that are **subgroups** = **t-groups**

4-groups do not exist, and the Clifford group is unique maximal group design.

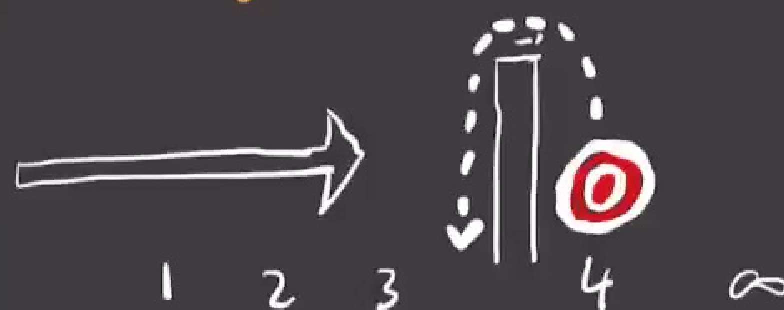


[Banai, Navarro, Riso, Tiep '18] & [Sawicki, Karnas '17], summarised in [HMHEGR'20]

PLAN I:

"Breaking through" the wall by

moving the target.



Done

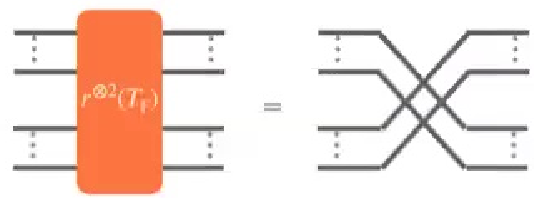
Commutant of the Clifford group

$$\operatorname{comm} \Delta^t_{\mathrm{Cl}(2^n)} \supset \operatorname{comm} \Delta^t_{U(2^n)} \cong \mathbb{C}[S_t]$$
$$= \langle r(T_\pi)^{\otimes n} \mid \pi \in S_t \rangle$$

For **permutation** $\pi \in S_t$ define

$$T_\pi = \{(\pi(x), x) \mid x \in \mathbb{Z}_2^t\} \quad \text{and} \quad r(T_\pi) = \sum_{(x,y)\in T_\pi} |x\rangle\langle y|$$

$r(T_\pi)^{\otimes n}$ simultaneously permutation n qubits.



Definition

$T \subset \mathbb{Z}_2^t \times \mathbb{Z}_2^t$ is a **stochastic Lagrangian subspace** if

1. $x \cdot x = y \cdot y \mod 4 \qquad \forall (x,y) \in T$
2. $\dim(T) = t$
3. $(1,\ldots,1) \in T$

Theorem [Gross, Nezami, Walter '17]

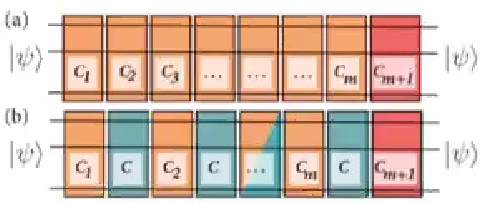
$$\operatorname{comm} \Delta^t_{\mathrm{Cl}(2^n)} = \langle \{\text{Lagrangian subspaces}\} \rangle = \langle \Sigma_t \rangle$$

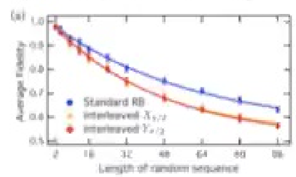
[IR, Kueng, Kimmel, Liu, Gross, Eisert, Kliesch '18]

SPAM-Robust tomography of unitary gates

Randomised benchmarking allows for **SPAM-robust extraction** of $F_{\mathrm{avg}}(\mathcal{U}, \mathcal{X})$ with **Clifford gates** $\mathcal{U}$

[. . . Magesan& '12, . . .]





$$F^{(0)}_{\mathrm{seq}}(m, \psi) = A_0 p^m + B_0$$

Randomised benchmarking: SPAM-robust

&

Compressed sensing: exploits low-rank structure

Unitary gates

Choose Haar random set of Clifford gates $\{\mathcal{C}_i\}_{i=1}^m$

Data: $f_i = F_{\mathrm{avg}}(\mathcal{C}_i, \mathcal{X}) + \epsilon_i$

Algorithm:

Efficient in d

$$\text{minimise} \quad \sum_{i=1}^{m} (F_{\mathrm{avg}}(\mathcal{C}_i, \mathcal{Z}) - f_i)^2$$

subject to $\mathcal{Z}$ is a unital quantum channel

Theorem (Main result)

Fix $d = 2^n$

$$m \geq c d^2 \log(d)$$

the result of the algorithm $\mathcal{Z}^\sharp$ *fulfils with high probability*

$$\|\mathcal{X} - \mathcal{Z}^\sharp\| \leq C \frac{d^2}{\sqrt{m}} \|\epsilon\|_{\ell_2}$$

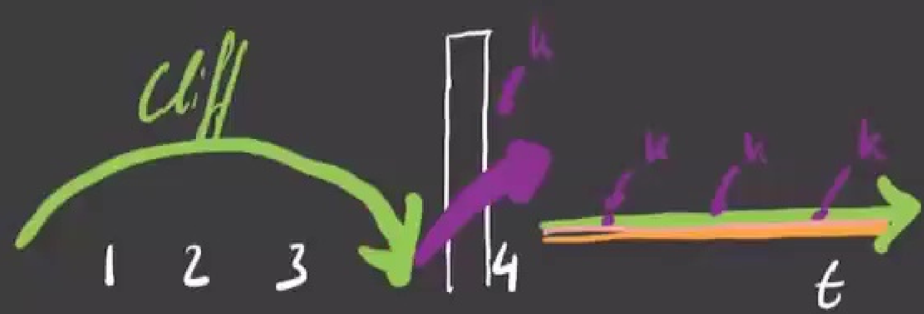
Optimal sampling complexity for separate POVMs: $\tilde{\mathcal{O}}(d^4)$ **channel uses**

. . . derive bound on 4th moment ✓ . . . sufficient for compressed sensing ✓

PLAN II:

Really "breaking through" with

a pinch of universality



T-designs via quantum circuits

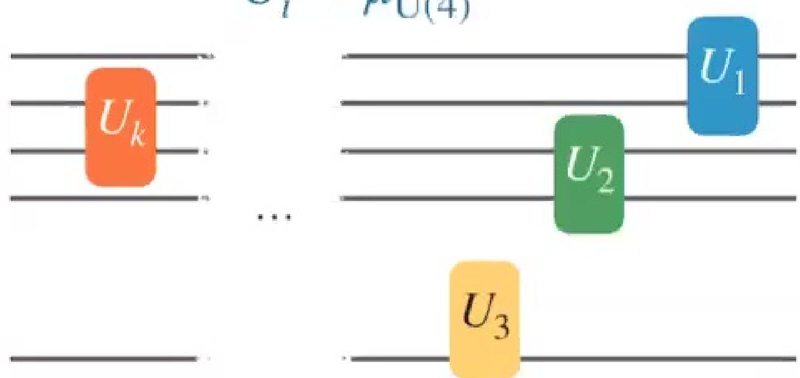
Definition

A measure σ on $U(d)$ is an ϵ-**approximate unitary t-design** if

$$\|E_\sigma - E_{\mu_{U(d)}}\|_\diamond \leq \epsilon$$

Theorem [Brandão, Harrow, Horodecki, 2016]

$$U_i \sim \mu_{\mathrm{U}(4)}$$



is an ϵ-approximate

unitary t-design

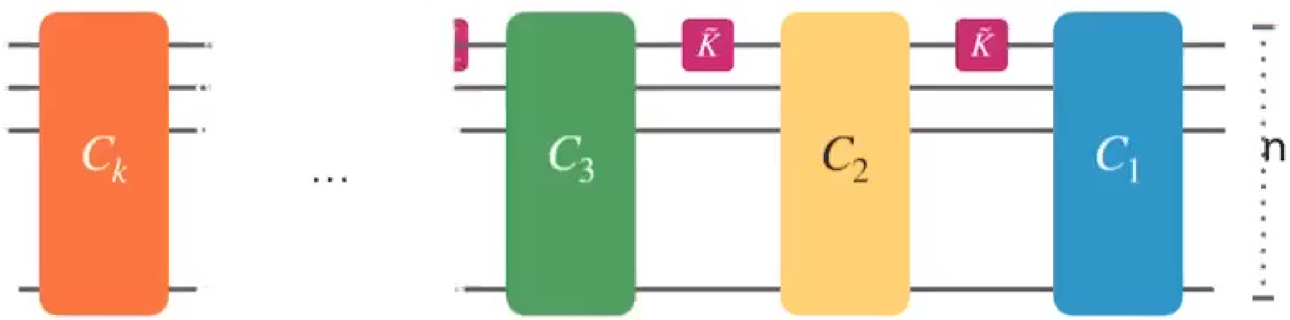
for

$$k \gtrsim n^2 t^9 \log^2 t \log \frac{1}{\epsilon}$$

T-designs via (mainly) Clifford circuits

Theorem (Main Result)

$C_i \sim \mu_{\mathrm{Cl}(2^n)}$ $\tilde{K} \sim \mu_{\{\mathrm{Id}, K, K^\dagger\}}$



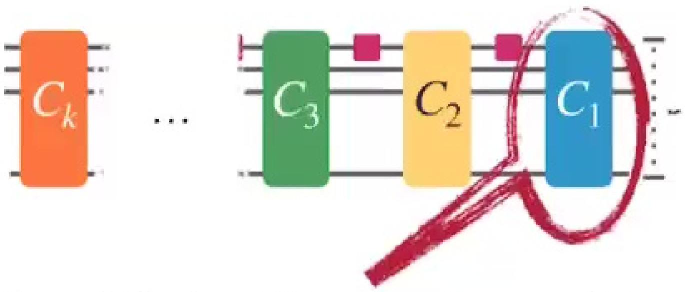
is an ϵ-approximate

unitary t-design

for

$$k \gtrsim t^4 \log^2 t \, \log \frac{1}{\epsilon} \qquad n \gtrsim t^2$$

Comparing Local Circuits



Every Clifford n-qubit gate can be compiled by

$O\left(\frac{n^2}{\log(n)}\right)$ local gates. [Aaronson & Gottesman '04]

Corollary 2

An interleaved Clifford quantum circuit with

$$k_{\text{overall}} \gtrsim \frac{n^2}{\log n} t^4 \log^2 t \, \log \frac{1}{\epsilon}$$

gates is an ϵ-approximate

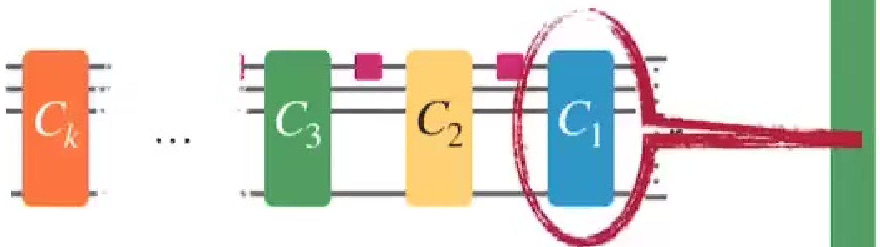
unitary t-design

Theorem

is an ϵ-approximate **unitary t-design** for $k \gtrsim n^2 t^9 \log^2 t \, \log \frac{1}{\epsilon}$

Done

Random Clifford Circuits

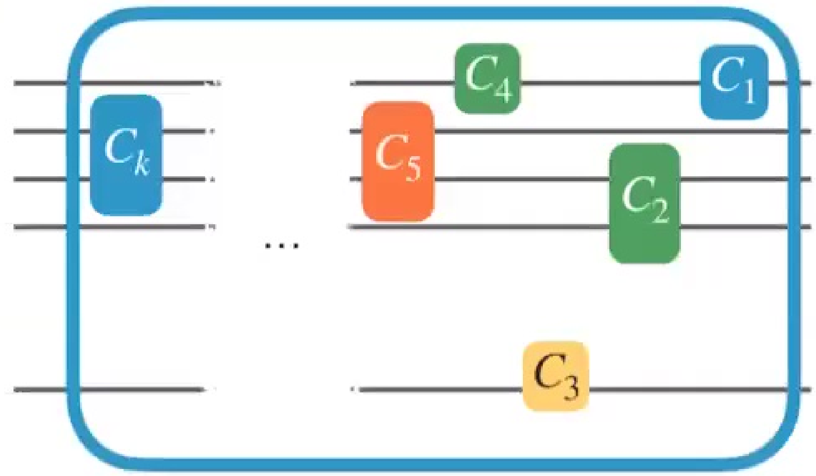


Definition

A measure σ on $U(d)$ is a **relative ϵ-approximate Clifford t-design** if

$$(1-\epsilon)E_{\mu_{Cl}} \leq E_{\mu_\sigma} \leq (1+\epsilon)E_{\mu_{Cl}}$$

Theorem 2



is an ϵ-approximate relative

Clifford t-design

for $n \gtrsim t$

$$k \gtrsim n \log^{-2}(t) t^8 \left(2nt + \log \frac{1}{\epsilon}\right)$$

Done

PROOF SKETCH

i) Rewrite moment operators as deviations from the Haar unitary case

$$E_\sigma - E_{\mu_H} = \underbrace{\left[(E_{\mu_H} - E_{H})K\right]^{k}}_{\text{projector onto } \langle \Sigma_t \setminus S_t \rangle}$$

ii) construct the projector onto $\langle \Sigma_t \setminus S_t \rangle$ using Gram-Schmidt

iii) carefully bound all terms

$$\| E_\sigma - E_{\mu_H} \|_\diamond \leq 2^{O(t^4) + t \log k} \left(1 + 2^{O(t^2) - n}\right)^{5k} \eta^{k-1}$$

Done

Lemma:

$$\eta := \max_{T \in \Sigma_t \setminus S_t} \left| \mathrm{Tr}\left[\tilde{r}(T)\, K\, r(T') \right] \right| \leq 1 - c(K) \log^{-2} t$$

Varju's (2013) bound on restricted spectral gaps.

Lemma:

$$\mathrm{Tr}\left[\tilde{r}(T)\, E_{\mu_u}\, \tilde{r}(T) \right] \leq \frac{7}{8}$$

PROOF SKETCH

i) Rewrite moment operators as deviations from the Haar unitary case

$$E_\sigma - E_{\mu_H} = \left[(E_{\mu_{CE}} - E_{\mu_H})K\right]^k$$

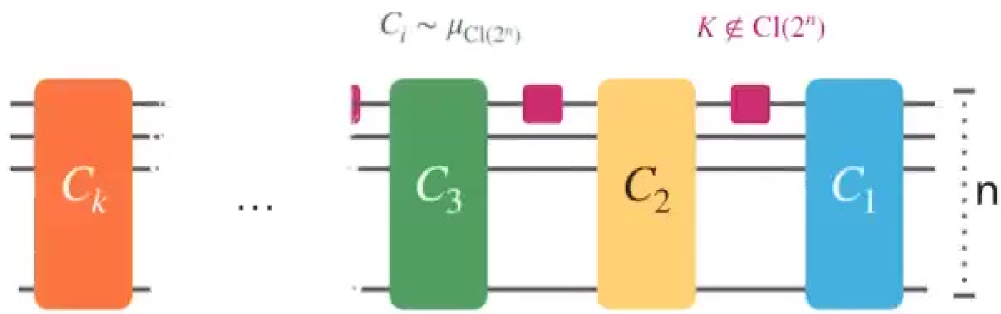
projector onto $\langle \Sigma_t \setminus S_t \rangle$

ii) construct the projector onto $\langle \Sigma_t \setminus S_t \rangle$ using Gram-Schmidt

iii) carefully bound all terms

$$\| E_\sigma - E_{\mu_H} \|_\diamond \leq 2^{O(t^4) + t \log k} \left(1 + 2^{O(t^2)} n\right)^{5k} \eta^{k-1}$$

Summary

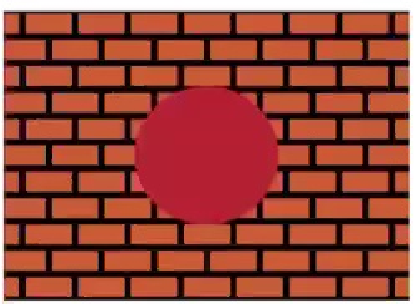


arXiv: 2002.09524

- Adding $\tilde{O}\left(t^4 \log \frac{1}{\epsilon}\right)$ non-Clifford gates yields an ϵ-approximate t-design.
  Independent of the system size!
- Convergence bound for local random Clifford
- New tools for studying the commutant of the diagonal action of the Clifford group

Unitary t-design that are **subgroups** = **t-groups**

4-groups do not exist, and the Clifford group is unique maximal group design.



[Banai, Navarro, Riso, Tiep '18] & [Sawicki, Karnas '17], summarised in [HMHEGR'20]