Title: Summer Undergrad 2020 - Symmetries (A) - Lecture 3

Speakers: Giuseppe Sellaroli

Collection: Summer Undergrad 2020 - Symmetries

Date: May 29, 2020 - 2:00 PM

URL: http://pirsa.org/20050047

Abstract: Noetherâ€™s theorem

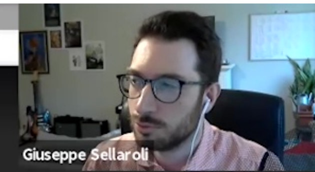
Today:
- symmetries of an action
- Noether's theorem

"Extra" folder: I will put additional notes in here, in case you are interested. Feel free to make suggestions!

Lecture 3 - Google Drive | Socrative

drive.google.com/drive/folders/1Kq_BrlXqeF08va0KuLmBV1g-Wh99Fp9F

Exercises.pdf



Exercise

$L(q,\dot q,t) = \frac{1}{2}(m\dot q^2 - \kappa q^2)\, e^{\alpha t} \quad (\alpha \in \mathbb{R}) \qquad T_\varepsilon(t) = t+\varepsilon \qquad Q_\varepsilon(q) = q\, e^{-\frac{\varepsilon\alpha}{2}}$

$\tilde t = T_\varepsilon(t)$

$\tilde q(\tilde t) = Q_\varepsilon(q(t))$

1. Show that T_ε and Q_ε are one-parameter subgroups
2. show that $\delta S = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde S[\tilde q] = 0$ (without using E-L eqs)
3. Find the associated conserved quantity
4. Find the Euler-Lagrange equations
5. Show that the conserved quantity is indeed conserved if the E-L eqs hold

Note: for point 2 use the fact that

$$\delta S = \int_a^b \left[L(q,\dot q,t)\frac{d}{dt}\delta t + \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\left(\frac{d}{dt}\delta q - \dot q \frac{d}{dt}\delta t\right)\right] dt$$

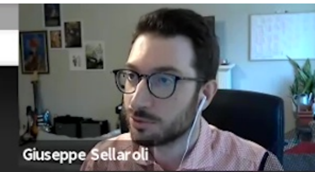
and substitute L, δt, δq

Page 1 / 1

Lecture 3 - Google Drive | Socrative

drive.google.com/drive/folders/1Kq_BrlXqeF08va0KuLmBV1g-Wh99Fp9F

Matrix Lie group & Lie algebras (not today).pdf



Def: A matrix Lie group is a closed subgroup $G \leq GL(n,\mathbb{C})$ for some $n \in \mathbb{N}$ (closed w.r.t. the topology induced from $M_n(\mathbb{C})$)

using operator norm
$$\|A\| = \sup\left\{ \frac{\|Ax\|}{\|x\|} \;\middle|\; x \in \mathbb{C}^n \setminus \{0\} \right\}$$

examples

- $GL(n,\mathbb{C})$ general linear group over $\mathbb{C}$
- $SL(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) \mid \det A = 1\}$ special linear group over $\mathbb{C}$
- $GL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{C}) \mid \bar{x} - x = 0\}$ general linear group over $\mathbb{R}$
- $SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid \det A = 1\}$ special linear group over $\mathbb{R}$
- $O(n) = \{A \in GL(n,\mathbb{R}) \mid A^t A = \mathbb{1}\}$ orthogonal group
- $SO(n) = \{A \in O(n) \mid \det A = 1\}$ special orthogonal group
- $U(n) = \{A \in GL(n,\mathbb{C}) \mid A^* A = \mathbb{1}\}$ unitary group
- $SU(n) = \{A \in U(n) \mid \det A = 1\}$ special unitary group

real Lie groups despite having complex matrices!

Page 1 / 2

Lecture 3 - Google Drive | Socrative

drive.google.com/drive/folders/1Kq_BrlXqeF08va0KuLmBV1g-Wh99Fp9F

Proof of Noether's theorem.pdf

Perimeter Institute Summer School 2020 > Symmetries

Giuseppe Sellaroli

<u>Proof of Noether's theorem</u>

$\tilde{t} = T_\varepsilon(t) \qquad \tilde{q}(\tilde{t}) = Q_\varepsilon(q(t)) = Q_\varepsilon(q(T_{-\varepsilon}(\tilde{t})))$

with T_ε and Q_ε smooth one-parameter subgroups ($T_0 = id \quad T_{-\varepsilon} = T_\varepsilon^{-1} \quad T_{\varepsilon+\varepsilon'} = T_\varepsilon \circ T_{\varepsilon'}$)

→ i.e. $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} T_\varepsilon$ makes sense

$S[q] = \int_a^b L(q(t), \dot{q}(t), t)$

$$\tilde{S}[\tilde{q}] = \int_{T_\varepsilon(a)}^{T_\varepsilon(b)} L(\tilde{q}(\tilde{t}), \underbrace{\dot{\tilde{q}}(\tilde{t})}_{\frac{d\tilde{q}}{d\tilde{t}}}, \tilde{t})\, d\tilde{t} = \int_a^b \frac{dT_\varepsilon}{dt} L(\tilde{q}(T_\varepsilon(t)), \dot{\tilde{q}}(T_\varepsilon(t)), T_\varepsilon(t))\, dt$$

suppose that $\tilde{S}[\tilde{q}] = S[q]$ for all paths (even those that are not solutions of E-L eqs.) and for all choiches of a, b.

Then $\delta S := \frac{d}{d\varepsilon}\big|_{\varepsilon=0} \tilde{S}[\tilde{q}] = 0$

Some notation:

- $\delta t := \frac{d}{d\varepsilon}\big|_{\varepsilon=0} T_\varepsilon$ (function of t)
- $\delta q := \frac{d}{d\varepsilon}\big|_{\varepsilon=0} Q_\varepsilon$ (function of q)

} generators of T_ε and Q_ε (see Lie algebras later)

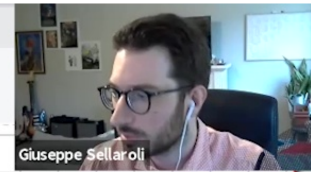
Let's apply $D := \frac{d}{d\varepsilon}\big|_{\varepsilon=0}$ everywhere we can!

- $D \frac{dT_\varepsilon}{dt} = \frac{d}{dt} DT_\varepsilon = \frac{d}{dt}\delta t$
- $D\tilde{q}(\tilde{t}) = D\,Q_\varepsilon(q(t)) = \delta q(q(t))$
- $\frac{d\tilde{q}}{d\tilde{t}}(\tilde{t}) = \frac{d}{d\tilde{t}} Q_\varepsilon(q(t)) = \frac{\partial Q_\varepsilon}{\partial q}(q(t))\,\dot{q}(t)\,\frac{dt}{d\tilde{t}}(\tilde{t})$ → $= \frac{1}{\frac{d\tilde{t}}{dt}(t)} = \frac{1}{\frac{dT_\varepsilon}{dt}(t)}$

$$\Rightarrow \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \dot{\tilde{q}}(\tilde{t}) = \frac{\partial}{\partial q} DQ_\varepsilon(q(t))\,\dot{q}(t)\,\frac{1}{\underbrace{\frac{dT_0}{dt}}_{=1}} + \underbrace{\frac{\partial}{\partial q} Q_0(q(t))}_{=1}\,\dot{q}(t)\,\underbrace{D\frac{1}{\frac{dT_\varepsilon}{dt}}}_{= -\frac{d}{dt}\delta t}$$

$$\Rightarrow D\dot{\tilde{q}}(\tilde{t}) = \frac{\partial}{\partial q}\delta q\,\dot{q}(t) - \dot{q}(t)\frac{d}{dt}\delta t = \frac{d}{dt}\delta q - \dot{q}(t)\frac{d}{dt}\delta t$$

Page 1 / 2



Symmetry of an action

$L(q, \dot{q}, t)$ Lagrangian

action $S[q] = \int_a^b L(q(t), \dot{q}(t), t)\, dt$

symmetry: transform $\underbrace{\text{something}}_{t,\, q}$ $\rightarrow$ keep action invariant

continuous (smooth) symmetries

$T_0 = id \quad T_{-\varepsilon} = T_{\varepsilon}^{-1} \quad T_{\varepsilon+\varepsilon'} = T_{\varepsilon} \circ T_{\varepsilon'}$

$\rightarrow$ one-parameter subgroup

$\tilde{t} = T_{\varepsilon}(t) \qquad \tilde{q}(\tilde{t}) = Q_{\varepsilon}(q(t)) = Q_{\varepsilon}(q(T_{-\varepsilon}(\tilde{t})))$

t, q

$T_0 = id \qquad T_{-\varepsilon} = T_\varepsilon^{-1}$

continuous (smooth) symmetries

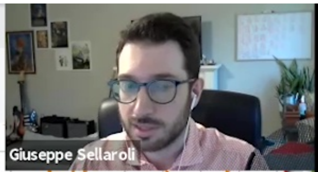
→ one-parameter subgroup

$$\tilde{t} = T_\varepsilon(t) \qquad \underset{\text{new time}}{\tilde{q}(\tilde{t})} = Q_\varepsilon(q(\underset{\text{old time}}{t})) = Q_\varepsilon(q(T_{-\varepsilon}(\tilde{t})))$$

how does S change?

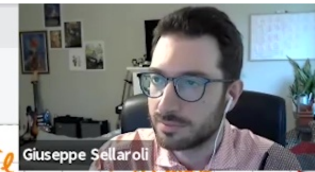
$$\tilde{S}[\tilde{q}] = \int_{T_\varepsilon(a)}^{T_\varepsilon(b)} d\tilde{t}\, L\left(\tilde{q}(\tilde{t}), \dot{\tilde{q}}(\tilde{t}), \tilde{t}\right) \qquad \dot{\tilde{q}} = \frac{d\tilde{q}}{d\tilde{t}}$$

S invariant if $\underbrace{\tilde{S}[\tilde{q}] = S[q]}$ → we have a symmetry



*notes afternoon.xopp - Xournal++

Continuous (smooth) symmetries → one-paramete



$$\tilde{t} = T_\varepsilon(t) \qquad \tilde{q}(\tilde{t}) = Q_\varepsilon(q(t)) = Q_\varepsilon(q(T_{-\varepsilon}(\tilde{t})))$$

new time   old time

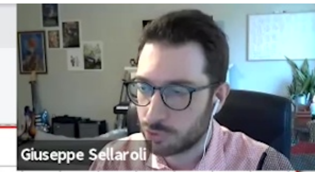
how does S change?

$$\tilde{S}[\tilde{q}] = \int_{T_\varepsilon(a)}^{T_\varepsilon(b)} d\tilde{t}\, L\left(\tilde{q}(\tilde{t}), \dot{\tilde{q}}(\tilde{t}), \tilde{t}\right) \qquad \dot{\tilde{q}} = \frac{d\tilde{q}}{d\tilde{t}}$$

S invariant if $\underbrace{\tilde{S}[\tilde{q}] = S[q]}$ → we have a symmetry

---

there is also a notion of quasi-symmetry



Noether's theorem (1918)

Suppose $\tilde t = T_\varepsilon(t)$ $\quad \tilde q(\tilde t) = Q_\varepsilon(q(t))$ is a symmetry of

$$S[q] = \int_a^b L(q(t),\dot q(t),t)\,dt \qquad \left(\underline{\tilde S[\tilde q] = S[q]}\right)$$

↳ for all a, b

for all q's

→ not just solutions to EL eqs.

Then if we denote

$$\delta t = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon \qquad\qquad \delta q = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Q_\varepsilon$$

then if we define

$$\delta t = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon \qquad \delta q = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Q_\varepsilon$$

the quantity $\left[L(q(t), \dot q(t), t)\, \delta t + \frac{\partial L}{\partial \dot q}\left(\delta q - \dot q\, \delta t\right) \right]$ is

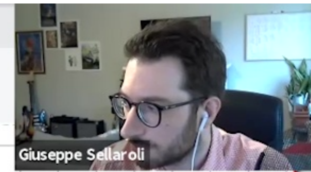
conserved if $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot q}$ (on physical trajectories)

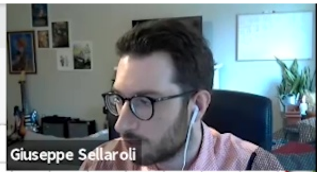
↳ constant in time

---

ex: $L = L(q, \dot q)$ time independent

$$\left.\begin{aligned} \tilde t &= T_\varepsilon(t) = t + \varepsilon \quad \text{(time translation)} \\ \tilde q(\tilde t) &= q(t) \;\rightarrow\; Q_\varepsilon = \mathrm{id} \end{aligned}\right\} \rightarrow \text{1-parameter sy}$$





ex: $L = L(q, \dot q)$ time independent

$\tilde t = T_\varepsilon(t) = t + \varepsilon$ (time translation)

$\tilde q(\tilde t) = q(t)$ $\rightarrow$ $Q_\varepsilon = id$

$\rightarrow$ 1-parameter subgroups

$$\tilde S[\tilde q] = \int_{a+\varepsilon}^{b+\varepsilon} L(\underbrace{\tilde q(\tilde t)}_{q(t)}, \underbrace{\dot{\tilde q}(\tilde t)}_{\dot q(t)})\, d\tilde t \qquad \leftarrow \frac{d\tilde t}{dt}\, dt \text{ “=” } d\tilde t$$

$\rightarrow \frac{d}{d\tilde t} = \frac{d}{dt}$

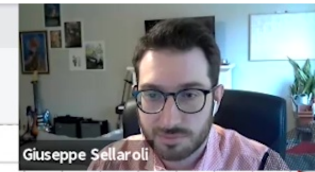
$$= \int_a^b L(q(t), \dot q(t))\, dt = S[q]$$

$\rightarrow$ symmetry!

$$\delta t = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon(t) = 1$$

$$\delta q = 0$$

conserved: $L(q, \dot q)\, \overset{\text{“}1\text{”}}{\delta t} + \frac{\partial L}{\partial \dot q}(\delta q - \dot q\, \delta t)$



ex: $L = L(q, \dot q)$ time independent

$\tilde t = T_\varepsilon(t) = t + \varepsilon$ (time translation)

$\tilde q(\tilde t) = q(t) \quad \rightarrow \quad Q_\varepsilon = id$

$\rightarrow$ 1-parameter subgroups

$$\tilde S[\tilde q] = \int_{a+\varepsilon}^{b+\varepsilon} L(\underbrace{\tilde q(\tilde t)}_{q(t)}, \underbrace{\dot{\tilde q}(\tilde t)}_{\dot q(t)})\, d\tilde t$$

$\frac{d\tilde t}{dt}\, dt$ "=" $d\tilde t$

$\dot q(t) \rightarrow \frac{d}{d\tilde t} = \frac{d}{dt}$

$$= \int_a^b L(q(t), \dot q(t))\, dt = S[q]$$

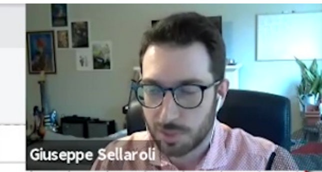
$\rightarrow$ symmetry!

$\delta t = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon(t) = 1$

$\delta q = 0$

conserved: $L(q, \dot q) \overset{=1}{\delta t} + \frac{\partial L}{\partial \dot q}(\overset{=0}{\delta q} - \dot q \overset{=1}{\delta t})$

$\rightarrow \left| L(q, \dot q) - \frac{\partial L}{\partial \dot q} \dot q \right|$



ex: $L = L(q, \dot q)$ time independent

$\tilde t = T_\varepsilon(t) = t + \varepsilon$ (time translation)

$\tilde q(\tilde t) = q(t) \rightarrow Q_\varepsilon = id$

$\rightarrow$ 1-parameter subgroups

$$\tilde S[\tilde q] = \int_{a+\varepsilon}^{b+\varepsilon} L(\underbrace{\tilde q(\tilde t)}_{q(t)}, \underbrace{\dot{\tilde q}(\tilde t)}_{\dot q(t)})\, d\tilde t$$

$\frac{d\tilde t}{dt} dt$ "=" $d\tilde t$

$\rightarrow \frac{d}{d\tilde t} = \frac{d}{dt}$

$$= \int_a^b L(q(t), \dot q(t))\, dt = S[q]$$

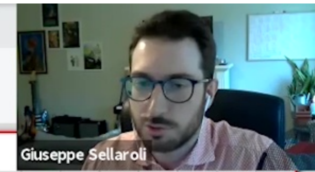
$\rightarrow$ symmetry!

$\delta t = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon(t) = 1$

$\delta q = 0$

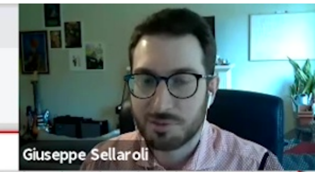
conserved: $L(q,\dot q)\, \overset{1}{\delta t} + \frac{\partial L}{\partial \dot q}\left(\overset{0}{\delta q} - \dot q\, \overset{1}{\delta t}\right)$

$\rightarrow \boxed{L(q,\dot q) - \frac{\partial L}{\partial \dot q}\dot q}$ energy



Activity

- In each breakout room, discuss with your peers the following topic: "reasons why Noether's theorem is so cool"
- In each room, go to socrative.com (SYMMETRIES) and submit the answers you (collectively) come up with.
    - You can answer multiple times.
    - One reason per answer, please!



Proof of Noether's theorem

$\tilde{t} = T_\varepsilon(t)$ $\qquad$ $\tilde{q}(\tilde{t}) = Q_\varepsilon(q(t)) = Q_\varepsilon(q(T_{-\varepsilon}(\tilde{t})))$

with T_ε and Q_ε smooth one-parameter subgroups ($T_0 = id$ $\quad T_{-\varepsilon} = T_\varepsilon^{-1}$ $\quad T_{\varepsilon+\varepsilon'} = T_\varepsilon \circ T_{\varepsilon'}$)

→ i.e. $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon$ makes sense

$$S[q] = \int_a^b L(q(t), \dot{q}(t), t)$$

$$\frac{d\tilde{q}}{d\tilde{t}}$$

$$\tilde{S}[\tilde{q}] = \int_{T_\varepsilon(a)}^{T_\varepsilon(b)} L(\tilde{q}(\tilde{t}), \dot{\tilde{q}}(\tilde{t}), \tilde{t})\, d\tilde{t} = \int_a^b \frac{dT_\varepsilon}{dt} L(\tilde{q}(T_\varepsilon(t)), \dot{\tilde{q}}(T_\varepsilon(t)), T_\varepsilon(t))\, dt$$

suppose that $\tilde{S}[\tilde{q}] = S[q]$ for all paths (even those that are not solutions of E-L eqs.) and for all choiches of a, b.

Then $\delta S := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{S}[\tilde{q}] = 0$

Some notation:

suppose that $\tilde{S}[\tilde{q}] = S[q]$ for all paths (even those that are not solutions of E-L eqs.) and for all choices of a, b.

Then $\delta S := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{S}[\tilde{q}] = 0$ → can use this as assumption

Some notation:

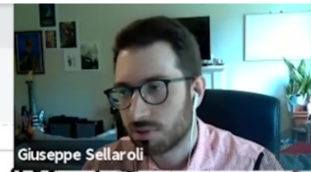
- $\delta t := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon$ (function of t)
- $\delta q := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Q_\varepsilon$ (function of q)

generators of T_ε and Q_ε (see Lie algebras later)

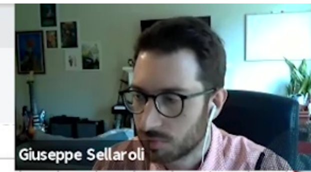
Let's apply $D := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}$ everywhere we can!

- $D \frac{dT_\varepsilon}{dt} = \frac{d}{dt} DT_\varepsilon = \frac{d}{dt}\delta t$
- $D\,\tilde{q}(\tilde{t}) = D\, Q_\varepsilon(q(t)) = \delta q(q(t))$

$\rightarrow = \dfrac{1}{\frac{d\tilde{t}}{dt}(t)} = \dfrac{1}{\frac{dT_\varepsilon}{dt}(t)}$



*noether.xopp - Xournal++



Some notation:

- $\delta t := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon$ (function of t)
- $\delta q := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Q_\varepsilon$ (function of q)

generators of T_ε and Q_ε (see Lie algebras later)

Let's apply $D := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}$ everywhere we can!

- $D\frac{dT_\varepsilon}{dt} = \frac{d}{dt} DT_\varepsilon = \frac{d}{dt}\delta t$
- $D\tilde{q}(\tilde{t}) = D Q_\varepsilon(q(t)) = \delta q(q(t))$
- $\frac{d\tilde{q}}{d\tilde{t}}(\tilde{t}) = \frac{d}{d\tilde{t}} Q_\varepsilon(q(t)) = \frac{\partial Q_\varepsilon}{\partial q}(q(t))\,\dot{q}(t)\,\frac{dt}{d\tilde{t}}(\tilde{t})$

$$\frac{dt}{d\tilde{t}}(\tilde{t}) = \frac{1}{\frac{d\tilde{t}}{dt}(t)} = \frac{1}{\frac{dT_\varepsilon}{dt}(t)}$$

$$\Rightarrow \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\dot{\tilde{q}}(\tilde{t}) = \frac{\partial}{\partial q} DQ_\varepsilon(q(t))\,\dot{q}(t)\,\frac{1}{\underbrace{\frac{dT_0}{dt}}_{=1}} + \underbrace{\frac{\partial}{\partial q}Q_0(q(t))}_{=1}\,\dot{q}(t)\,D\frac{1}{\frac{dT_\varepsilon}{dt}}$$

$$D\frac{1}{\frac{dT_\varepsilon}{dt}} = -\frac{d}{dt}\delta t$$

Page 1 of 2 Layer Layer1

Let's apply $D := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}$ everywhere we can:

- $D\,\frac{dT_\varepsilon}{dt} = \frac{d}{dt}\,DT_\varepsilon = \frac{d}{dt}\,\delta t$

double chain rule

- $D\,\tilde q(\tilde t) = D\,Q_\varepsilon(q(t)) = \delta q\,(q(t))$

$= \frac{1}{\frac{d\tilde t}{dt}(t)} = \frac{1}{\frac{dT_\varepsilon}{dt}(t)}$ $\Leftarrow$ derive of inverse function

- $\frac{d\tilde q}{d\tilde t}(\tilde t) = \frac{d}{d\tilde t}\,Q_\varepsilon(q(t)) = \frac{\partial Q_\varepsilon}{\partial q}(q(t))\,\dot q(t)\,\frac{dt}{d\tilde t}(\tilde t)$

$-\frac{d}{dt}\,\delta t$

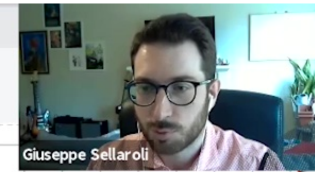
$\rightarrow$ chain rule

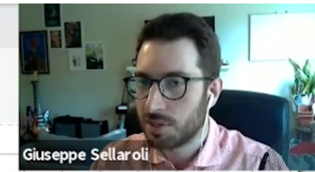
$$\Rightarrow \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\dot{\tilde q}(\tilde t) = \frac{\partial}{\partial q}DQ_\varepsilon(q(t))\;\dot q(t)\;\frac{1}{\underbrace{\frac{dT_0}{dt}}_{=1}} + \underbrace{\frac{\partial}{\partial q}Q_0(q(t))}_{=1}\;\dot q(t)\;D\,\frac{1}{\frac{dT_\varepsilon}{dt}}$$

$\varepsilon = 0$

$$\Rightarrow D\,\dot{\tilde q}(\tilde t) = \frac{\partial}{\partial q}\delta q\;\dot q(t) - \dot q(t)\,\frac{d}{dt}\,\delta t = \frac{d}{dt}\,\delta q - \dot q(t)\,\frac{d}{dt}\,\delta t$$

Now suppose that q satisfies $\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = 0$





Now suppose that q satisfies $\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = 0}$ EL eqs

$$\underbrace{0 = D\tilde S[\tilde q]}_{\delta S = 0} = \int_a^b dt\, D\left[\frac{dT_\varepsilon}{dt} L(\tilde q(\tilde t), \dot{\tilde q}(\tilde t), \tilde t)\right]$$

$D\frac{dT_\varepsilon}{dt}$

$$= \int_a^b dt \left\{ \frac{d\delta t}{dt} L(\underbrace{q,\dot q,t}_{\varepsilon=0}) + \underbrace{\frac{dT_0}{dt}}_{=1} D L(\tilde q(\tilde t), \dot{\tilde q}(\tilde t),\tilde t)\right\}$$

$$D L(\tilde q(\tilde t), \dot{\tilde q}(\tilde t),\tilde t) = \frac{\partial L}{\partial t} D\tilde t + \frac{\partial L}{\partial q} D\tilde q(\tilde t) + \frac{\partial L}{\partial \dot q} D\dot{\tilde q}(\tilde t)$$

$$= \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\left(\frac{d}{dt}\delta q - \dot q \frac{d}{dt}\delta t\right)$$

$$= \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q}\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q\right) - \delta q \frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\dot q\,\delta t\right) + \delta t \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\dot q\right)$$

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] + \frac{\partial L}{\partial t}\delta t + \delta t\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\dot q + \delta t \frac{\partial L}{\partial \dot q}\ddot q$$

*noether.xopp - Xournal++

$$D L(\tilde q(\tilde t), \dot{\tilde q}(\tilde t), \tilde t) = \frac{\partial L}{\partial t} D\tilde t + \frac{\partial L}{\partial q} D\tilde q(\tilde t) + \frac{\partial L}{\partial \dot q} D\dot{\tilde q}(\tilde t)$$

chain rule again

$$A\frac{dB}{dt} = \frac{d}{dt}(AB) - B\frac{dA}{dt}$$

$$= \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\left(\frac{d}{dt}\delta q - \dot q \frac{d}{dt}\delta t\right)$$

$$= \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q}\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q\right) - \delta q \frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\dot q\,\delta t\right) + \delta t\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\dot q\right)$$

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] + \frac{\partial L}{\partial t}\delta t + \delta t\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\dot q + \delta t\frac{\partial L}{\partial \dot q}\ddot q$$

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] + \delta t\left[\frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot q + \frac{\partial L}{\partial \dot q}\ddot q\right]$$

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] + \delta t\frac{d}{dt}L(q(t),\dot q(t),t)$$

$$\Rightarrow\ 0 = \int^b \left\{ L(q(t),\dot q(t),t)\frac{d}{dt}\delta t + \delta t\frac{d}{dt}L(q(t),\dot q(t),t) + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right]\right\} dt$$

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}\left(\delta q - \dot q\,\delta t\right)\right] + \frac{\partial L}{\partial t}\delta t + \delta t\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\dot q + \delta t\,\frac{\partial L}{\partial \dot q}\ddot q$$

EL

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}\left(\delta q - \dot q\,\delta t\right)\right] + \delta t\,\underbrace{\left[\frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot q + \frac{\partial L}{\partial \dot q}\ddot q\right]}$$

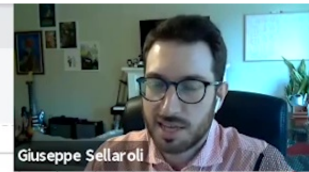
$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}\left(\delta q - \dot q\,\delta t\right)\right] + \delta t\,\frac{d}{dt}L(q(t),\dot q(t),t)$$

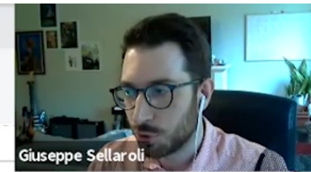
$$\Rightarrow \overset{\delta S}{0} = \int_a^b\left\{\underbrace{L(q(t),\dot q(t),t)\frac{d}{dt}\delta t + \delta t\frac{d}{dt}L(q(t),\dot q(t),t)}_{\text{product rule}} + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}\left(\delta q - \dot q\,\delta t\right)\right]\right\}dt$$

$$= \int_a^b \frac{d}{dt}\left[L\,\delta t + \frac{\partial L}{\partial \dot q}\left(\delta q - \dot q\,\delta t\right)\right]dt \qquad \text{zero for all } a, b$$

$$\Rightarrow 0 = \frac{d}{dt}\left[L\,\delta t + \frac{\partial L}{\partial \dot q}\left(\delta q - \dot q\,\delta t\right)\right] \quad \text{or} \quad L\,\delta t + \frac{\partial L}{\partial \dot q}\left(\delta q - \dot q\,\delta t\right) \ \underline{\text{conserved}}$$

<u>Note:</u> for this to work we need to check that $\tilde S[\tilde q] = S[q]$ or $D\tilde S[\tilde q] = 0$





$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] + \delta t \,\frac{d}{dt} L(q(t),\dot q(t),t)$$

δS

$$\Rightarrow 0 = \int_a^b \Big\{ \underbrace{L(q(t),\dot q(t),t)\,\frac{d}{dt}\delta t + \delta t\,\frac{d}{dt}L(q(t),\dot q(t),t)}_{\text{product rule}} + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right]\Big\}\,dt$$

$$= \int_a^b \frac{d}{dt}\left[L\,\delta t + \frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right]dt$$

zero for all a, b

$$\Rightarrow 0 = \frac{d}{dt}\left[L\,\delta t + \frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] \quad \text{or} \quad \boxed{L\,\delta t + \frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)} \ \text{conserved}$$

Note: for this to work we need to check that $\tilde S[\tilde q] = S[q]$ or $D\tilde S[\tilde q] = 0$

whithout using E-L eqs!

Exercise

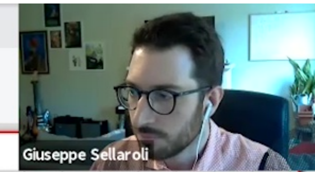
$$L(q,\dot q,t) = \frac{1}{2}\left(m\dot q^2 - \kappa q^2\right) e^{\alpha t} \quad (\alpha \in \mathbb{R}) \qquad T_\varepsilon(t) = t+\varepsilon \qquad Q_\varepsilon(q) = q\, e^{-\frac{\varepsilon\alpha}{2}}$$

$$\tilde t = T_\varepsilon(t)$$

$$\tilde q(\tilde t) = Q_\varepsilon(q(t))$$

1. Show that T_ε and Q_ε are one-parameter subgroups
2. Show that $\delta S = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde S[\tilde q] = 0$ (without using E-L eqs)
3. Find the associated conserved quantity
4. Find the Euler-Lagrange equations
5. Show that the conserved quantity is indeed conserved if the E-L eqs hold

Note: for point (2) use the fact that



Exercises.xopp - Xournal++

File Edit View Navigation Journal Tools Plugin Help

Sans Regular 19

Giuseppe Sellaroli

① Show that T_ε and Q_ε are one-parameter subgroups

② Show that $\delta S = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{S}[\tilde{q}] = 0$ (without using E-L eqs)

③ Find the associated conserved quantity

④ Find the Euler-Lagrange equations

⑤ Show that the conserved quantity is indeed conserved if the E-L eqs hold

<u>Note</u>: for point ② use the fact that

$$\delta S = \int_a^b \left[L(q,\dot{q},t) \frac{d}{dt}\delta t + \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\left(\frac{d}{dt}\delta q - \dot{q}\frac{d}{dt}\delta t \right) \right] dt$$

and substitute L, δt, δq

Page 1 of 1 Layer Layer 1